A simple proof of Shapiro's Theorem

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Abstract
We prove Shapiro's Theorem by applying the well known bijection between Catalan trees and trivalent plane rooted trees, and using a simple symmetry argument.

Key words: Catalan trees, trivalent plane rooted trees, terminal vertices

1 Shapiro's Theorem

For $n \in \mathbb{N}_0$, let $C_n$ denote the set of planted planar trees\footnote{i.e. planar trees with a root $r$ of degree 1} with $n + 1$ edges, sometimes called Catalan trees. Figure 1 shows $C_3$. Terminal edges which are not incident with the root are called leaves. Shapiro observed the following:

Theorem
For $n > 0$ exactly half of the edges of the planted planar trees in $C_n$ are leaves.

Shapiro presented a proof of this result using generating functions in [4], but finding it so attractive, and believing that there must be other, neater, more insightful proofs, offered it also as a problem in The American Mathematical Monthly [3]. A detailed history of Shapiro’s Theorem, and additional bibliographic remarks on Catalan Problems can be found in [2].

2 A simple proof of Shapiro’s Theorem

We recall that there is a bijection between the sets $C_n$ of Catalan trees and $T_n$, the sets of planar rooted trivalent trees with $n + 1$ leaves (Figure 2 shows $T_3$). This bijection between $C_n$ and $T_n$ can be described as follows: First we bring a trivalent rooted planar tree in a special position—starting from the root, all edges run from bottom to top or from left to right. Then we contract the
Figure 2: $T_3 = \{\text{trivalent planar rooted trees with 4 leaves}\}$

Figure 3: Bijection between $T_n$ and $C_n$

horizontal edges and obtain the corresponding Catalan tree. See Figure 3 for convenience, and [1] for how to find this bijection. So, in particular, $|T_n| = |C_n|^2$.

The argument of the proof is now simply the following: For $n > 0$ we observe that, by symmetry, just as many leaves in $T_n$ are oriented to the right as to the left. Now, since exactly the edges that go to the right are contracted, there are $\frac{|T_n|(n+1)}{2}$ leaves in $C_n$. This is, indeed, half of the $|C_n|(n+1)$ edges in $C_n$!

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References


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And of course $|C_n|$ is the $n$-th Catalan number.