Packings in Complete Graphs

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Abstract

We deal with the concept of packings in graphs, which may be regarded as a generalization of the theory of graph design. In particular we construct a vertex- and edge-disjoint packing of K_n (where $\frac{n}{2}$ mod 4 equals 0 or 1) with edges of different cyclic length. Moreover we consider edge-disjoint packings in complete graphs with uniform linear forests (and the resulting packings have special additional properties). Further we give a relationship between finite geometries and certain packings which suggests interesting questions.

1 Introduction

In geometry the concept of packing may be described as follows: Given a closed set $A \subset \mathbb{R}^n$ and a family $\{B_i\}_{i\in\Lambda}$ of closed subsets of A, e.g. $A = \mathbb{R}^2$ and $B_{x,r} = \{y \in \mathbb{R}^2 : |x-y| \leq r\}$, $(x,r) \in \mathbb{R}^2 \times \mathbb{R}_+$. A packing in A by the family $\{B_i\}_{i\in\Lambda}$ is an almost disjoint subset $\{B_i\}_{i\in\lambda} \subset \{B_i\}_{i\in\Lambda}$, i.e. $B_i \cap B_j$ is a zero-set in $\mathbb{R}^n$ for $i,j \in \lambda$, $i \neq j$. The density σ_λ of a packing is defined by $\sigma_\lambda = \frac{1}{\mu(A)} \sum_{i\in\lambda} \mu(B_i)$ if A has finite volume $\mu(A)$ and else $\sigma_\lambda = \lim_j \frac{1}{\mu(A_j)} \sum_{i\in\lambda} \mu(B_i \cap A_j)$, where the family $\{A_j\}_{j\in\mathbb{N}}$ of subsets of A of finite measure is exhausting A in a regular way. The typical question is to ask for the densest packing under eventual some restrictions on the admissible subset $\{B_i\}_{i\in\lambda}$: e.g. the densest packing in the plane $\mathbb{R}^2$ by circles of radius 1 (see [9]) or the densest packing in the unit square by ten circles of equal radius (see [7]).

It is known, that the concept of geometric packing has discrete analogues (see [10]). Here we deal with packings in (finite) graphs: Given a (finite) graph $G = (V,E)$, V the set of vertices and E the set of edges, and a family $\{B_i\}_{i\in\Lambda}$ of partial subgraphs $B_i = (V_i, E_i)$ of G. A packing in G by the family $\{B_i\}_{i\in\Lambda}$ is a subset $\{B_i\}_{i\in\lambda} \subset \{B_i\}_{i\in\Lambda}$ such that either the condition

(C1) $B_i \cap B_j \subset V$ for $i,j \in \lambda$, $i \neq j$

or the condition

(C2) $B_i \cap B_j = \emptyset$ for $i,j \in \lambda$, $i \neq j$

holds. If, in the (C1)-case, the packing $\{B_i\}_{i\in\lambda}$ in (V,E) has the additional property that there exists an $m \in \mathbb{N}$ such that every pair x_l, x_k of distinct vertices of V occurs for m or $m+1$ indices $i \in \lambda$ in a connected component of B_i, then we call it *homogeneous* (C1)*-packing. So, homogeneous (C1)*-packings are particularly regular or "well-balanced" (C1)-packings. This will become more clear in the examples we consider below.

There is always a good chance to find in the set of (C1)-packings of maximal cardinality a (C1)*-representative. The number m is determined by a diophantic equation and also the number of pairs of vertices occurring $m+1$ times in a connected component of B_i (this number may happen to be zero).

Now we may ask for the optimal packing in the sense that the density $\sigma_\lambda = \frac{\text{card}(\{\cup_{i\in\lambda} E_i\})}{\text{card}(E)}$ is maximal under eventual some restrictions on the admissible subset $\{B_i\}_{i\in\lambda}$.

In the words of graph design we have the following:

A (C1)-packing of a complete graph with density $\sigma_\lambda = 1$ such that all the B_i's are isomorphic to a given graph G is a G-design. A (C1)-packing of a complete graph with density $\sigma_\lambda = 1$ such that all the B_i's are isomorphic to a complete graph may be regarded as a balanced incomplete block design. Further a (C2)-packing with $\sigma_\lambda = 1$ such that all the B_i's are isomorphic to a complete graph on 2 vertices is a 1-factor. (For the definitions see [6].) In this sense, our concept of packings is more general than graph design.

2 Notations and Definitions

We use the standard notation of [1].

Let K_n denote the complete, simple graph on n vertices.
A tree T is called a *linear tree,* if each vertex of T has degree 1 or 2.
The *length of a linear tree* $T = (V_T, E_T)$ is the cardinality of V_T.
A *linear forest* is a set of linear trees satisfying condition (C2).
A *uniform forest* F is a linear forest such that all linear trees of F have the same length, the *height* of the forest.
The *size of a forest* F is the cardinality of F.

Given a complete graph $K_n = (V_n, E_n)$ and $h > 1$ a divisor of n. Let $\mathcal{B}_{n,h}$ denote the family

$$\mathcal{B}_{n,h} := \{B_i = (V_i, E_i) : B_i \text{ a uniform forest of heigth } h \text{ and size } \tfrac{n}{h}\} \tag{1}$$

of subgraphs of K_n. We are interested in packings $\mathcal{A}_{n,h} \subset \mathcal{B}_{n,h}$ in K_n by the family $\mathcal{B}_{n,h}$ such that condition (C1) or (C1)* (as in Section 4) or condition (C2) and some additional restrictions hold (as in Section 3). In the language of graph design, a (C1)-packing $\mathcal{A}_{n,h} \subset \mathcal{B}_{n,h}$ in K_n with density $\sigma_\lambda = 1$ is a resolvable, balanced path design (cf. [6]). In the (C1)-case it is easy to see that for a packing of K_n by $\mathcal{B}_{n,h}$ there holds

$$\text{card}(\lambda) \leq \frac{n(n-1)/2}{(h-1)n/h}$$

and because card(λ) is an integer we get

$$\text{card}(\lambda) \leq \left\lfloor \frac{h(n-1)}{2(h-1)} \right\rfloor \tag{2}$$

(where $\lfloor x \rfloor$ is the nearest integer less or equal than x).

On the other hand if we consider packings which respect (C2) we trivially have card(λ) ≤ 1: So here the question is whether a packing *exists* or not.

3 Packings in complete graphs by edges of different length

Let K_n be the complete graph with vertices $\{x_i\}_{1\leq i\leq n}$. We define the *cyclic length* of an edge $[x_i, x_j]$ joining x_i and x_j as

$$l([x_i, x_j]) := \min\{|i-j|, n-|i-j|\}$$

See also Figure 1 for the geometric meaning of the cyclic length. Then there holds

Theorem 1 *If n is even then there exists a (C2)-packing in K_n by the family $\mathcal{B}_{n,2}$ such that only edges of different cyclic length occur, if and only if $\frac{n}{2}$ mod4 equals 0 or 1.*

Remark 1: If n is odd the corresponding problem is trivial.

Proof: (i) Consider a (C2)-packing in K_{2m} by $\mathcal{B}_{2m,2}$ such that every cyclic length $1, 2, \ldots, m$ occurs. Let $P := \{x_i : i \text{ odd}\} \subset V_{2m}$ and $Q := \{x_i : i \text{ even}\} \subset V_{2m}$. If an edge of the packing has odd cyclic length it is joining the sets P and Q, else it is joining two vertices of P or of Q. Hence the number of edges of the packing having even cyclic length must be even. Now, if m is even the even cyclic lengths occurring in the packing are $\{2, 4, \ldots, m\}$ and this set is even if and only if $m \equiv 0 \pmod 4$. If on the other hand m is odd the even cyclic lengths occurring in the packing are $\{2, 4, \ldots, m-1\}$ and this set is even if and only if $m \equiv 1 \pmod 4$.

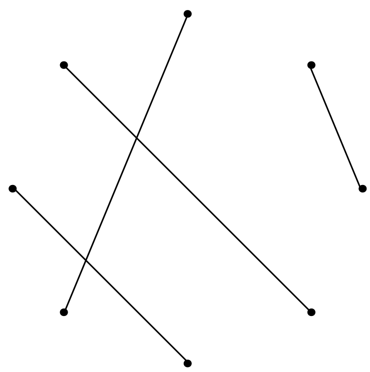

Figure 1: (C2)-packing in K_8 by edges such that every cyclic length occurs.

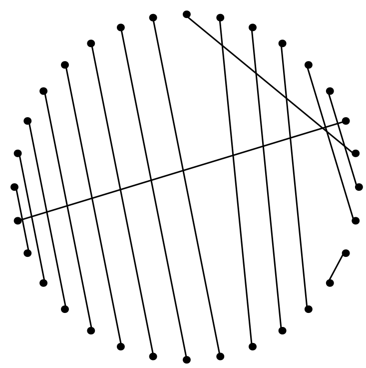

Figure 2: (C2)-packing in K_{32} by edges such that every cyclic length occurs.

(ii) For the other direction we consider two cases.

Case 1. $m \equiv 0 \pmod 4$:
If $m = 4$ then $\mathcal{A}_{8,2} := \{[x_1, x_8], [x_2, x_5], [x_3, x_7], [x_4, x_6]\}$ is a packing in K_{2m} such that every cyclic length $1, 2, \ldots, m$ occurs (see Figure 1).

If $m = 4k$ ($k > 1$) then it is easy to check that

$$\mathcal{A}_{2m,2} := \Big\{[x_1, x_{2k}], [x_2, x_{4k+1}], [x_{7k+2}, x_{7k+1}], \{[x_i, x_{8k+1-i}]\}_{k<i<2k}, \\ \{[x_i, x_{8k+2-i}]\}_{2k<i\leq 4k}, \{[x_i, x_{8k+3-i}]\}_{3\leq i\leq k}\Big\}$$

is a packing in K_{2m} with the desired properties. Figure 2 shows the resulting packing for $n = 32$.

Case 2. $m \equiv 1 \pmod 4$:
If $m = 1$ then $\mathcal{A}_{2,2} := \{[x_1, x_2]\}$ is a packing in K_{2m} such that the cyclic length 1 occurs.

If $m = 5$ then $\mathcal{A}_{10,2} := \{[x_1, x_2], [x_3, x_9], [x_4, x_7], [x_5, x_{10}], [x_6, x_8]\}$ is a packing in K_{2m} such that every cyclic length $1, 2, \ldots, m$ occurs (see Figure 3).

If $m = 4k + 1$ ($k > 1$) then it is easy to check that

$$\mathcal{A}_{2m,2} := \Big\{[x_1, x_{4k+1}], [x_{2k}, x_{4k+2}], [x_{7k+2}, x_{7k+1}], \{[x_i, x_{8k+2-i}]\}_{k+2\leq i<2k}, \\ \{[x_i, x_{8k+3-i}]\}_{2k<i\leq 4k}, \{[x_i, x_{8k+4-i}]\}_{2\leq i\leq k+1}\Big\}$$

is a packing in K_{2m} with the desired properties. Figure 4 shows the resulting packing for $n = 34$.

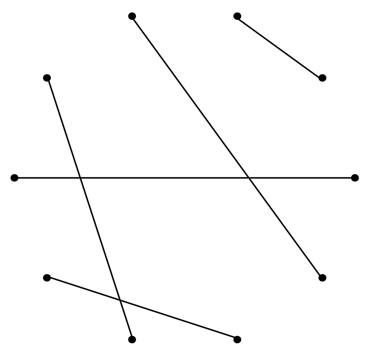

Figure 3: (C2)-packing in K_{10} by edges such that every cyclic length occurs.

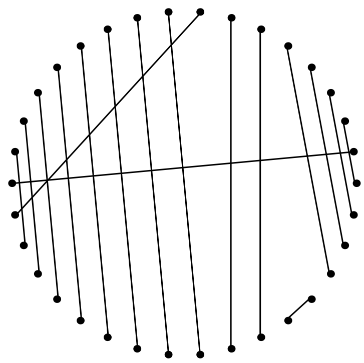

Figure 4: (C2)-packing in K_{34} by edges such that every cyclic length occurs.

□

Remark 2: Although it was quite hard to find a packing in a complete graph by edges of different cyclic length, there exist in fact *many* solutions for large m:

$$\begin{array}{rcl} K_2 & : & 1 \text{ solution} \\ K_8 & : & 1 \text{ solution} \\ K_{10} & : & 2 \text{ solutions} \\ K_{16} & : & 128 \text{ solutions} \\ \ldots & : & \ldots \end{array}$$

Of course, congruent solutions are identified.

Remark 3: These packings are in fact very special 1-factorizations of K_{2m}. Note that in general 1-factorizations of K_{2m} always exist (cf. [4] p. 85).

4 High, large and balanced forests

In this section we will consider (C1) and (C1)*-packings in K_n by the family $\mathcal{B}_{n,h}$. We are interested in the cases $h = n$ (hence the corresponding forests are of maximal possible height), $2h = n$ (the corresponding forests contain exactly two trees), $h = 2$ (the corresponding forests are as large as possible) and $h^2 = n$ (the corresponding forests are as large as high). We show in most of the mentioned cases that estimate (2) is sharp.

Notation: If σ is a permutation of the set $\{1, \ldots, n\}$ and $H = (V_H, E_H)$ a partial subgraph of K_n, then $\sigma[H] = (V_{\sigma[H]}, E_{\sigma[H]})$ where $V_{\sigma[H]} := \{x_{\sigma(i)} : x_i \in V_H\}$ and $E_{\sigma[H]} := \{[x_{\sigma(i)}, x_{\sigma(j)}] : [x_i, x_j] \in E_H\}$ (see also Figure 5). Further let σ^0 be the identity and $\sigma^{n+1} := \sigma(\sigma^n)$.

4.1 High forests: $h = n$

For $h = n > 1$ we obtain by estimate (2) that a maximal packing is of cardinality less or equal than $\lfloor \frac{n}{2} \rfloor$. And indeed we find:

Theorem 2 *In K_n there exists a (C1)*-packing $\mathcal{A}_{n,n}$ by $\mathcal{B}_{n,n}$ of cardinality $\lfloor \frac{n}{2} \rfloor$.*

Proof: Let

$$A := \{[x_1, x_n], [x_1, x_{n-1}], [x_2, x_{n-1}], [x_2, x_{n-2}], \ldots, [x_{\lfloor \frac{n}{2} \rfloor}, x_{\lfloor \frac{n}{2} \rfloor + 1}]\}$$

and

$$\sigma := \begin{pmatrix} 1 & 2 & 3 & \ldots & i & \ldots & n \\ 2 & 3 & 4 & \ldots & i+1 & \ldots & 1 \end{pmatrix}.$$

Then $\mathcal{A}_{n,n} := \{B_i : B_i = \sigma^i[A],\ 1 \le i \le \lfloor \frac{n}{2} \rfloor\}$ is a (C1)-packing of cardinality $\lfloor \frac{n}{2} \rfloor$ (see Figure 5). Because all pairs of vertices x_k, x_l belong to every $B_i \in \mathcal{A}_{n,n}$ and since every B_i is connected, the packing is trivially (C1)*. $\square$

In fact Theorem 2 follows also from [4] p. 89.

Remark 4: If n is even, the density of the packing constructed above is 1. Hence, it can be regarded as a path design (in contrast to the case n odd).
At this stage we get, as a byproduct which will be useful afterwards, also an optimal (C1)*-packing in K_{n+1} by cycles of length $n + 1$: Just introduce a new point x_{n+1} and close every tree constructed above by joining both ends with x_{n+1} (see Figure 6). The cardinality of this packing is $\lfloor \frac{n}{2} \rfloor$, thus it is optimal. If n is even its density is 1 and hence we get a 2-factorization of K_{n+1} (see [4] p. 89).

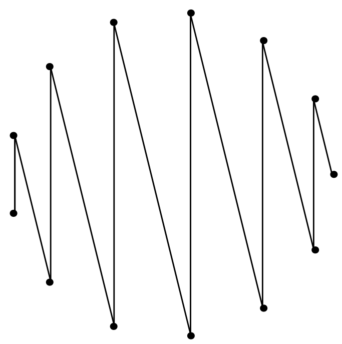

Figure 5: Generation of a maximal (C1)-packing in K_{13} by trees of length 13.

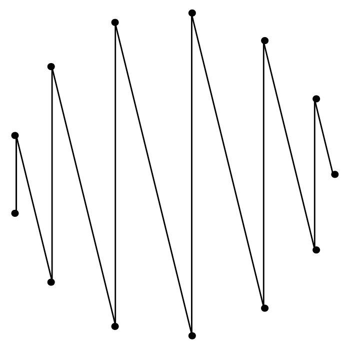

Figure 6: Generation of a maximal (C1)-packing in K_{14} by cycles of length 14.

If $n = 2k$ and if we consider each linear forest occurring in the packing $\mathcal{A}_{n,n}$ (constructed in the proof of Theorem 2) as a row of a matrix, we get a $k \times n$-matrix which yields in an natural way a horizontally complete $k \times n$ latin rectangle (cf. [3]).

4.2 The case $2h = n$

The second highest forests appear if $2h = n > 2$. In this case estimate (2) says, that a maximal packing is of cardinality less or equal than $\lfloor \frac{h(2h-1)}{2h-2} \rfloor$ which is h $(= \frac{n}{2})$ for $h > 2$. We find:

Theorem 3 *In K_n (with $n = 2h$) there exists a (C1)-packing $\mathcal{A}_{2h,h}$ by $\mathcal{B}_{2h,h}$ of cardinality $\lfloor \frac{h(2h-1)}{2h-2} \rfloor$ (and hence this packing is optimal), whereas a (C1)*-packing of this cardinality only exist for $h = 2$.*

Proof: The case $h = 2$ is trivial, so let us assume $h > 2$. By Section 4.1 we can find a packing for $h' = n$ $(= 2h)$ of cardinality $\frac{n}{2}$ $(= h)$. Canceling an edge of each linear tree of this packing such that both parts are of length h we get a packing $\mathcal{A}_{2h,h}$ of cardinality h. Thus (2) is sharp also in case $2h = n$.

To see that for $h > 2$ no (C1)*-packing of the mentioned cardinality exists we proceed by contradiction. Suppose there is such a packing $\mathcal{A}_{2h,h} = \{B_i \in \mathcal{B}_{2h,h} : i = 1, \ldots, h\}$. Consider the sets $S_i = \{j : x_i$ and x_{2h} are in the same connected component of $B_j\}$ for $i = 1, \ldots, 2h-1$. Since $\mathcal{A}_{2h,h}$ is a (C1)*-packing the sets S_i are all of "almost equal size" or more precisely there exists $m \in \mathbb{N}$ such that every set S_i has cardinality m or $m+1$, say $|S_1| = \ldots = |S_x| = m$ and $|S_{x+1}| = \ldots = |S_{2h-1}| = m+1$. By counting edges we obtain:

$$m = \begin{cases} \frac{h}{2} - 1 & \text{if } h \text{ even} \\ \frac{h-1}{2} & \text{if } h \text{ odd} \end{cases} \qquad x = \begin{cases} \frac{h}{2} & \text{if } h \text{ even} \\ \frac{3h-1}{2} & \text{if } h \text{ odd} \end{cases}$$

To continue we have to distinguish the four cases $h \equiv \iota \mod 4$, $\iota = 0, 1, 2, 3$. We only carry out $\iota = 1$ (the other cases are similar). For $h = 4k + 1$ we obtain that $|S_1 \cap S_j| = k$ for $j = 2, \ldots, x$. It follows that x_1 and x_j, $j = 2, \ldots, x$, are $m + 1$ times in the same connected component of a V_i. But since $x - 1 = \frac{3h-1}{2} - 1 > \frac{h-1}{2} = 2h - 1 - x$ this is impossible. (If $\iota = 3$, consider S_1 and S_j for $j = x + 1, ldots, 2h - 1$.)

An alternative proof is based upon the observation that the (C1)*-packing considered above would induce a partition of the set $\{1, \ldots, h\}$ into x subsets S_i of cardinality m having the property that their intersection is of cardinality k. It is quite easy to see that there is no such partition. $\square$

4.3 Large forests: $h = 2$

If $h = 2$, then because h is a divisor of n, n has to be even and of the form $n = 2m$ (for an $m > 0$). Estimate (2) says, that in this case a maximal packing is of cardinality less or equal than $\frac{2(n-1)}{2(2-1)} = n - 1$. In fact there holds:

Theorem 4 *If n is even then there exists a (C1)*-packing $\mathcal{A}_{n,2}$ in K_n of cardinality $n-1$.*

Proof: Let $n = 2m$. We consider two cases.

Case 1. m is odd, hence of the form $m = 2k + 1$:
Let $A_1 := \{[x_1, x_2], [x_3, x_4], \ldots, [x_{n-1}, x_n]\}$ and $\sigma_1 := \begin{pmatrix} 2 & 4 & \ldots & 2i & \ldots & 2m \\ 4 & 6 & \ldots & 2i+2 & \ldots & 2 \end{pmatrix}$, further $A_2 := \{[x_1, x_n]\} \cup \{[x_2, x_{n-2}], [x_3, x_{n-1}], [x_4, x_{n-4}], [x_5, x_{n-3}], \ldots, [x_m, x_{m+2}]\}$ and $\sigma_2 := \begin{pmatrix} 1 & 2 & \ldots & i & \ldots & n-1 & n \\ 3 & 4 & \ldots & i+2 & \ldots & 1 & 2 \end{pmatrix}$.

Then

$$\mathcal{A}_{n,2} := \{B_i : B_i = \sigma_1^{i-1}[A_1] \text{ for } 1 \leq i \leq 2k \text{ and } B_i = \sigma_2^{i-2k-1}[A_2] \text{ for } 2k < i < n\}$$

is a (C1)-packing of cardinality $n - 1$.

Case 2. m is even, hence of the form $m = 2k$. Here we give the proof by induction on k. Let $P := \{x_i : i \text{ is odd}\}$ and $Q := \{x_i : i \text{ is even}\}$. By induction there are packings $\mathcal{A}^P_{2k,2} = \{A^P_i : 1 \leq i < m\}$ and $\mathcal{A}^Q_{2k,2} = \{A^Q_i : m \leq i < n - 1\}$ in P (respectively Q) both of cardinality $m - 1$.
Then with $A := \{[x_1, x_2], [x_3, x_4], \ldots, [x_{n-1}, x_n]\}$ and

$$\sigma := \begin{pmatrix} 2 & 4 & \ldots & 2i & \ldots & 2k \\ 4 & 6 & \ldots & 2i+2 & \ldots & 2 \end{pmatrix},$$

define

$$\mathcal{A}_{2m,2} := \{B_i : B_i = \sigma^i[A] \text{ for } 0 \leq i < m \text{ and } B_i = A^P_i \cup A^Q_i \text{ for } m \leq i < n - 1\}$$

which is a (C1)-packing of cardinality $n - 1$.

In both cases, the packing is trivially (C1)* since every pair of vertices is exactly once in the same connected component of a forest. □

Remark 5: In fact we proved that if n is even, then K_n has a 1-factorization (cf. [4] Theorem 9.1).

4.4 Balanced forests: $h^2 = n$

For $h^2 = n$ the estimate (2) says, that a maximal packing is of cardinality less or equal than $\binom{h+1}{2}$.

Lemma *If h is odd and $n = h^2$, then there is a (C1)-packing $\mathcal{A}_{n,h}$ in K_n of cardinality $\frac{n-1}{2}$.*

Proof: Use the Remark 4 to construct in K_n $\frac{n-1}{2}$ many pairwise edge disjoint cycles of length n. By canceling suitable edges in each cycle, we get a set of uniform edge disjoint forests of height h, thus a (C1)-packing of cardinality $\frac{n-1}{2}$. □

Note that the difference between $\frac{n+h}{2}$ (the upper bound for the cardinality of a (C1)-packing which is given by estimate (2)) and $\frac{n-1}{2}$ is only $\frac{h+1}{2}$, hence a (C1)-packing in K_n of cardinality $\frac{n-1}{2}$ looks almost optimal. However the next Theorem shows, that there are always (C1)-packings, such that estimate (2) is sharp and that in some cases we can even find a (C1)*-packing of density 1.

Theorem 5 *For any $h > 1$ there exists a (C1)-packing $\mathcal{A}_{n,h}$ in K_n of cardinality $\binom{h+1}{2}$ and hence of density 1. Moreover, if h is of the form $h = p^m$ (where p is a prime number and $m \in \mathbb{N}$), there exists a (C1)*-packing $\mathcal{A}_{n,h}$ in K_n of the same cardinality and density.*

Proof: The first part of the theorem, namely that there exist (C1)-packings $\mathcal{A}_{n,h}$ in K_n of cardinality of density 1 follows quite easily from the results of [5], [6] and [2] (see also the interpretation of the packing as solution of the well-known "handcuffed prisoner problem"). Nevertheless, the packings constructed in the cited papers are not (C1)* as one easily checks (two prisoners may walk quite often in the same row whereas others only once). So, we have to show that for h being a power of a prime, a (C1)*-packing (and hence a particularly regular solution of the problem) of density 1 exists.

For even h we can give a shorter construction of a (C1)-packing than in the mentioned papers, so let us start with

Case 1. h is an even number, hence of the form $h = 2k$.
First we take the (C2)-packing $\mathcal{A}_{n,n}$ of cardinality $\frac{h^2}{2}$ constructed in the proof of Theorem 2. Now if we cancel in each linear tree all edges of cyclic length 0 (mod h), we get a (C2)-packing $\mathcal{A}_{n,h}$ of the same cardinality.
The canceled edges form h disjoint complete graphs $\{K_h^i\}_{1\le i\le h}$. Again by Theorem 2 we find a (C2)-packing $\mathcal{A}_{h,h}^i$ of cardinality k in each such graph. Choosing one linear tree

(of length h) in each $\mathcal{A}^i_{h,h}$ we get a uniform forest of height h and size h. We repeat this procedure k times and end up with the k missing uniform forests: $\frac{h^2}{2} + k = \binom{h+1}{2}$.

Case 2. h is of the form $h = p^m$, where p is a prime number and $m \in \mathbb{N}$. We will give the proof of this case in three steps.

1^{st} *step:* We identify the vertices of K_n with the points (i, j), $i, j \in F$, of the plane of the coordinate geometry over a Galois field F with $h = p^m$ elements (as a general reference for finite geometry see [8]). In this plane we are given $h + 1$ bundles of parallels, each bundle consisting of h nonintersecting straight lines. One bundle is consisting of the lines $l_{\infty i} = \{(i, j)\}_{j\in F}$, the other bundles are $l_{si} = \{(j, sj + i)\}_{j\in F}$ (where $s \in F$). Each bundle of parallels may be considered as a partition of V_n, the vertices of K_n.

2^{nd} *step:* It is easy to see that for any two partitions $P_1 = \{v^1_k : 1 \le k \le h\}$ and $P_2 = \{v^2_k : 1 \le k \le h\}$ constructed in step 1 there is a $(h \times h)$-matrix $A = a_{ij}$ such that $\{a_{ij} : i = k\} = v^1_k$ and $\{a_{ij} : j = k\} = v^2_k$. With the $h + 1$ partitions constructed in step 1 we obtain in this way $\frac{h+1}{2}$ many $(h \times h)$-matrices.

3^{rd} *step:* Now we take a matrix $A = a_{ij}$ constructed in step 2 and show that it yields a packing in K_n of cardinality h. Combining the h packings given by each of the $\frac{h+1}{2}$ matrices we obtain a packing in K_n of cardinality $\frac{h(h+1)}{2} = \binom{h+1}{2}$.

(a) First consider the h linear trees $[a_{i,i}, a_{i+1,i}, a_{i+1,i-1}, a_{i+2,i-1}, \ldots, a_{i+\frac{h-1}{2},i-\frac{h-1}{2}}]$, where all indices are taken modulo h and $i = 1, \ldots, h$. Those trees form a uniform forest F in K_n of height h and size h.

(b) According to Theorem 2 it is—after a suitable rearrangement of the vertices—possible to construct $\frac{h-1}{2}$ linear trees of length h in each row or column such that all these trees are pairwise edge-disjoint and also edge-disjoint with each linear tree belonging to the forest F. Therefore we get $\frac{h-1}{2}$ uniform forests of height h and size h coming from the rows of A and the same number coming from the columns. Altogether we obtain $1 + \frac{h-1}{2} + \frac{h-1}{2} = h$ uniform forests of height h and size h which are by construction edge-disjoint.

Thus we get a (C1)-packing $\mathcal{A}_{n,h}$ in K_n of cardinality $\frac{h(h+1)}{2} = \binom{h+1}{2}$, which is by construction even a (C1)*-packing.

□

Example: To illustrate the construction above we consider the case $h = 3$.

1^{st} *step:* Figure 7 shows the coordinateplane $F \times F$ for the finite field $F = F_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ and the bundles of parallels. We identify $x_1 \equiv 1 \equiv (\bar{0}, \bar{2})$, $x_2 \equiv 2 \equiv (\bar{1}, \bar{2})$ etc.

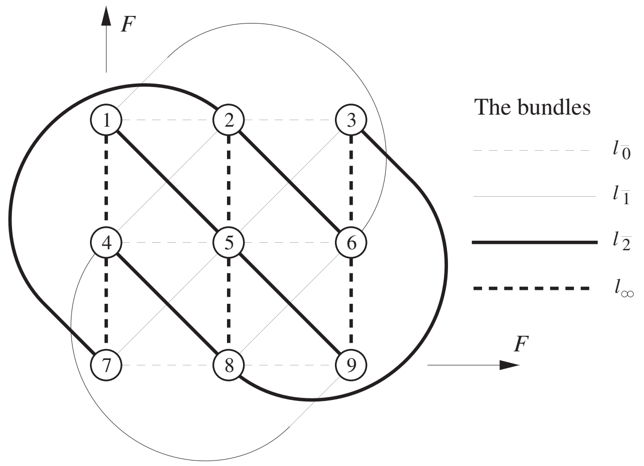

Figure 7

2^{nd} *step:* The partitions given by the bundles of parallels of step 1 give rise to the following 2 matrices having the property that each bundle occurs in exactly one of the matrices either in the rows or in the columns:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 5 & 7 \\ 8 & 1 & 6 \\ 4 & 9 & 2 \end{pmatrix}$$

The first matrix is built of $l_{\bar{0}}$ and l_∞, the second of $l_{\bar{1}}$ and $l_{\bar{2}}$ (other choices are also possible).

3^{rd} *step:* By each of the two matrices of step 2 we construct packings in K_9 of cardinality 3. The combination gives the packing of cardinality 6.

(a) By the construction given in the proof we first get the two uniform forests $\{[1,4,6]$, $[5,8,7]$, $[9,3,2]\}$ and $\{[3,8,6]$, $[1,9,4]$, $[2,7,5]\}$.

(b) At least we get the four uniform forests $\{[2,1,3]$, $[4,5,6]$, $[7,9,8]\}$, $\{[1,7,4]$, $[5,2,8]$, $[3,6,9]\}$, $\{[5,3,7]$, $[8,1,6]$, $[4,2,9]\}$ and $\{[3,4,8]$, $[1,5,9]$, $[7,6,2]\}$ where the first two come from the first matrix and the last two from the second matrix.

Remark 6: P. Hell and A. Rosa have shown in [5] that a (C1)-packing $\mathcal{A}_{h^2,h}$ of K_{h^2} with density $\sigma_\lambda = 1$ always exists. The difference between our solution and the solution given in [5] for $h = p^m$ (where p is a prime number) is, that our solution is homogeneous, i.e. if we take two arbitrary distinct vertices of K_{h^2}, then they appear in the same tree exactly $\frac{p^m-1}{2}$ or $\frac{p^m+1}{2}$ times if p is odd and $\frac{p^m}{2}$ times if $p = 2$. The solution given in [5] is far away from being (C1)*. In the language of graph design we may summarize the results as follows.

Summary: If $n = h^2$, then there exists a resolvable balanced path design of type (n,h,1). Furthermore, if $h = 2^k$, then we can choose this resolvable balanced path design such that it is *at the same time* a balanced incomplete block design (the blocks being the vertices of the trees) with every pair of vertices occurring 2^{k-1} times in a block. If $h = p^m$, p an odd prime number, then for diophantic reasons, there is no m such that every pair of vertices occurs exactly m times in the same tree. Therefore, in this case, the (C1)*-packing we constructed is the most balanced solution one can think of.

We close with the following question.

Does a (C1)*-packing of K_{36} by $\mathcal{B}_{36,6}$ with density $\sigma_\lambda = 1$ exist?

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