A simple proof of Shapiro's Theorem

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Abstract

We prove Shapiro's Theorem by applying the well known bijection between Catalan trees and trivalent plane rooted trees, and using a simple symmetry argument.

Key words: Catalan trees, trivalent plane rooted trees, terminal vertices

1 Shapiro's Theorem

For $n \in \mathbb{N}_0$, let C_n denote the set of planted planar trees[1] with $n+1$ edges, sometimes called *Catalan trees*. Figure 1 shows C_3. Terminal edges which are not incident with the root are called *leaves*. Shapiro observed the following:

Theorem *For $n > 0$ exactly half of the edges of the planted planar trees in C_n are leaves.*

Shapiro presented a proof of this result using generating functions in [4], but finding it so attractive, and believing that there must be other, neater, more insightful proofs, offered it also as a problem in *The American Mathematical Monthly* [3]. A detailed history of Shapiro's Theorem, and additional bibliographic remarks on Catalan Problems can be found in [2].

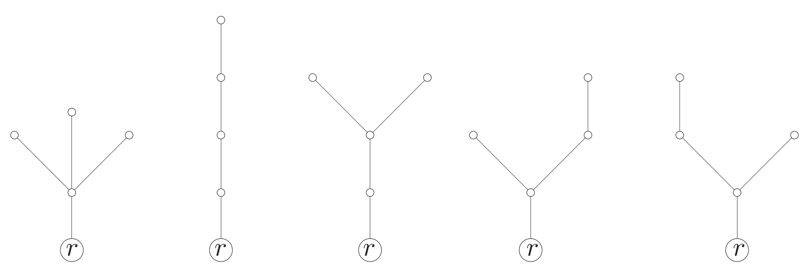

Figure 1: $C_3 = \{$Catalan trees with 4 edges$\}$. 10 among the altogether 20 edges are leaves.

2 A simple proof of Shapiro's Theorem

We recall that there is a bijection between the sets C_n of Catalan trees and T_n, the sets of planar rooted trivalent trees with $n+1$ leaves (Figure 2 shows T_3). This bijection between C_n and T_n can be described as follows: First we bring a trivalent rooted planar tree in a special position—starting from the root, all edges run from bottom to top or from left to right. Then we contract the

[1] i.e. planar trees with a root r of degree 1

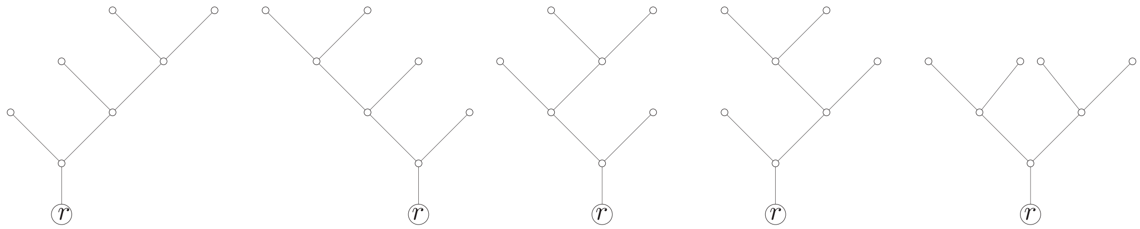

Figure 2: $T_3 = \{$trivalent planar rooted trees with 4 leaves$\}$

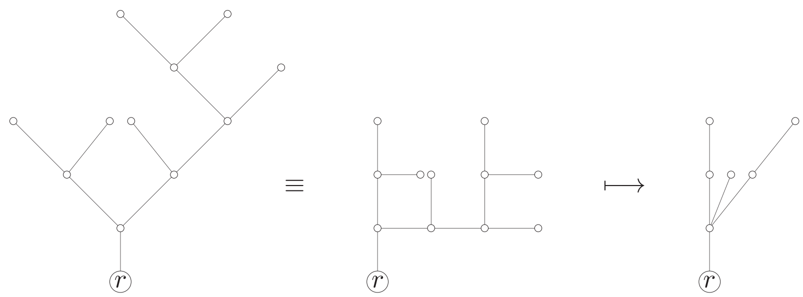

Figure 3: Bijection between T_n and C_n

horizontal edges and obtain the corresponding Catalan tree. See Figure 3 for convenience, and [1] for how to *find* this bijection. So, in particular, $|T_n| = |C_n|$.[2]

The argument of the proof is now simply the following: For $n > 0$ we observe that, by symmetry, just as many leaves in T_n are oriented to the right as to the left. Now, since exactly the edges that go to the right are contracted, there are $\frac{|T_n|(n+1)}{2}$ leaves in C_n. This is, indeed, half of the $|C_n|(n+1)$ edges in C_n!

3 Acknowledgment

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References

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[2] And of course $|C_n|$ is the n-th Catalan number.