An Algebraic Method for Eigenvalue Problems

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Abstract

If for a linear symmetric (unbounded) operator F and a linear operator S holds

$$FSq(F) = Sp(F)$$

on the span of the eigenspaces of F for two polynomials p and q, then S is a raising operator. This means roughly that if $Fy_i = \lambda_i y_i$ then $y_{i+1} := Sy_i$ is an eigenvector of F with eigenvalue $\lambda_{i+1} = \frac{p(\lambda_i)}{q(\lambda_i)}$. Also an inverse statement of this kind holds true. We use this technique in order to discuss several eigenvalue problems. Similarly, we consider lowering operators T and discuss commutator relations between S and T.

1 Introduction

The relation between a linear operator and its spectrum has many aspects which have been extensively investigated: e.g. it is well known that the eigenvalues of a matrix depend in an unstable way on the coefficients (see e.g. [12]). Another question is how far the spectrum determines the operator (see [8], [5], [2], [6]). Other aspects are important from the physical point of view:

Example 1 (Dirac [1]) As an example how algebraic methods may help to solve eigenvalue problems we recall the factorization method for the operator φ of the harmonic oscillator

$$\varphi := -D^2 + x^2$$

where $D := \frac{d}{dx}$ with independent variable $x \in \mathbb{R}$. We consider φ as an unbounded symmetric operator defined on the Schwartz space $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. For the operators $S := -D + x$ and $T := D + x$ we have

$$ST = \varphi - I \quad \text{and} \quad TS = \varphi + I, \tag{1}$$

where I denotes the identity and hence

$$\varphi S - S\varphi = 2S \tag{2}$$

and

$$T\varphi - \varphi T = 2T\,. \tag{3}$$

Observe that

$$Ty = 0$$

has the (up to multiplication with a constant) unique solution $y_1(x) = \exp(-\frac{x^2}{2})$ in $\mathscr{S}(\mathbb{R})$ and hence from (1) it follows that y_1 is an eigenfunction of φ with eigenvalue $\lambda_1 = 1$. Then we have by (2) for $y_2 := Sy_1 \in \mathscr{S}(\mathbb{R})$

$$\begin{aligned} \varphi y_2 &= \varphi S y_1 = (S\varphi + 2S)y_1 = \\ &= (\lambda_1 + 2)y_2\,, \end{aligned}$$

i.e. y_2 is an eigenfunction for eigenvalue $\lambda_2 = \lambda_1 + 2$. By iteration we obtain a sequence $y_i \in \mathscr{S}(\mathbb{R})$ of eigenfunctions generated by $y_{i+1} = Sy_i$ with eigenvalues λ_i which form an arithmetic sequence $\lambda_{i+1} = \lambda_i + 2$.

On the other hand, if y denotes *any* eigenfunction of φ, i.e. $\varphi y = \lambda y$, then, by (3), $\varphi T y = (\lambda - 2)y$. Hence, either $Ty = 0$ and thus $y = y_1$, or Ty is an eigenfunction of φ with eigenvalue $\lambda - 2$. Since the spectrum of the positive definite operator φ is positive, iteration of this argument shows, that $\lambda = \lambda_i$ for some i. Hence, no other eigenvalues than the arithmetic sequence λ_i that we found exist. By the same argument we have that, since the kernel of T consists of the one dimensional eigenspace spanned by y_1, all eigenspaces must be one dimensional. Of course, the eigenfunctions y_i are L^2-orthogonal and in fact they form an orthogonal base of $L^2(\mathbb{R})$. ○

For a description of the general factorization method see [4], [13], [7], [3] and [9]–[11].

The aim of this paper is to present an algebraic method which has some common features with the factorization method and which allows to generate eigenfunctions and eigenvalues in some more general situations.

2 Theoretical foundation of the method

The motivation in this section is to treat generalizations of the commutator equation (2) (see also [14]–[16]):

Let L be a complex vector space equipped with a non-degenerate sesquilinear form $(\cdot,\cdot)$, i.e. for all $x, y, z \in L$ and all $\lambda \in \mathbb{C}$ holds

$$\begin{aligned} (x + \lambda y, z) &= (x, z) + \lambda (y, z) \\ (x, y) &= \overline{(y, x)} \\ x \neq 0 &\Longrightarrow (x, x) \neq 0 \end{aligned}$$

Remark: The reader may think of a complex Hilbert space. Notice, that if $(\cdot,\cdot)$ is continuous on finite dimensional subspaces, then the sesquilinear form is either positive or negative definite, and hence, up to a change of the sign if necessary, an inner product: to see this consider the real function $\lambda \mapsto (\lambda y + (1-\lambda)x, \lambda y + (1-\lambda)x)$. However, we will not assume this additional property.

Let $F : L \supset D(F) \to L$ be a linear (bounded or unbounded) operator, where the domain $D(F)$ of F is a linear subspace of L. F is supposed to be symmetric, i.e. $(Fx, y) = (x, Fy)$ holds for all $x, y \in D(F)$. By considering (Fx, x), it follows as usual that the eigenvalues of F are real. Moreover, two eigenvectors x and y belonging to two distinct eigenvalues are orthogonal with respect to the given sesquilinear form, i.e. $(x, y) = 0$; in multidimensional eigenspaces we will always choose an orthogonal basis.

Remark: If F is positive with respect to $(\cdot,\cdot)$, i.e. $(Fx, x) \geq 0$ for all $x \in D(F)$, and if $(\cdot,\cdot)$ is positive, i.e. $(x, x) \geq 0$ for all $x \in L$, then the eigenvalues of F are positive real numbers. But this is not assumed in the sequel.

Now, let $\{y_i\}_{i \in \mathbb{N}_0}$ be a sequence of normed eigenvectors of F such that either

(i) $(y_i, y_j) = \delta_{ij}$

or

(ii) $(y_i, y_j) = 0$ for $y_i \neq y_j$ and $y_i = y_j \iff i = j \pmod n$ for a given $n \in \mathbb{N}$.

(With the second case we take care of the finite dimensional situation.) For $i \in \mathbb{N}_0$ let λ_i denote the eigenvalue to y_i, i.e. $Fy_i = \lambda_i y_i$. Let $A = \operatorname{span}\{y_i \,:\, i \in \mathbb{N}_0\}$ be the linear subspace of all finite linear combinations of eigenvectors y_i.

Suppose that there exist polynomials p, q such that

(1) $q(\lambda_i) \neq 0$ for all $i \in \mathbb{N}_0$,

(2) $\lambda_{i+1} q(\lambda_i) = p(\lambda_i)$ for all $i \in \mathbb{N}_0$.

Finally, let $\{\alpha_i\}_{i\in\mathbb{N}_0}$ and $\{\beta_i\}_{i\in\mathbb{N}_0}$ be sequences of complex numbers, which in case (ii) are supposed to satisfy $\alpha_i = \alpha_j \iff i = j \pmod n$ and $\beta_i = \beta_j \iff i = j \pmod n$.

Theorem 1 *With the notations above the following is true: There exists a linear operator $S : A \to A$ such that*

(a) $Sy_i = \alpha_{i+1} y_{i+1}$ *for all* $i \in \mathbb{N}_0$.

(b) $FSq(F) = Sp(F)$ *on* A.

(c) If $|\alpha_i| = 1$ *for all* $i \in \mathbb{N}_0$, *then* S *is an isometry.*

(d) S *is nilpotent if,*

- *in case (i),* $\exists \alpha_k = 0 \wedge \exists n : \forall i(\alpha_i = 0 \implies \min\{j > i \,:\, \alpha_j = 0\} < i + n)$.
- *in case (ii), at least one* $\alpha_i = 0$.

Furthermore there exists a linear operator $T : A \to A$ such that

(e) $Ty_{i+1} = \beta_{i+1} y_i$ *for all* $i \in \mathbb{N}_0$ *and* $Ty_0 = \begin{cases} 0 & \text{in case (i),} \\ \beta_0 y_{n-1} & \text{in case (ii).} \end{cases}$

(f) y_i *is an eigenvector of* TS *to eigenvalue* $\alpha_{i+1}\beta_{i+1}$ *for all* $i \in \mathbb{N}_0$. y_i *is an eigenvector of* ST *to eigenvalue* $\alpha_i\beta_i$ *for all* $i \geq 1$ *and in case (ii) also for* $i = 0$.

(g) S *and* T *are adjoint in the sense that* $(Sx, y) = (x, Ty)$ *holds on* A *provided* $\alpha_i = \beta_i$ *for all* $i \in \mathbb{N}_0$.

Proof
Define $S : A \to A$ as

$$S : \sum_i c_i y_i \mapsto \sum_i c_i \alpha_{i+1} y_{i+1} \,. \tag{4}$$

Notice, that $y_i \neq y_j \implies (y_i, y_j) = 0$, and hence that S is well defined. (a), (c) and (d) follow immediately. Because of the linearity, it is sufficient to show (b) for all y_i:

$$\begin{aligned} FSq(F)y_i &= FSq(\lambda_i)y_i \\ &= q(\lambda_i)F\alpha_{i+1}y_{i+1} \\ &= q(\lambda_i)\alpha_{i+1}\lambda_{i+1}y_{i+1} \\ &= p(\lambda_i)\alpha_{i+1}y_{i+1} \\ &= p(\lambda_i)Sy_i = Sp(F)y_i \,. \end{aligned}$$

The operator $T : A \to A$ is defined by

$$T : \sum_i c_i y_i \mapsto \sum_i c_i \beta_i y_{i-1}$$

where we set $y_{-1} := 0$ in case (i) and $y_{-1} := y_{n-1}$ in case (ii) respectively. Then (f) and (g) follow by simple calculation. $\square$

Remark: If L is a Hilbert space we have for (4) explicitly

$$Sy = \sum \frac{\langle y, y_n \rangle}{\|y_n\|^2} \alpha_{n+1} y_{n+1}. \tag{5}$$

Note that for $y \in A$ only finitely many terms in the sum are non-zero. If S is bounded then (5) defines the continuous closure of S on $\bar{A}$. Moreover if the inner product on the Hilbert space $L = L^2(\Omega, d\mu)$, $\Omega \subset \mathbb{R}^n$, is given by

$$\langle f, g \rangle = \int_\Omega fg \, d\mu$$

then (5) is just

$$Sy(x) = \int_\Omega G(x,z)\, y(z)\, d\mu(z) \tag{6}$$

with a kernel

$$G(x,z) = \sum \frac{y_i(z) y_{i+1}(x)}{\|y_i\|^2} \alpha_{i+1}$$

provided the α_n respect a suitable growth condition such that summation and integration commute. All this is of course true without most of the assumptions made for Theorem 1, in fact, only an orthogonal base of eigenvectors y_i has to be given.

Now we consider an inverse form of Theorem 1:

Let $\{\lambda_i\}_{i \in I}$ ($I = \mathbb{N}_0$ or $I = \{1, 2, \ldots, m\}$) be the set of eigenvalues of F (F as above) and let Y_i denote the eigenspace to λ_i. Let $A = \operatorname{span}\{Y_i : i \in I\}$.

Theorem 2 *With the above notations the following holds: If p and q are polynomials such that q does not vanish in eigenvalues of F and if*

$$FSq(F) = Sp(F)$$

holds on A for some linear operator $S : A \to A$ then there exist linear subspaces $A_j = \operatorname{span}\{Y_k : k \in I_j\}$ of A, such that $A = \times_j A_j$ and

$$S|_{A_j} : A_j \to A_j \quad \textit{for all } j.$$

Furthermore, if $Y_k \subset A_j$ then there holds $S(Y_k) \subset Y_m$ for $Y_m \subset A_j$ with

$$\lambda_m = \frac{p(\lambda_k)}{q(\lambda_k)} \tag{7}$$

provided $S(Y_k) \neq 0$.

The following figure shows a typical situation:

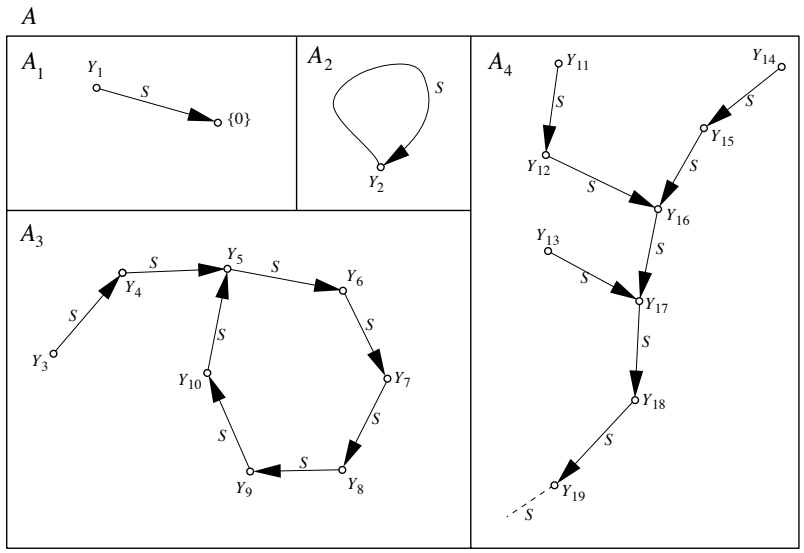

Figure 1: Interpretation. Suppose for the moment that all eigenspaces Y_k are one-dimensional. Then, picking up $y_{14} \neq 0$ in Y_{14} we see that iterated application of S generates an infinite sequence $y_{i+1} := Sy_i \in Y_{i+1}$ of eigenfunctions whose eigenvalues are given by (7).

Proof
For $y \in Y_k$ we have $q(\lambda_k)F(Sy) = p(\lambda_k)(Sy)$. Thus, if $Sy \neq 0$, Sy is an eigenvector to the eigenvalue $\frac{p(\lambda_k)}{q(\lambda_k)}$ and hence $Sy \in Y_m$ for some m only depending on k. We say then, that Y_k and Y_m are in relation. The transitive hull of this relation is an equivalence relation with equivalence classes $\{Y_i : i \in I_j\}$. Thus the assertions follow. □

Remark: Most of the theory remains valid for an arbitrary linear operator F in a normed vector space L without a sesquilinear form (just replace orthogonality by linear independence). Since the theorems do not require a lot of structure the field of application is wide open. Notice also that the assertions are purely algebraic such that every topological aspect will depend on the concrete problem one works with. On the other hand we will see in the applications below that this low level tool may be very useful in practice.

3 Examples

Let us review some familiar examples in the light of the previous section.

3.1 The Laguerre polynomials

Let H denote the Hilbert space $L^2([0,\infty[; e^{-x}dx)$. Consider the differential operator

$$F := -D(xD) + xD$$

defined on

$$D(F) := \{y \in C^\infty([0,\infty[: \exists c, n, |y''(x)| < c + x^n \text{ on } [0,\infty[\} \subset H$$

with values in H. An easy calculation shows that F is symmetric and positive (and of course F is not bounded). If we define the operator S on $D(F)$ with values in H by

$$Sy(x) := y(x) - \int_0^x y(\xi)\, d\xi$$

we have

$$FS - SF = S \tag{8}$$

on $D(F)$. The first eigenvalue of F is $\lambda_0 = 0$ with eigenfunction $y_1(x) = 1$. Observing (8) we choose $p(x) = 1 + x$ and $q(x) = 1$ and obtain, by applying Theorem 2, an infinite sequence of orthogonal eigenfunctions $y_{n+1} = Sy_n$ with eigenvalues $\lambda_{n+1} = p(\lambda_n)/q(\lambda_n) = \lambda_n + 1 = n + 1$. One can check that the spectrum of F in fact is $\mathbb{N} \cup \{0\}$ and that no other eigenfunctions of F exist. The y_n are known as the Laguerre polynomials and form an orthogonal base of H.

Remark: The operator $T : D(F) \subset H \to H$ defined as

$$Ty(x) := \int_0^x e^{x-\xi} y'(\xi)\, d\xi$$

satisfies $FT - TF = -T$ on $D(F)$. Thus with $p(x) = -1 + x$ and $q(x) = 1$ we conclude by Theorem 2 that $T : y_{n+1} \mapsto \mu_{n+1} y_n$. And in fact, $\mu_n \neq 0$ for all $n > 0$ and $\mu_0 = 0$.

3.2 The vibrating string

The vibrating string is described by

$$-y'' = \lambda y \qquad \text{on } [0, \pi] \tag{9}$$

$$y(0) = y(\pi) = 0. \tag{10}$$

This is the eigenvalue problem for the unbounded symmetric differential operator $F = -D^2$ defined on the Sobolev space $D(F) = H_0^{2,2}([0, \pi]) \subset L^2([0, \pi]) =: H$ with values in H. Of course, the eigenfunctions of F are $y_n = \sin(nx)$ with eigenvalues $\lambda_n = n^2$, $n \in \mathbb{N}$. Here the uniquely determined operator S which maps y_n to y_{n+1} is most easily described in terms of the Fourier transform $\mathscr{F}$:

$$\begin{aligned} \mathscr{F} : L^2([0, \pi]) &\to \ell^2 \\ f &\mapsto (a_n)_{n \in \mathbb{N}} \end{aligned}$$

with

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx$$

and

$$\begin{aligned} \mathscr{F}^{-1} : \ell^2 &\to L^2([0, \pi]) \\ (a_n)_{n \in \mathbb{N}} &\mapsto f(x) = \sum_{n \geq 1} a_n \sin(nx). \end{aligned}$$

Let $s : \ell^2 \to \ell^2$ be the right shift operator, i.e. $(sa)_n := a_{n-1}$ for $n > 1$ and $(sa)_1 = 0$. Then

$$S := \mathscr{F}^{-1} s \mathscr{F},$$

or explicitly

$$Sy(x) = \frac{2}{\pi} \sum_{n \geq 1} \int_0^{\pi} y(t) \sin(nt) \sin((n+1)x) dt,$$

which is essentially (5). (Note that since S is here an isometry the closure of S in H is well defined.) Another way to describe S is based on the elementary formula $\sin((n+1)x) = \sin(nx)\cos(x) + \cos(nx)\sin(x)$ and the fact that sin and cos are conjugate harmonic functions which allows to express one in terms of the other by using Poissons formula. In fact one obtains

$$\cos(nx) = \frac{1}{\sqrt{2\pi}} \, \mathrm{pV} \int_0^{2\pi} \sin(nt) \cot(\frac{t-x}{2}) \, dt$$

(pV denotes the principle value of the singular integral) such that we can define

$$Sy(x) = y(x)\cos(x) + \frac{\sin(x)}{\sqrt{2\pi}} \, \mathrm{pV} \int_0^{2\pi} y(t) \cot(\frac{t-x}{2}) \, dt \, .$$

Given this operator, one verifies easily that

$$FS^2 - 2SFS + S^2F = kS^2$$

with $k = 2$ holds, and that $Sy_1 = y_2$. Thus, by an obvious modification of Theorem 2, we recover that

$$\lambda_{n+1} - 2\lambda_n + \lambda_{n-1} = k.$$

3.3 Mixed arithmetic sequences as eigenvalues

We consider the operator

$$F := -D^2 + \Big(\frac{k^2x^2}{16} - \frac{2\gamma}{x^2}\Big) I,$$

where I denotes the identity, and where $k > 0$ and $\gamma \leq \frac{1}{8}$ are fixed real constants. (For a physical interpretation of this operator see [4].) D denotes the distributional derivative and x is the independent variable. The spaces are chosen to be as follows:

$$F : D(F) \subset L^2(\mathbb{R}, |x|^r dx) \to L^2(\mathbb{R}, |x|^r dx) := H$$

with

$$D(F) := \{ f \in H^{2,2}(\mathbb{R}, |x|^r dx) \mid x^{\pm 2} f \in L^2(\mathbb{R}, |x|^r dx), xf' \in L^2(\mathbb{R}, |x|^r dx) \}.$$

The weight $r \geq 0$ is chosen below. Furthermore, we define the operators $S : D(F) \subset H \to H$ and $T : D(F) \subset H \to H$ by

$$\begin{aligned} S &= D^2 - \frac{k}{2} xD + \Big(\frac{k^2x^2}{16} + \frac{2\gamma}{x^2} - \frac{k}{4}\Big) I \\ T &= D^2 + \frac{k}{2} xD + \Big(\frac{k^2x^2}{16} + \frac{2\gamma}{x^2} + \frac{k}{4}\Big) I. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} FS - SF &= kS \\ FT - TF &= -kT \end{aligned}$$

on $D(F)$ in the sense of distributions. By Theorem 2 it follows that S is a raising and T a lowering operator. Since F is a positive operator, the eigenvalues of F are non-negative and hence, if there

exists an eigenvalue at all, there must be one, say y_0, which satisfies $Ty_0 = 0$. Adding the two equations $Fy_0 = \lambda_0 y_0$ and $Ty_0 = 0$ gives:

$$\frac{k}{2}xy_0' + \Big(\frac{k^2x^2}{8} + \frac{k}{4} - \lambda_0\Big)y_0 = 0$$

and hence

$$-\frac{y_0'}{y_0} = \frac{kx}{4} + \frac{k - 4\lambda_0}{2kx}.$$

This differential equation is easily integrated, the solution is (formally)

$$y_0 = x^{-\frac{k-4\lambda_0}{2k}} e^{-\frac{kx^2}{8}}.$$

Writing $Fy_0 = \lambda_0 y_0$ explicitly, we find the following quadratic equation for λ_0:

$$\lambda_0^2 - k\lambda_0 + \frac{k^2}{16}(3 + 8\gamma) = 0.$$

Solving this yields

$$\lambda_{0,i} = \frac{k}{4}(2 \pm \sqrt{1 - 8\gamma}),$$

with $i = 1, 2$. So, formally, we have the following two eigenfunctions of F:

$$y_{0,i}(x) = e^{-kx^2/8} x^{(1\pm\sqrt{1-8\gamma})/2}.$$

If we choose $r > 2 + \sqrt{1 - 8\gamma}$, it follows that these functions belong to $D(F)$. It is easy to see that $y_{j,i} := S^j y_{0,i} \not\equiv 0$, $j \in \mathbb{N}$, $i = 1, 2$, belong to $D(F)$ and are of the form $y_{0,i}p_j$, where p_j are polynomials of increasing degree. Hence, by Theorem 2, we have two mixed arithmetic sequences of eigenvalues:

$$\lambda_{j,i} = \lambda_{0,i} + jk, \qquad j \in \mathbb{N},\ i = 1, 2.$$

(Notice, that $\lambda_{0,1} - \lambda_{0,2}$ may be a multiple of k and hence that multiple eigenvalues are possible.)

Remark: The ground state eigenvalue can alternatively be obtained from the relation

$$ST = F^2 - kF + (\frac{3}{16} + \frac{\gamma}{2})k^2 I$$

by applying it to y_0: This immediately yields the quadratic equation for λ_0. In the very same way it is possible to deal with raising and lowering operators of higher order. This leads to higher order equations for the ground state eigenvalue, and hence to the mixing of more then two arithmetic sequences in the spectrum.

3.4 Connection to partial differential equations

Suppose $F := a(x)D^2 + b(x)D + c(x)$ is defined on $D(F) = H_0^{2,2}(I)$ with values in $L^2(I)$, where $I \subset \mathbb{R}$ is a closed interval and a, b, c are given smooth functions on I. Suppose that S is given on $D(F)$ by

$$Sf(x) := \int_I f(s)g(s, x)ds$$

for a kernel g satisfying $g(x, y) = 0$ whenever $x \in \partial I$ or $y \in \partial I$, and that

$$FS - SF = kS \tag{11}$$

holds on $D(F)$ for a constant $k > 0$. Expanding (11), we obtain

$$\begin{aligned} 0 &= \int_I f(s)((-k + c(x) - c(s))g(s,x) + b(x)g_x(s,x) + a(x)g_{xx}(s,x))ds \\ &\quad \underbrace{-\int_I b(s)g(s,x)f'(s)ds}_{=\int_I f(s)\frac{\partial}{\partial s}(b(s)g(s,x))ds} - \underbrace{\int_I a(s)g(s,x)f''(s)ds}_{=\int_I f(s)\frac{\partial 2}{\partial s^2}(a(s)g(s,x))ds} \end{aligned}$$

for all $f \in H_0^{2,2}(I)$. Hence, it is equivalent to (11), that $g \in H_0^{2,2}(I \times I)$ is a nontrivial solution of the following partial differential equation

$$\begin{aligned} 0 =& (-k + c(x) - c(s) + b'(s) - a''(s))g(s,x) + b(x)g_x(s,x) + \\ &+ a(x)g_{xx}(s,x) + (b(s) - 2a'(s))g_s(s,x) - a(s)g_{ss}(s,x) \qquad \text{on } I \times I. \end{aligned}$$

In other words: If g is a solution of the previous two dimensional eigenvalue problem, then the integral transformation S with kernel g is a raising operator for the one dimensional eigenvalue problem for the operator F.

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