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MEAN CURVATURE FLOW SOLITONS

by

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Abstract. — We consider the mean curvature flow $F_t : M \to N$ of hypersurfaces in a Riemannian manifold N. The stationary solutions of this flow are the minimal surfaces in N. Other interesting solutions are those, which move along the integral curves of a smooth vector field X of N. In this way conformal vector fields X give raise to self-similarly shrinking solutions of the mean curvature flow. If X is even parallel then the corresponding solutions of the mean curvature flow are called isometric solitons or just solitons. Soliton solutions have attracted increasing attention in the past years since they are interesting objects for a number of reasons: solitons appear as blow ups of singularities and exhibit interesting geometric and analytic properties. They serve as tailor-made comparison solutions and allow a certain insight into the behaviour of the mean curvature flow viewed as a dynamical system.

***Résumé* (Solitons issus du flot par la courbure moyenne).** — Nous considérons le flot de la courbure moyenne $F_t : M \to N$ d'hypersurfaces dans une variété riemannienne N. Les solutions stationnaires de ce flot sont les surfaces minimales dans N. D'autres solutions intéressantes sont celles qui se déplacent le long de courbes intégrales d'un champ de vecteur lisse X dans N. De cette manière les champs de vecteurs conformes X engendrent des solutions autosimilaires contractantes du flot de la courbure moyenne. Si X est parallèle alors les solutions correspondantes au flot de la courbure moyenne sont appelées solitons isométriques ou juste solitons. Il y a un intérêt croissant ces dernières années pour les solutions solitons car ce sont des objets intéressants pour diverses raisons: les solitons apparaissent comme des éclatements de singularités et font apparaître des propriétés géométriques et analytiques intéressantes. Elles servent comme des solutions de comparaison sur mesure et donnent une certaine idée du comportement du flot de la courbure moyenne vu comme un système dynamique.

1. Introduction

Physicists investigated in the fifties of the twentieth century the annealing process of aluminum. They observed, that in melted aluminum, at random points the material

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starts to crystallize spontaneously, as the temperature reaches a critical level. In these points, homogeneous crystals with face centered cubic lattice start to grow. These grains finally touch each other and fill the space (see Figure 1). However, this is not the end of the process: atoms sitting at the edge of a grain are integrated in their atomic

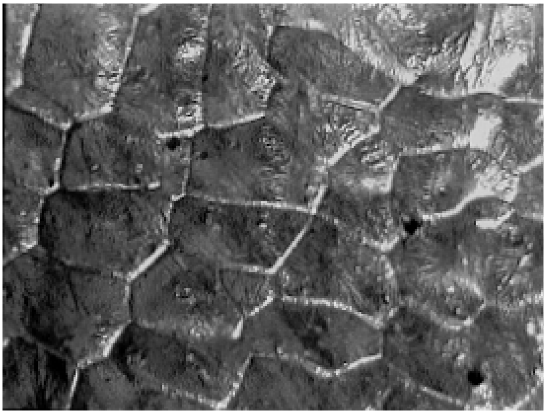

FIGURE 1. Grains in aluminum: a typical grain size is around 10 micrometer, the lattice parameter of aluminum amounts to 4.05 10^{-7}m.

crystal lattice only to one side and are therefore in a slightly elevated energy state. On account of this, such an atom can spontaneously jump to the neighboring lattice. This change is the more likely, the more convex the boundary at this point is: if, e.g., the atom is sitting at a cusp, it is surrounded almost entirely by a "foreign" crystal grid and will therefore easily change its affiliation. By the described mechanism, the grain boundaries keep moving even after the metal has solidified. It has been observed, that the velocity of a grain boundary is proportional to its mean curvature. This is plausible, if we assume that the elevated energy state of the boundary atoms amounts to a surface energy which is isotropic and proportional to the surface area. The system, trying to minimize its energy, will therefore reduce this surface, and the first variation of the area functional corresponds just to the mean curvature vector field. This means, the system reduces its energy by moving the grain boundaries with a velocity which is (proportional to) the mean curvature in each point. This is the *mean curvature flow*.

Mathematically, the mean curvature flow has first been investigated 1978 by Brakke (see [**4**]), later by Huisken (see [**12**]). Brakke used geometric measure theory, Huisken a more classical, differential geometric approach. In order to describe singularities of the flow, Osher-Sethian introduced a level-set formulation for the mean curvature flow (see [**18**]), which was investigated later by Evans-Spruck (see [**6**], [**7**], [**8**], [**9**]) and Chen-Giga-Goto (see [**5**]) in detail. Ilmanen revealed in [**14**] the relation between the level-set formulation and the geometric measure theory approach.

In this article, we use the following model of the mean curvature flow: let N be a n-dimensional Riemannian manifold with a metric $\bar{g}$ and M a differentiable, connected m-dimensional manifold with $m = n - 1$. Let $F_t : M \to N$, $t \in [0, T[, T > 0$, be a smooth family of immersions from M to N. Then we say:

Definition 1.1. — The family F_t is a solution of the mean curvature flow on $[0, T[$, $T > 0$, if

$$\begin{aligned} \frac{d}{dt} F_t &= -H\nu \quad \text{on } M \times]0, T[\\ F_0 &= f \qquad\quad \text{on } M, \end{aligned} \tag{1}$$

where $f : M \to N$ describes a given initial hypersurface M_0. H denotes the mean curvature of $F_t(M)$ with respect to the unit normal vector field ν on $F_t(M)$. ◊

The minus sign in (1) causes the flow to decrease area (or arc length in the case of curves). We also remark that the product $H\nu$ is independent of the chosen orientation of ν (see (6)–(7) below). H can be interpreted as the trace of the second fundamental form of the immersion, and $H\nu$ as the first variation of the area functional. The term $H\nu$ can also be written as $\Delta_{F_t(M)} F_t$, which is the Laplace-Beltrami operator on M with respect to the pull back by F_t of the metric on N. In this form, the parabolic nature of the equation becomes apparent. However, the operator evolves in time together with the solution. Nonetheless, classical solutions of the mean curvature flow, inherit a parabolic comparison principle:

Comparison principle. — *If the initial surfaces $F_0(M)$ and $G_0(M')$ are disjoint, so are the solutions $F_t(M)$ and $G_t(M')$ as long as they exist classically.*

This comparison principle allows already to make some qualitative statements regarding the behaviour of solutions of the mean curvature flow. The following example in $N = \mathbb{R}^3$ is due to Angenent: the two spheres S in Figure 2 have the same radius

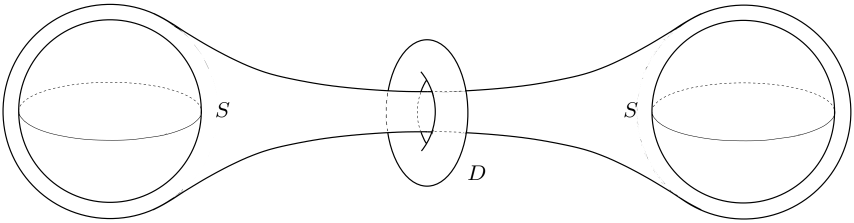

FIGURE 2. Four initial surfaces

R. Subject to the mean curvature flow, they shrink in time $T_S = \frac{R^2}{2\dim S}$ to a point. Between the two spheres, there is a special torus D which has the property, that it shrinks self similarly to a point. Such a torus has been found in 1992 by Angenent (Angenent's doughnut, see [**3**]). Choosing the torus small enough to be enclosed by a sphere of radius $r < R$, its vanishing time T_D is strictly less than T_S. Finally, we thread a dumbbell surface around the two spheres S through the torus D. The comparison principle guarantees that the configuration stays disjoint during its evolution under the mean curvature flow. Therefore, after a certain time, the solution looks as indicated in Figure 3. At the latest at time T_D the torus strangles the neck of the

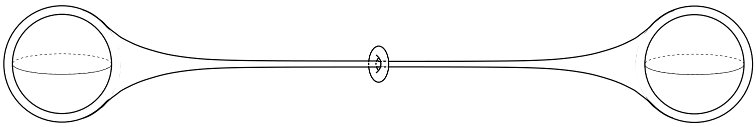

FIGURE 3. Solution at time $t < T_D$

dumbbell (see Figure 4) and a singularity must occur for this surface. (To continue

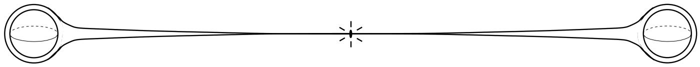

FIGURE 4. A singularity occurs

the flow past such a singularity see [**4**] and [**5**], or [**6**]–[**9**].)

If the initial surface is convex, the situation is better: Huisken proved 1984, that the solution stays convex and shrinks in finite time to a round point. This means, that if one rescales the solution suitably (e.g., by keeping the area constant), it converges in finite time uniformly to a round sphere.

In $\mathbb{R}^2$, the situation is even better: if one starts the (mean) curvature flow (also called curve shortening flow in this case) with an embedded closed curve, the solution stays embedded and converges in finite time to a round point. This is a result of Grayson (see [**11**]).

Nonetheless, the curve shortening flow can develop singularities also in the plane, if one starts with a curve that is not initially embedded. The example in Figure 5 is due to Angenent (see [**2**]): here, the inner loop suffers from its higher curvature and therefore shrinks faster than the outer loop. A singularity forms in finite time. By rescaling the solution suitably, e.g., by keeping the maximal curvature constant, the rescaled solution converges to a very particular limit, namely the curve $x = -\log\cos y$ (see Figure 6): Angenent has shown, that the blow-up of every so called type II

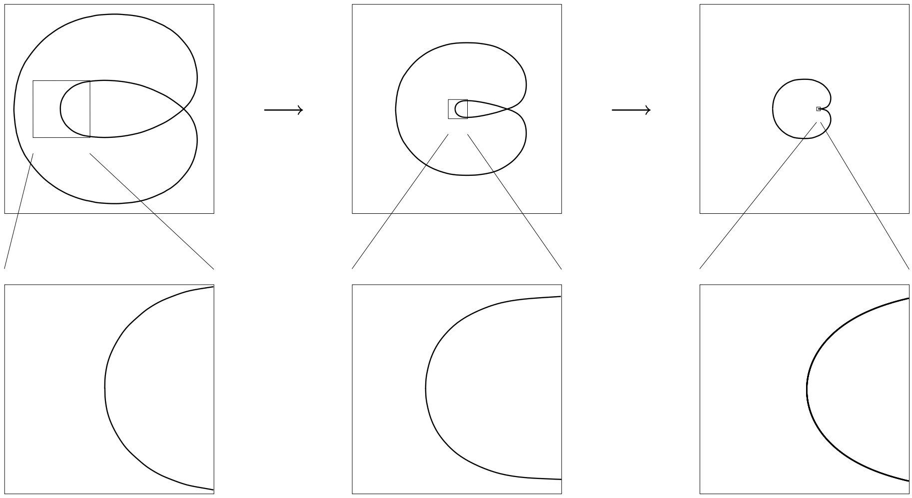

FIGURE 5. Singularity formation

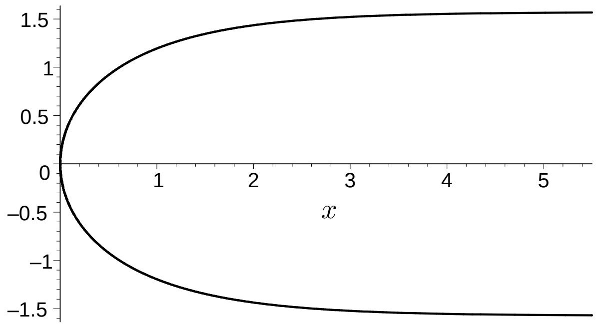

FIGURE 6. Grim Reaper

singularity of convex [1] plane curves generates exactly this limit. This limit curve now turns out to have a very special property: if we let it evolve under the curvature flow, it keeps its shape and evolves by translating with constant speed to the right. Since the form and motion of the curve reminds of a scythe, Hamilton called it the "Grim Reaper". Generalizing this example, we will call every solution that evolves under the mean curvature flow by the motion of a one parameter group of isometries of the ambient manifold, a soliton. Note, that it is equally interesting to consider the motion of solutions along integral curves of vector fields other than Killing fields, e.g., conformal vector fields, which give raise to self similarly shrinking solutions of the

[1] Convex means here, that the curvature does not change its sign along the curve.

mean curvature flow (see, e.g., [**16**], Angenent's doughnut in [**3**], or the self-similarly moving curves of Abresch-Langer in [**1**]). In [**13**] it has been shown, that other soliton solutions occur as parabolic blow-up of type II singularities. However, the Grim Reaper and its higher dimensional relatives have long been the only examples of solitons. A more general study of solitons of the mean curvature flow finally started in [**17**].

2. Soliton solutions

The word soliton was coined by Kruskal and Zabusky in their fundamental work [**21**] on the Korteweg-de Vries equation. There, they observed, that this non-linear equation possesses so called traveling wave solutions which superimpose almost like solutions of linear wave equations. Such solutions were also called "*soli*tary waves". Since the interaction of these solutions reminds of the behaviour of elementary particles (like a "pro*ton*"), Kruskal and Zabusky created the made-up word "soli-ton". This denomination has been transfered to geometry for solutions that move under the action of isometry groups, even though there is usually no interaction property.

Soliton solutions of the mean curvature flow are interesting objects for several reasons:

- they appear, as we mentioned in Section 1, as blow ups of singularities of the mean curvature flow
- they have interesting geometric properties (as we will see in Section 3.2)
- they have interesting analytic properties (e.g., certain stability properties: see [**17**], [**20**] and [**15**])
- they serve as tailor-made comparison solutions (see, e.g., [**17**, Section 5])
- they allow a certain insight in the behaviour of the mean curvature flow viewed as a dynamical system.

To formulate the definition of a soliton, let N be a n-dimensional Riemannian manifold with metric $\bar{g}$, equipped with a Killing vector field X related to an isometry group $\varphi : N \times \mathbb{R} \to N$ by

$$\frac{d\varphi(x,t)}{dt} = X(\varphi(x,t)) \qquad \text{on } N \times \mathbb{R} \tag{2}$$

$$\varphi(x,0) = x \qquad \text{on } N. \tag{3}$$

Furthermore, M is a differentiable, connected m-dimensional manifold, $m = n - 1$. Let $F_t : M \to N, t \geq 0$ be a solution of the mean curvature flow on $[0,T[$, $T > 0$. Then, we say:

Definition 2.1. — F_t is a *soliton solution* of the mean curvature flow with respect to the Killing field X, if $\widetilde{F}_t := \varphi^{-1}(F_t, t)$ is stationary in normal direction, i.e., if $\widetilde{F}_t(M) = F_0(M)$ for all $t \in [0, T[$. In this case, the initial state $\tilde{F}_0 = F_0 = f$ will also be called a soliton. ♢

To better understand this definition, let us consider Figure 7: we fix a point $x \in M$

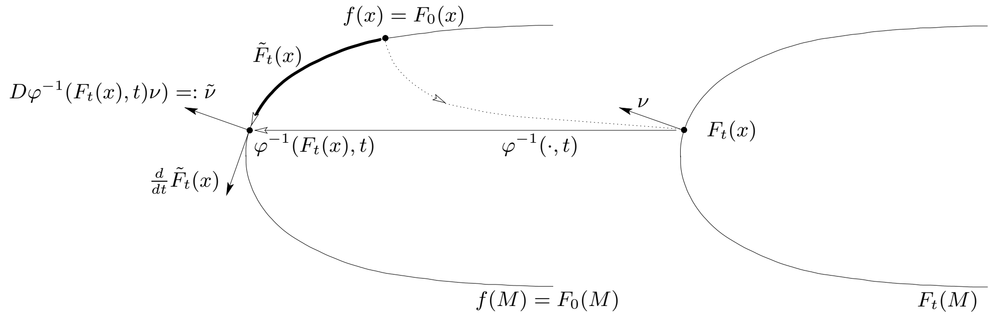

FIGURE 7. The definition of soliton solutions

and consider its image $f(x) \in f(M) = F_0(M)$. At time $t > 0$ this point has moved by the mean curvature flow to the place $F_t(x) \in F_t(M)$. Then, we move it back by $\varphi^{-1}(\cdot, t)$ to obtain the point $\tilde{F}_t(x) := \varphi^{-1}(F_t(x), t)$. The definition now requires, that $\tilde{F}_t(x) \in F_0(M)$. Geometrically, this means, that $F_t(M) = \varphi(F_0(M), t)$, i.e., the initial surface $F_0(M)$ moves under the mean curvature flow exactly like under the action of the isometry group $\varphi(\cdot, t)$. Stated otherwise: $\tilde{F}_t(x)$ moves points only within the surface $F_0(M)$ (it is stationary in normal direction). In order to investigate solitons, we now need the describing equation. We compute, by using (1) and (2),

$$\begin{aligned} \frac{d}{dt}\widetilde{F}_t(x) &= \frac{\partial \varphi^{-1}(F_t(x), t)}{\partial t} + D\varphi^{-1}(F_t(x), t)\frac{\partial F_t(x)}{\partial t} \\ &= -X(\widetilde{F}_t(x)) - D\varphi^{-1}(F_t(x), t)H\nu, \end{aligned} \tag{4}$$

where D denotes the spacial derivative. Since X is a Killing field, $\widetilde{\nu}(x, t) = D\varphi^{-1}(F_t(x), t)\nu$ is a unit normal vector field to $\widetilde{F}_t(M)$ (see Figure 7). Definition 2.1 now requires, that $\langle \frac{d}{dt}\widetilde{F}_t(x), \widetilde{\nu}(x, t)\rangle = 0$. Hence, by (4), we get

$$-\langle X(\widetilde{F}_t), \widetilde{\nu}\rangle = H\langle D\varphi^{-1}\nu, \widetilde{\nu}\rangle = H\langle \widetilde{\nu}, \widetilde{\nu}\rangle = H.$$

This holds in particular for $t = 0$, i.e., for $\tilde{F}_0 = F_0 = f$. Therefore, a soliton solution $f : M \to N$ of the mean curvature flow satisfies

$$-\langle X(f), \nu(f)\rangle = H(f). \tag{5}$$

Vice versa, it is easy to see, that every solution f of (5) induces a soliton solution with respect to X.

By multiplication by ν one can reformulate (5) equivalently:

$$X^{\perp} + \vec{H} = 0,$$

i.e., soliton surfaces are characterized by the property, that in every point the mean curvature vector $\vec{H} = H\nu$ annihilates the normal component of the Killing field.

For concrete calculations, we need to express (5) in local coordinates. Let $\bar{g}_{\alpha\beta}$ denote the metric, and $\bar{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\gamma\varepsilon}(\bar{g}_{\alpha\varepsilon,\beta} + \bar{g}_{\beta\varepsilon,\alpha} - \bar{g}_{\alpha\beta,\varepsilon})$ the corresponding Christoffel symbols on N, with Greek indices between 1 and n. The induced metric and the Christoffel symbols on the immersed manifold $M \overset{f}{\hookrightarrow} N$ is denoted by g_{ij} and Γ^k_{ij}, with Latin indices running from 1 to $m = n-1$. To compute the mean curvature, we use the Gauss equation

$$\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j} = -h_{ij}\nu^\alpha. \tag{6}$$

Here, h_{ij} is the second fundamental form which can be determined by multiplying (6) by $\nu_\alpha = \bar{g}_{\alpha\beta}\nu^\beta$. Then, the mean curvature is given by

$$H = g^{ij}h_{ij}. \tag{7}$$

Observe, that the middle term in (6) vanishes under this operation, since $\langle \frac{\partial f}{\partial x^k}, \nu\rangle_{\bar{g}} = 0$.

3. Geometric aspects of soliton curves

3.1. Soliton curves on geodesically complete Riemannian surfaces. — In this section, let N be a two-dimensional smooth Riemannian manifold with metric $\bar{g}$. We assume, that there exists a smooth Killing vector field $X : N \to TN$. Since we are dealing with curves in this section, M is just an open interval of $\mathbb{R}$. For concrete calculations, we introduce local coordinates on N: let $\phi : U \subset N \to V \subset \mathbb{R}^2$ be a coordinate chart. By $\gamma : M \to N$ we denote the soliton solution of the curve shortening flow we are looking for. In local coordinates we consider $f := \phi \circ \gamma : \gamma^{-1}(U) \to V,\ s \mapsto \big(x(s), y(s)\big)$. Vice versa γ is locally given by $\gamma = \phi^{-1} \circ f$ in the considered chart.

The curve $\gamma : M \to N$ is a soliton solution, if its local representation f satisfies the soliton equation

$$-\langle X(f), \nu(f)\rangle_{\bar{g}} = H(f). \tag{8}$$

Now, we need an explicit form of this equation, i.e., expressions for $X(f)$, $\nu(f)$, and $H(f)$ in local coordinates: to ease notation, we write $\bar{g}_{\alpha\beta} = \bar{g}_{\alpha\beta}\big(x(s), y(s)\big)$, $\alpha, \beta \in \{1, 2\}$, and $f' = \frac{d}{ds}f(s) = \big(x'(s), y'(s)\big) = (x', y')$. It turns out to be convenient to parametrize the curve γ by arc length, i.e.,

$$\langle f', f'\rangle_{\bar{g}} = (x')^2\bar{g}_{11} + 2x'y'\bar{g}_{12} + (y')^2\bar{g}_{22} = 1. \tag{9}$$

Let (η, ξ) denote the coordinates of the Killing vector field X. For the regularity of X it suffices in the sequel that X be at least locally Lipschitz continuous. A unit normal vector ν to the curve γ is given by

$$\nu = \sqrt{\det \bar{g}_{\alpha\beta}}\ \bar{g}^{-1} \begin{pmatrix} y' \\ -x' \end{pmatrix}. \tag{10}$$

Thus, we obtain

$$\langle X\,,\,\nu\rangle_{\bar{g}} = \sqrt{\det \bar{g}_{\alpha\beta}}(\eta y' - \xi x').$$

By formula (7) the (mean) curvature H with respect to ν is given by $H = g^{11}h_{11}$. In view of (9), the induced metric g on M by f is just the Euclidean metric. Hence, $H = h_{11}$, where h_{11} is given by the Gauss equation (6) which here takes the form

$$\frac{\partial^2 f^\alpha}{\partial s^2} - \Gamma^1_{11}\frac{\partial f^\alpha}{\partial s} + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{df^\beta}{ds}\frac{df^\gamma}{ds} = -h_{11}\nu^\alpha\,. \tag{11}$$

A simple, but lengthy calculation yields then

$$\begin{aligned} -H \;=\;& (x''y' + y''x')\sqrt{\det \bar{g}_{\alpha\beta}} \\ &+\frac{1}{2}\frac{1}{\sqrt{\det \bar{g}_{\alpha\beta}}}\Big((x')^3\big(\bar{g}_{12}\bar{g}_{11,1} + \bar{g}_{11}(-2\bar{g}_{12,1} + \bar{g}_{11,2})\big) \\ &+(x')^2y'\big(\bar{g}_{22}\bar{g}_{11,1} + \bar{g}_{12}(-2\bar{g}_{12,1} + 3\bar{g}_{11,2}) - 2\bar{g}_{11}\bar{g}_{22,1}\big) \\ &+x'(y')^2\big(2\bar{g}_{22}\bar{g}_{11,2} + \bar{g}_{12}(2\bar{g}_{12,2} - 3\bar{g}_{22,1}) - \bar{g}_{11}\bar{g}_{22,2}\big) \\ &+(y')^3\big(\bar{g}_{22}(2\bar{g}_{12,2} - \bar{g}_{22,1}) - \bar{g}_{12}\bar{g}_{22,2}\big)\Big). \end{aligned}$$

Thus, in local coordinates, equation (5) reads as follows:

$$\begin{aligned} \sqrt{\det \bar{g}_{\alpha\beta}}(\eta y' - \xi x') \;=\;& (x''y' + y''x')\sqrt{\det \bar{g}_{\alpha\beta}} \\ &+\tfrac{1}{2}\tfrac{1}{\sqrt{\det \bar{g}_{\alpha\beta}}}\Big((x')^3\big(\bar{g}_{12}\bar{g}_{11,1} + \bar{g}_{11}(-2\bar{g}_{12,1} + \bar{g}_{11,2})\big) \\ &+(x')^2y'\big(\bar{g}_{22}\bar{g}_{11,1} + \bar{g}_{12}(-2\bar{g}_{12,1} + 3\bar{g}_{11,2}) - 2\bar{g}_{11}\bar{g}_{22,1}\big) \\ &+x'(y')^2\big(2\bar{g}_{22}\bar{g}_{11,2} + \bar{g}_{12}(2\bar{g}_{12,2} - 3\bar{g}_{22,1}) - \bar{g}_{11}\bar{g}_{22,2}\big) \\ &+(y')^3\big(\bar{g}_{22}(2\bar{g}_{12,2} - \bar{g}_{22,1}) - \bar{g}_{12}\bar{g}_{22,2}\big)\Big). \end{aligned} \tag{12}$$

We remark, that (12) simplifies if one chooses special coordinates. However, for concrete (numerical) computations, the general form (12) is quite practical, since one is not forced to first look for, e.g., isothermal coordinates.

Together with condition (9) and equation (12) we get the following system describing local soliton curves:

$$\bar{g}_{11}(x')^2 + 2\bar{g}_{12}x'y' + \bar{g}_{22}(y')^2 = 1 \tag{13}$$

$$\begin{aligned} x''y' - x'y'' = &-\frac{1}{2}\frac{1}{\det \bar{g}_{\alpha\beta}}\Big((x')^3\big(\bar{g}_{12}\bar{g}_{11,1} + \bar{g}_{11}(\bar{g}_{11,2} - 2\bar{g}_{12,1})\big) \\ &+ (x')^2y'\big(\bar{g}_{22}\bar{g}_{11,1} + \bar{g}_{12}(3\bar{g}_{11,2} - 2\bar{g}_{12,1}) - 2\bar{g}_{11}\bar{g}_{22,1}\big) \\ &+ x'(y')^2\big(2\bar{g}_{22}\bar{g}_{11,2} + \bar{g}_{12}(2\bar{g}_{12,2} - 3\bar{g}_{22,1}) - \bar{g}_{11}\bar{g}_{22,2}\big) \\ &+ (y')^3\big(\bar{g}_{22}(2\bar{g}_{12,2} - \bar{g}_{22,1}) - \bar{g}_{12}\bar{g}_{22,2}\big)\Big) \\ &- x'\xi + y'\eta\,. \end{aligned} \tag{14}$$

In the chosen chart we give ourselves the initial conditions

$$\begin{aligned} \big(x,y\big)(s_0) &= \big(x_0,y_0\big) \\ \big(x',y'\big)(s_0) &= \big(u_0,v_0\big) \end{aligned} \tag{15}$$

where we impose

$$\|(u_0,v_0)\|_{\bar{g}} = 1. \tag{16}$$

In order to rewrite (13)–(14) as a first order system, we differentiate (13). This yields, together with (14), two linear equations for x'' and y''. By (9), the determinant of the corresponding 2×2 system turns out to equal $-2 \neq 0$, and it can therefore be solved for x'' and y''. By setting $u := x'$ and $v := y'$ we obtain in this way the system

$$\begin{aligned} x' &= u \\ y' &= v \\ u' &= -\frac{1}{2}\frac{1}{\det \bar{g}_{\alpha\beta}}\Big(u^2(\bar{g}_{22}\bar{g}_{11,1} + \bar{g}_{12}(\bar{g}_{11,2} - 2\bar{g}_{12,1})) \\ &\quad + uv(2\bar{g}_{22}\bar{g}_{11,2} - 2\bar{g}_{12}\bar{g}_{22,1}) \\ &\quad + v^2(\bar{g}_{22}(2\bar{g}_{12,2} - \bar{g}_{22,1}) - \bar{g}_{12}\bar{g}_{22,2})\Big) \\ &\quad - u^2\bar{g}_{12}\xi - uv(\bar{g}_{22}\xi - \bar{g}_{12}\eta) + v^2\bar{g}_{22}\eta \\ v' &= -\frac{1}{2}\frac{1}{\det \bar{g}_{\alpha\beta}}\Big(u^2(\bar{g}_{12}\bar{g}_{11,1} + \bar{g}_{11}(\bar{g}_{11,2} - 2\bar{g}_{12,1})) \\ &\quad + uv(2\bar{g}_{12}\bar{g}_{11,2} - 2\bar{g}_{11}\bar{g}_{22,1}) \\ &\quad + v^2(\bar{g}_{12}(2\bar{g}_{12,2} - \bar{g}_{22,1}) - \bar{g}_{11}\bar{g}_{22,2})\Big) \\ &\quad + u^2\bar{g}_{11}\xi - uv(\bar{g}_{12}\xi - \bar{g}_{11}\eta) - v^2\bar{g}_{12}\eta \end{aligned} \tag{17}$$

with initial condition

$$(x, y, u, v)(s_0) = (x_0, y_0, u_0, v_0) \tag{18}$$

and (16).

We write the system (17) in compact form as $(x, y, u, v)' =: \Theta(x, y, u, v)$.

Therefore, every solution (x, y) of (13)–(14) with initial conditions (15)–(16) corresponds to a solution (x, y, x', y') of (17) with initial conditions (18) and (16). Vice versa, it is easy to check that every solution (x, y, u, v) of (17) with (18) and (16) corresponds to a solution (x, y) of (13)–(14) with (15)–(16).

Now, we show that in the chosen chart the system (17) has a maximal solution (x, y, u, v) and that this solution can be continued by adding other charts. Actually, we will see, that the solution can be continued arbitrarily far if N is geodesically complete.

Definition 3.1. — A C^1 curve $\gamma :]a, b[\to N$ is called a *solution* of

$$\langle X \, , \, \nu \rangle_{\bar{g}} \quad = \quad -H \tag{19}$$

if it satisfies the following condition:

For all $s_0 \in]a, b[$ there exists a chart $\phi : U \subset N \to V \subset \mathbb{R}^2$ with $\gamma(s_0) \in U$ such that for an $\varepsilon > 0$

(i) $\gamma|_{U_\varepsilon(s_0)} = \phi^{-1} \circ f$ for a C^1 curve

$$\begin{array}{rcl} f : U_\varepsilon(s_0) \subset]a, b[& \to & V \\ s & \mapsto & \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \end{array}$$

(ii) $\big(x(s), y(s), x'(s), y'(s)\big)$ is a solution of (17). ♢

For fixed $p \in N,\ v \in T_pN$ with $\|v\|_{\bar{g}} = 1$, we say, $[a, b] \subset \mathbb{R}$ is an existence interval, if there is $\gamma \in C^1([a, b], N)$ such that γ is a solution in the sense of Definition 3.1 on $]a, b[$ and $\gamma(0) = p,\ \gamma'(0) = v$. Then

$$I \quad := \quad \bigcup \{ \, [a, b] \subset \mathbb{R} \mid [a, b] \text{ is an existence interval } \}$$

is the maximum interval of existence.

Lemma 3.2. — *I is open and not empty.*

Proof. — The function $\Theta(x, y, u, v)$ in (17) is locally Lipschitz continuous with respect to x, y, u and v. Therefore, by the Theorem of Picard-Lindelöf, there exists a unique local C^1-solution $\big(x(s), y(s), u(s), v(s)\big)$ of (17) on an interval $] -\varepsilon, \varepsilon[$. Hence, I is not empty.

Now, we show, that for all $s \in I$ there exists a neighbourhood of s contained in I. By the definition of I there exists an existence interval $[a, b]$ such that $s \in [a, b]$.

If $s \in]a,b[$, we have nothing to show. So let us assume $s = b$ (the case $s = a$ is similar). We choose a chart $\phi : U \subset N \to V \subset \mathbb{R}^2$ with $\gamma(b) \in U$. Thus, there exists an $\varepsilon > 0$ such that $\gamma(]b-\varepsilon, b]) \subset U$ and $\phi \circ \gamma(]b-\varepsilon, b]) \subset V$. By the Definition 3.1 the map $f :]b-\varepsilon, b[\to V,\ s \mapsto \phi \circ \gamma(s)$ is a solution of (17). Since $\gamma \in C^1([a,b], N)$, $\gamma(b)$ and $\gamma'(b)$ are well defined. By the Theorem of Picard-Lindelöf there exists for a $\delta \in]0,\varepsilon[$ a unique map $\tilde{f} :]b-\delta, b+\delta[\to V$ such that $\tilde{f}$ is a solution of (17) and such that $\tilde{f}(b) = \phi \circ \gamma(b)$ and $\tilde{f}'(b) = \frac{d}{ds}\phi \circ \gamma(b)$. Due to the uniqueness it follows that $\tilde{f}|_{]b-\delta,b[} = f|_{]b-\delta,b[}$.

Therefore, all $s \in \mathbb{R}$ with $b - \delta < s < b + \delta$ belong to I, hence, I is open. □

The previous consideration basically shows, that a soliton curve can always be continued in an inner point of N. It is therefore natural to expect that

Lemma 3.3. — *If N is geodesically complete, then I is closed.*

Proof. — Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in I. We show: if $\lim_{n\to\infty} s_n = s$, then $s \in I$. It suffices to show this for $0 < s_n \nearrow s$. We have

$$0 \le \mathrm{d}(\gamma(s_m), \gamma(s_n)) \le \text{length of the curve } \gamma([s_n, s_m]) = |s_m - s_n| \tag{20}$$

where d is the geodesic distance on N. Hence, $\big(\gamma(s_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence on N. By the Theorem of Hopf-Rinow $\gamma(s_n)$ converges in N, hence, $\lim_{n\to\infty} \gamma(s_n) =: A \in N$.

If $s \in I$, there is nothing to show. So suppose otherwise $s \notin I$. Then we extend γ by setting $\gamma(s) := A$.

This extension is continuous in s, i.e., $\gamma(\tau_i) \to A$ $(i \to \infty)$ for all sequences $\tau_i \nearrow s$. Indeed $(s_1, \tau_1, s_2, \tau_2, \dots)$ is evidently a Cauchy sequence in $\mathbb{R}$. By (20) it follows that $\big(\gamma(s_1), \gamma(\tau_1), \gamma(s_2), \gamma(\tau_2), \dots\big)$ is a Cauchy sequence in N. Since $\gamma(s_n)$ converge to A, the whole sequence converge to A, in particular $\lim_{i\to\infty} \gamma(\tau_i) = A$.

Next, we show that $\gamma \in C^1([0,s], N)$. First of all, $\gamma \in C^1([0,s_n], N)$ for all n and thus $\gamma \in C^1([0,s[, N)$. Further, we consider a chart $\phi : U \subset N \to V \subset \mathbb{R}^2$ where $\gamma(s) \in U$ and with V bounded. Then $f := \phi \circ \gamma$ is on $]s-\varepsilon, s]$ well defined for some $\varepsilon > 0$ and on $]s-\varepsilon, s[$ it is a solution of (17) in the sense of Definition 3.1. In particular this means, that $f \in C^1(]s-\varepsilon, s[, V)$. So we have to show that $f \in C^1([s-\varepsilon, s], V)$. On $]s-\varepsilon, s[$ the functions x, y, u, and v are continuous and bounded. (The boundedness of u and v follows from (13).)

Thus, $\Theta\big(x(\cdot), y(\cdot), u(\cdot), v(\cdot)\big)$ is also continuous on the interval $]s-\varepsilon, s[$. Since Θ is Lipschitz continuous, $\Theta\big(x(\cdot), y(\cdot), u(\cdot), v(\cdot)\big)$ is moreover bounded on $]s-\varepsilon, s[$. Then, by (17), the functions x', y', u' and v' are on $]s-\varepsilon, s[$ continuous and bounded. In particular u and v are Lipschitz continuous on $]s-\varepsilon, s[$. Therefore, the functions $u = x'$ and $v = y'$ (and consequently f' and finally γ') can be continuously extended to the point s. Therefore, $s \in I$. □

The last two lemmas yield the following Theorem.

Theorem 3.4. — *Let N be a two-dimensional smooth, geodesically complete Riemannian manifold, which carries a smooth Killing vector field $X : N \to TN$. Then, for each point $p \in N$ and every direction $v \in T_pN$ with $v \neq 0$ there exists a unique soliton curve through p which is tangent to v and which can be extended arbitrarily far to both sides.*

From the proof of Lemma 3.2, it follows in particular, that through every fixed point in N in every direction a unique local solution of the soliton equation exists. We now want to have a closer look at the local behaviour and start by introducing some notation.

Let $v \in T_pN$. The curve γ_v denotes the unique solution of the soliton equation with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. Further, we consider the set $\Omega := \{\, v \in TN \mid \gamma_v \text{ is defined on } [0,1]\,\}$. We notice that $\Omega \cap T_pN$ is star-shaped. Now, we define the analogue of the exponential map for geodesics.

Definition 3.5. — The *soliton exponential mapping* is defined by

$$\begin{aligned} \mathrm{solexp} : \Omega &\to N \\ v &\mapsto \gamma_v(1). \end{aligned}$$

To ease notation we write $\mathrm{solexp}_p := \mathrm{solexp}|_{T_pN}$. ♢

Proposition 3.6. —

1. $\mathrm{solexp} : \Omega \to N$ *is differentiable.*
2. *The map*

$$\begin{aligned} \Phi : \Omega &\to N \times N \\ v &\mapsto (\pi(v)\,,\, \mathrm{solexp}_{\pi(v)}\, v) \end{aligned}$$

is a local diffeomorphism from a neighbourhood of $(p_0, 0) \in \Omega$ to a neighbourhood of $\Phi(p_0, 0) = (p_0, p_0) \in N \times N$.

Proof. —

1. This follows from the theory of ordinary differential equations (differentiable dependence of the solution from the initial conditions).
2. Locally, $TN|_U \cong U \times \mathbb{R}^n$, where U is a sufficiently small neighbourhood of p_0 in N. In the sequel we consider the restriction of Φ to $\big(U \times \mathbb{R}^n\big) \cap \Omega$ and denote this map still by Φ. Now, we consider a local chart from a subset of $\big(U \times \mathbb{R}^n\big) \cap \Omega$ to $V \times V$, $V \subset \mathbb{R}^n$. Further we consider a chart from a neighbourhood of $\Phi(p_0, 0) = (p_0, p_0) \in N \times N$ to $V' \times V'$, $V' \subset \mathbb{R}^n$. Then Φ can be viewed as a map $V \times V \to V' \times V'$, and we have to show that $D\Phi(p_0, 0)$ is invertible. For this, we calculate the matrix representing $D\Phi(p_0, 0)$ in the local coordinates.

For this purpose, let α be a differentiable curve in N with $\alpha(0) = p_0$. Then we get

$$\frac{d}{dt}\bigg|_{t=0} \Phi\big(\alpha(t), 0\big) = \frac{d}{dt}\bigg|_{t=0} \big(\alpha(t), \alpha(t)\big) = \big(\dot\alpha(0), \dot\alpha(0)\big).$$

Otherwise, for an arbitrary $v \in T_{p_0}N$ we get

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \Phi\big(p_0, tv\big) &= \frac{d}{dt}\bigg|_{t=0} \big(p_0, \gamma_{tv}(1)\big) = \frac{d}{dt}\bigg|_{t=0} \big(p_0, \gamma_v(t)\big) \\ &= \big(0, \dot\gamma_v(0)\big) = (0, v). \end{aligned}$$

Thus, we obtain

$$D\Phi(p_0, 0) = \begin{pmatrix} \mathrm{Id} & 0 \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

and therefore the desired result follows by the inverse function theorem. □

Corollary 3.7. — 1. *For all $p_0 \in N$ there exists a neighbourhood U and $\varepsilon > 0$, such that every pair of points $p, q \in U$ can be joined by a unique soliton curve with length smaller then ε.*

2. *For $\varepsilon > 0$ small enough,* solexp_{p_0} *is a diffeomorphism from $B_\varepsilon(0) \subset T_{p_0}N$ to its image.*

Proof. — 1. Let $\Phi : U' \to V' \subset N \times N$ be the diffeomorphism from Proposition 3.6 with $(p_0, 0) \in U'$. We choose an open set $V \subset N$ with $p_0 \in V$ and $\varepsilon > 0$ such that

$$U'' := \bigcup_{p \in V} B_\varepsilon(p) \subset U'.$$

Here $B_\varepsilon(p)$ is the ε-ball in TN. Now, we consider the restriction $\Phi|_{U''} : U'' \to V'' := \Phi(U'')$. In particular $\Phi(p_0, 0) = (p_0, p_0) \in V''$. We choose an open neighbourhood U of p_0 in N such that $U \times U \subset V''$. Let $p, q \in U$ and $\Phi^{-1}(p, q) =: (p, v) \in U''$, i.e., $\Phi(p, v) = (p, q) \in U \times U$. This means, that p and q are connected by a soliton curve of length $\|v\| < \varepsilon$. If there would be an other soliton curve connecting p and q with length smaller than ε, then there would exist a $v' \in T_pN$, $v' \neq v$, with $\|v'\| < \varepsilon$ such that $\Phi(p, v') = (p, q) = \Phi(p, v)$. But this contradicts the injectivity of Φ.

2. This follows directly from Proposition 3.6. □

Remark 3.8. — 1. Theorem 3.4 generalizes the results in [**17**, Section 2.1–2.4].

2. Corollary 3.7 shows that, locally, two points can always be joined by a unique short soliton curve. On the other hand, Theorem 3.4 shows that, on a geodesically complete Riemannian surface, every soliton curve can be extended arbitrarily far. However, as the example of the Grim Reaper shows, two arbitrary points need not be joinable in general. It remains the question, whether for two

arbitrary points, the joining soliton curve is unique, if it exists. We will give an answer to this question in Corollary 3.17.

3.2. Geometry of soliton curves. — In this section, we detect some interesting geometric properties of soliton curves on Riemannian surfaces. At first sight, this may come as a surprise, since soliton curves are solutions of a highly non-linear differential equation. On the other hand, soliton curves are geometrically motivated such that geometrical properties can be expected (see [**17**, Proposition 1, Corollary 1]). It is, however, not so obvious how to work out these properties because of the non-linear nature of the equation. Theorem 3.9 will be a crucial observation which will enable us to find the generalization of the invariant in [**17**, Lemma 1] to arbitrary Riemannian surfaces. Then, by applying the Theorem of Gauss-Bonnet, it will be possible to work with soliton curves almost as with geodesics.

At first, we recall some basic facts.

Let $x : U \subset N \to \mathbb{R}^n$ be a chart and let $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ denote the local coordinates. For a differentiable function $F : N \to \mathbb{R}$, the gradient $\nabla_g F$ of F with respect to the basis $\partial_i := \frac{\partial}{\partial \xi^i}$ of T_pN is given by

$$\nabla_g F = g^{-1} \partial F,$$

where $\partial F := (\frac{\partial F \circ x^{-1}}{\partial \xi^1}, \ldots, \frac{\partial F \circ x^{-1}}{\partial \xi^n})^T$, and g^{-1} is the inverse of the matrix describing the metric g.

Now, let X be a smooth vector field on N. We say, X has locally a *conjugate potential* if on every simply connected chart $U \subset N$ there exists a function $\chi : U \to \mathbb{R}$ such that $X = J\,\nabla_g \chi$ on U. Here, J is a complex structure on U compatible with g, i.e., $J \in \Gamma(T^1_1 U)$ is such that

$$\begin{aligned} J_p^2 &= -\operatorname{Id}_{T_pU} && \text{for all } p \in U \text{ and} \\ g(X,Y) &= g(JX, JY) && \text{for all } X, Y \in \Gamma(TU), \end{aligned}$$

where $J_p : T_pU \to T_pU$ is the induced endomorphism on T_pU.

Recall that the vector field $X : N \to TN$ is a gradient field on a simply connected set $U \subset N$, iff X satisfies in local coordinates

$$X_{i,j} = X_{j,i} \qquad \forall\, i, j$$

where commas indicate partial derivatives.

Using this, it is easy to check, that the vector field X has a conjugate potential on a simply connected set V, iff in local coordinates one of the two equivalent conditions is satisfied

$$(X^i \sqrt{g})_{,i} = 0 \tag{21}$$

$$X^i{}_{;i} = 0\,. \tag{22}$$

Here, we denote the covariant derivative by a semicolon, i.e., $X^\ell{}_{;i} = X^\ell{}_{,i} + X^k \Gamma^\ell_{ki}$.

Recall furthermore, that X is a Killing field on N, iff in local coordinates

$$(L_X\, g)_{ij} = X^k{}_{,i}\, g_{kj} + X^k{}_{,j}\, g_{ik} + g_{ij,k} X^k = 0\,, \tag{23}$$

where $L_X\, g$ is the Lie derivative of g along X.

Using again the covariant derivative, we can rewrite (23) as

$$0 = X_{i;j} + X_{j;i}\,. \tag{24}$$

Now, we come to the theorem which we mentioned in the introduction above.

Theorem 3.9. — *Each Killing vector field on a two dimensional Riemannian manifold possesses locally a conjugate potential.*

Proof. — Let X be the Killing vector field and $K_{ij} := X_{i;j} + X_{j;i}$. Since X satisfies the Killing equation (24), we have

$$0 = K_{ij} g^{ij} = X_{i;j}\, g^{ij} + X_{j;i} \underbrace{g^{ij}}_{=g^{ji}} = 2X_{i;j}\, g^{ij} = 2X^j{}_{;j}\,. \tag{25}$$

Hence, by (22), X has locally a conjugate potential.

In the last equality of (25), we used the product rule for the covariant derivative and the Lemma of Ricci ($g^{ij}{}_{;k} = 0$):

$$X^j{}_{;k} = (X_i\, g^{ij})_{;k} = X_{i;k}\, g^{ij} + X_i\, g^{ij}{}_{;k} = X_{i;k}\, g^{ij}.$$

Alternatively, we can compute the trace of $L_X g$ which is , according to (23), equal to zero. A short calculation yields

$$0 = (L_X g)^i{}_i = \frac{2}{\sqrt{g}} (X^i \sqrt{g})_{,i}\,.$$

Therefore, by (21), X has locally a conjugate potential. □

Remark 3.10. — In general, the converse of Theorem 3.9 is wrong: not every conjugate potential field is a Killing field.

To fix notation, we recall the Theorem of Gauss-Bonnet in the following form: let N be an oriented surface and Ω be a polygon in N. Further, let γ denote the piecewise smooth boundary curve $\partial\Omega$ of Ω, parametrized by arc length. The curve $\gamma(s)$ is assumed to be positively oriented. The exterior angles are denoted by $\alpha_1, \ldots, \alpha_q$. Then

$$\iint\limits_{\Omega} K\, d\mu + \int\limits_{\gamma} k_g\, ds + \sum_{i=1}^{q} \alpha_i = 2\pi\,. \tag{26}$$

In this formula, K denotes the Gaussian curvature of N and k_g the geodesic curvature of γ with respect to the given orientation of N and γ. The situation, in particular the measuring of the exterior angles, is illustrated in Figure 8.

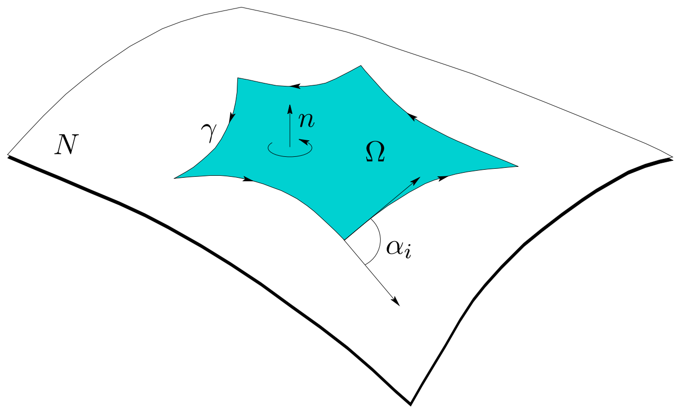

FIGURE 8. Polygon Ω with boundary curve γ and exterior angles on an oriented surface N

Now, using Theorem 3.9, we will be able to find an invariant quantity for soliton curves: let $\gamma : [s_0, s] \to N$ be a smooth soliton curve parametrized by arc length. Using the properties of J and Theorem 3.9, we can rewrite the soliton equation (1) locally as follows:

$$\begin{aligned} k_g\big(\gamma(t)\big) &= -\langle X\big(\gamma(t)\big), \nu\big(\gamma(t)\big)\rangle = -\langle JX\big(\gamma(t)\big), J\nu\big(\gamma(t)\big)\rangle \\ &= \langle \nabla_g \chi\big(\gamma(t)\big), \dot\gamma(t)\rangle = \frac{d}{dt}\chi\big(\gamma(t)\big). \end{aligned}$$

Now, we integrate both sides along the interval $[s_0, s]$ and obtain the difference of the potential in the starting and the end point

$$\int_{s_0}^{s} k_g\, dt = \chi\big(\gamma(s)\big) - \chi\big(\gamma(s_0)\big)\,. \tag{27}$$

Thus, the quantity

$$\chi\big(\gamma(s)\big) - \int_{s_0}^{s} k_g\, dt \tag{28}$$

is constant along a soliton curve. Now, using the invariant (28) together with (26), we can prove the following theorem about polygons whose sides are soliton curves.

Theorem 3.11. — *Let $U \subset N$ be simply connected and Ω be a polygon in U. Furthermore, let γ denote the piecewise smooth boundary curve $\partial\Omega$ of Ω, parametrized by* *arc length. The curve γ is assumed to be positively oriented and to consist of soliton curves. The exterior angles of the polygon are denoted by $\alpha_1, \ldots, \alpha_q$. Then*

$$\iint\limits_{\Omega} K\, d\mu + \sum_{i=1}^{q} \alpha_i = 2\pi\,. \tag{29}$$

Proof. — By the Gauss-Bonnet formula (26), we have only to show, that the integral of the geodesic curvature along the curve γ vanishes, i.e., $\int_\gamma k_g\, dt = \sum_{i=1}^{q} \int_{s_{i-1}}^{s_i} k_g\, dt = 0$. From (27) it follows, that each term in this sum is the difference of the conjugate potential in both endpoints. Therefore, since the boundary curve γ is closed, the sum must vanish indeed. □

Notice, that an alternative proof of Theorem 3.11 is possible by applying the divergence theorem. However, we preferred here to present a version which illustrates the use of the invariant quantity (28). This quantity can be used not only for closed curves and allows therefore to obtain quantitative information about soliton curves (see, e.g., Figure 4 in [**17**]).

Furthermore, notice, that the formula (29) also holds, if the boundary of Ω consists of geodesic curves: in this case the integrals $\int_{s_{i-1}}^{s_i} k_g\, ds$ vanish, since $k_g \equiv 0$.

Now, we apply Theorem 3.11 to obtain information about soliton n-gons. We start by closed soliton curves:

***Corollary 3.12* (zero-gon).** — *If the boundary of a bounded, simply connected domain $\Omega \subset N$ consists of a closed smooth soliton curve, then*

$$\iint\limits_{\Omega} K\, d\mu = 2\pi\,. \tag{30}$$

In particular, there do not exist any closed soliton curves which are boundary of a simply connected domain if one of the following three conditions is satisfied:

(i) $\iint\limits_{N} K^+\, d\mu < 2\pi$ *(where $K^+ := \max\{K, 0\}$)*

(ii) $K \geq 0$ *on N and* $\iint\limits_{N} K\, d\mu < 2\pi$

(iii) $K \leq 0$ *on N.*

Proof. — Formula (30) follows from Theorem 3.11, since there are no exterior angles. If we suppose that the boundary of a simply connected domain Ω is a closed soliton curve, then by (i) we have

$$\iint\limits_{\Omega} K\, d\mu \leq \iint\limits_{\Omega} K^+\, d\mu \leq \iint\limits_{N} K^+\, d\mu < 2\pi$$

which contradicts (30). Of course, (ii) and (iii) imply (i). □

Remark 3.13. — 1. In particular, from Corollary 3.12 it follows that in the Euclidean plane $\mathbb{R}^2$ there exist no closed soliton curves regardless of the Killing field.

2. Consider the unit sphere in $\mathbb{R}^3$ equipped with the symmetry group of rotations around the north-south axis. In this case, the equator is a closed soliton curve. Indeed, the area enclosed by this soliton curve is 2π, in accordance with (30).
3. On a one-sheet hyperboloid of revolution equipped with the natural isometry group, there exists a closed soliton curve, namely the gorge circle, although condition (iii) is fulfilled. This, of course, does not contradict the statement of Corollary 3.12, since the gorge circle does not enclose a simply connected domain.

A closed soliton curve can be seen as a zero-gon. Now, for a one-gon we have:

***Corollary 3.14* (one-gon).** — *If the boundary of a bounded, simply connected domain $\Omega \subset N$ consists of a soliton curve which is smooth except for at most one point, then*

$$\iint\limits_{\Omega} K\, d\mu \in \left]\pi, 3\pi\right[. \tag{31}$$

In particular, every soliton curve on a simply connected N is embedded if one of the following three conditions is satisfied:

(i) $\iint\limits_N K^+\, d\mu \leq \pi$

(ii) $K \geq 0$ *on* N *and* $\iint\limits_N K\, d\mu \leq \pi$

(iii) $K \leq 0$ *on* N.

Proof. — Since $q = 1$, formula (31) follows from Theorem 3.11: indeed, the only exterior angle belongs to $]-\pi, \pi[$. Notice, that the values π and $-\pi$ can be excluded due to the uniqueness of solutions through a point in a given direction (see Corollary 3.7).

Now, let N be simply connected and suppose that there is a soliton curve on N which is not embedded. Then, a suitable restriction of this soliton curve is the boundary of a simply connected domain Ω (for more details see [**19**, section 4.1]). Then, by (i) we have

$$\iint\limits_{\Omega} K\, d\mu \leq \iint\limits_{\Omega} K^+\, d\mu \leq \iint\limits_{N} K^+\, d\mu \leq \pi$$

which contradicts (31). Of course, (ii) and (iii) imply (i). □

Remark 3.15. — From this corollary it follows, that all soliton curves in the Euclidean plane $\mathbb{R}^2$ are embedded no matter what the Killing field is. In particular we recover the result [**17**, Proposition 1 (a)], namely that Yin-Yang curves are embedded. ♢

Example 3.16. — We consider a regular cone with half opening angle $\beta \in]0, \frac{\pi}{2}[$. For the symmetry group we take the rotation group around the cone axis. We are interested in the question, whether a global soliton curve on this cone is embedded or not. Such a soliton does not pass through the apex. To apply Corollary 3.14 we smooth the apex by attaching a spherical cap which is so small, that the soliton avoids it.

Now, we suppose that the soliton curve is not embedded. Then, a suitable restriction of this soliton curve is the boundary of a simply connected domain Ω. If Ω does not contain the spherical cap, then a contradiction follows from formula (31), since in this case $K = 0$ on Ω. If, on the other hand, Ω contains the spherical cap, then

$$\iint_{\Omega} K \, d\mu = 2\pi(1 - \sin\beta).$$

The term on the right hand side is less than or equal to π iff $\beta \geq \frac{\pi}{6}$. In other words, if the half opening angle of the cone is greater than or equal to $\frac{\pi}{6}$ then every soliton curve on the cone is embedded. Figure 9 shows projections of solitons on cones with three different half opening angles. In Figure 10 a soliton curve on a cone with half opening

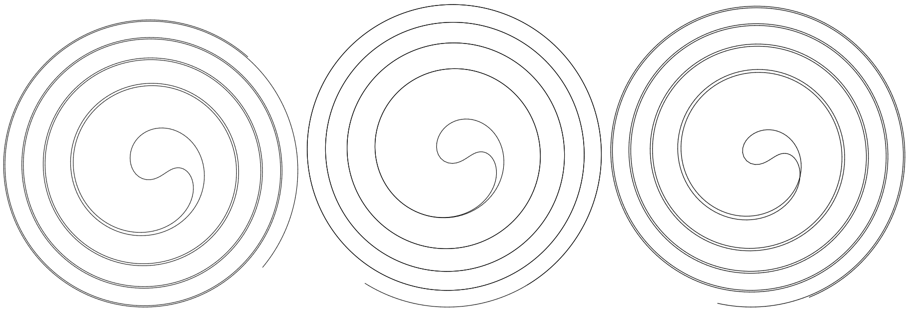

FIGURE 9. These three curves are projections of soliton curves in the direction of the cone axis. On the left, the half opening angle is $\frac{\pi}{6} + 0.05$, in the middle, we have the critical value $\frac{\pi}{6}$ and on the right $\frac{\pi}{6} - 0.05$. Corresponding to Example 3.16 the first two solitons are embedded. On the right, the soliton curve shows, at least numerically, a self-intersection.

angle $\frac{\pi}{6} + 0.05$ (which is therefore embedded) is displayed. For concrete formulas and calculations see [**19**, Section 4.3 and Section 4.3.1]. ♢

After considering one-gons we investigate soliton curves composing a two-gon. If we can exclude the existence of two-gons then in particular the soliton curves are globally unique. This means, that there is at most one soliton curve through two different points on a surface.

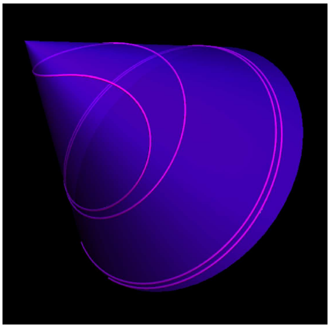

FIGURE 10. An embedded soliton curve on a cone with half opening angle $\frac{\pi}{6} + 0.05$

***Corollary 3.17* (two-gons).** — *Let $\Omega \subset N$ be a simply connected soliton two-gon. This means, its boundary consists of two different curves which are restrictions of two soliton curves to intervals. Then*

$$\iint_{\Omega} K \, d\mu \in \,]0, 4\pi[\, . \tag{32}$$

In particular, if N is simply connected and $K \leq 0$ on N, then two points on N can be connected by at most one soliton curve.

Proof. — Since $q = 2$, formula (32) follows from Theorem 3.11: indeed, the two exterior angles α_1 and α_2 belong to $]-\pi, \pi[$. Notice, that the values π and $-\pi$ can be excluded due to the uniqueness of solutions through a point in a given direction (see Corollary 3.7).

Now, let N be simply connected with $K \leq 0$. Furthermore, we suppose that there are two different soliton curves joining two points on N. From Corollary 3.14 it follows that both solitons are embedded. Then suitable restrictions of these two soliton curves are the boundary of a soliton two-gon (see [**19**, section 4.2]). A contradiction follows by applying formula (32) to this two-gon. □

Remark 3.18. —
1. In particular, by this corollary, we recover the result in [**17**, Proposition 1 (b)]: two different Yin-Yang curves (with respect to the same Killing field) intersect in at most one point.
2. Corollary 3.17 answers the previous question in Remark 3.8 about the uniqueness of solitons joining two points. ♢

Example 3.19. — Now, we consider the one-sheet hyperboloid of revolution $N := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1 \}$ equipped with the isometry group of rotations with respect to the z-axis. We claim that every soliton on N cuts the gorge circle at most once. To see this, we cannot directly use Corollary 3.17 since N is not simply connected. However, we can pass to the universal cover:

$$(\mathbb{R}^2, g(t,\varphi)) \to M, \quad (t,\varphi) \mapsto (\cosh t \cos\varphi, \cosh t \sin\varphi, \sinh t)$$

with

$$g(t,\varphi) = \begin{pmatrix} \cosh^2 t + \sinh^2 t & 0 \\ 0 & \cosh^2 t \end{pmatrix}.$$

The Killing field $X = (1,0)$ generates the rotation group. Now, consider two distinct points on the φ-axis (i.e., the images are situated on the gorge circle). The φ-axis is a soliton curve joining these two points. Since $K < 0$ and $\mathbb{R}^2$ is simply connected, it follows by Corollary 3.17, that there is no other soliton curve joining the two points. This implies the desired result on N. On the right hand side of Figure 11 there is an embedded soliton curve cutting the gorge circle once. On the left there is a soliton which avoids the gorge circle. Notice, that this soliton has a self intersection on the hyperboloid but, by Corollary 3.14, not on its universal cover. ♢

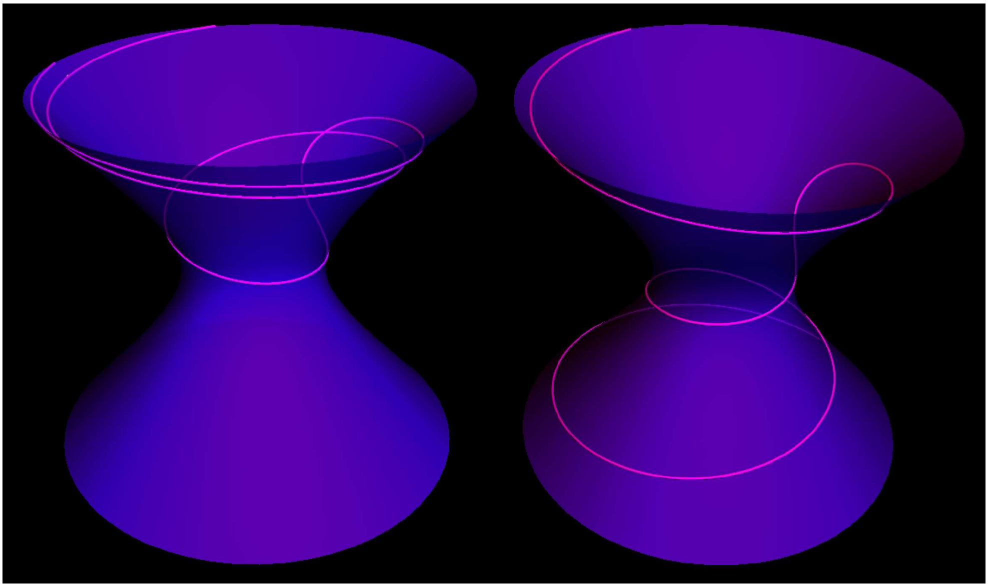

FIGURE 11. Two soliton curves on a one-sheet hyperboloid of revolution

Example 3.20. — We can now design situations where every soliton is embedded, but where two-gons may occur: consider a sheet N of a two-sheet hyperboloid of revolution. We choose for the half opening angle of the asymptotic cone a value in $]\frac{\pi}{6}, \frac{\pi}{2}[$. Then the curvature is

$$0 < \iint\limits_N K\, d\mu < \pi.$$

Therefore, by Corollary 3.14, it follows that the solitons with respect to the rotation group are embedded. Nonetheless, as we can see in Figure 12, two-gons occur. $\diamondsuit$

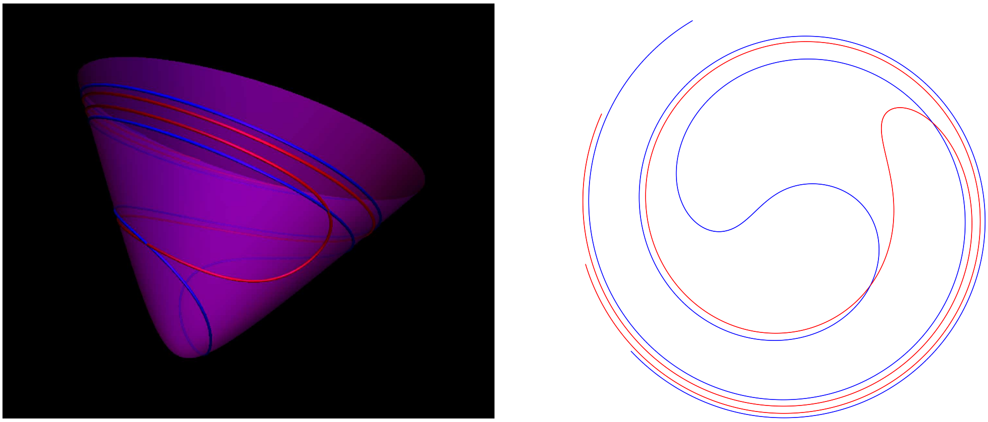

FIGURE 12. Embedded solitons on a sheet of a two-sheet hyperboloid of revolution which form a two-gon (on the right in a projection)

Example 3.21. — As in Example 3.19 we can identify two types of soliton curves on a standard helicoid: solitons that cut the axis exactly once (Figure 13, left) and solitons that avoid the axis (Figure 13, right). $\diamondsuit$

Remark 3.22. —

1. The isometry groups of the surfaces in the examples above were induced by isometry groups of the ambient Euclidean space (translation, rotation, screw motion). Of course there are examples, where the isometry group of the surface is intrinsic. For example the Enneper-minimal-surface features an isometry group generated by rotations in the standard parameter plane.
2. It is easy to see, that the geodesics with respect to the metric $g_{ij}(x,y) = e^{-2x}\delta_{ij}$ on $\mathbb{R}^2$ are the Grim Reaper curves. It is therefore natural to ask, whether one can find always (at least locally) a modified metric on N that has as geodesics precisely the solitons with respect to the original metric on N and the given Killing field. However, as new results by Thomas Mettler indicate, the answer to this question is, in general, negative. $\diamondsuit$

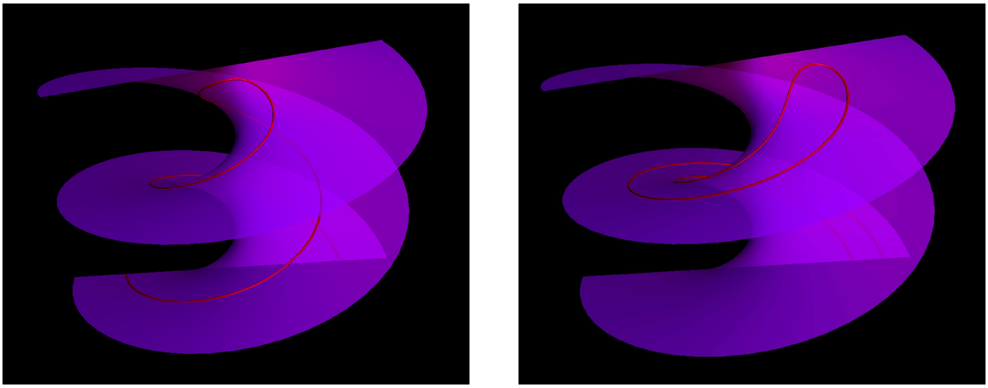

FIGURE 13. Two soliton curves on a standard helicoid

4. Local existence for higher dimensional solitons

4.1. Problem setting. — In this section we ask for local solutions of the soliton equation (5). More precisely, we will treat the boundary value problem. This corresponds to the following problem, e.g., for a 3-dimensional ambient manifold N: suppose, a closed curve γ is moving in N under the action of a one parameter group of isometries of N. Does there exist a surface whose boundary is, at some fixed time, the curve γ, and that satisfies (5)? This surface would then move under the mean curvature flow together with γ just as by the isometry group. Observe, that for the Killing field $X = 0$ this is the Plateau problem for minimal surfaces.

The general framework we want to work with is as follows: N continues to denote the ambient manifold of dimension n and we suppose $n \geq 3$, since the corresponding problem in two dimensions is treated in Corollary 3.7.1 (local existence and uniqueness) and in Corollary 3.17 (global uniqueness). We consider a chart $\psi : V \subset N \to U \subset \mathbb{R}^n$, such that for a non-empty open set $\Omega \subset \mathbb{R}^{n-1}$ we have $\bar{\Omega} \times \{0\} \subset U$. We take Ω as our reference manifold M and look for a soliton of the form

$$\phi : \Omega \to (U, \bar{g}), \ x \mapsto (x, u(x)). \tag{33}$$

For the Plateau problem (i.e., $X = 0$) it is well known, that, in general, we cannot expect a solution to exists for arbitrary boundary data. We will see in the sequel, what kind of conditions we have to impose in order to ensure existence.

We denote indices ranging in the set $\{1, \ldots, n-1\}$ by Latin, indices from the set $\{1, \ldots, n\}$ by Greek letters, and we adopt the Einstein summation convention. Let e_i end e_α be the standard basis in $\mathbb{R}^{n-1}$ and in $\mathbb{R}^n$ respectively. We write u_i for the partial derivative $\frac{\partial u}{\partial x_i}$. Then, the soliton equation (5) transforms into an equation for the unknown function u:

$$\langle X(x,u(x)), \nu(x,u(x)) \rangle = -H(x,u(x)). \tag{34}$$

First, we need to make this equation explicit. Let us start with the left hand side. To determine ν we consider the map $F : \Omega \times \mathbb{R} \subset \mathbb{R}^n \to \mathbb{R}$, $(x_1,\dots,x_n) \mapsto u(x_1,\dots,x_{n-1}) - x_n$, having the graph of u (i.e., the soliton) as a level set. Then, since the gradient

$$\nabla_{\bar g} F = \bar g^{\alpha\beta} \frac{\partial F}{\partial x^\alpha}\, e_\beta$$

is perpendicular to the level set, we get the (downward-pointing) normal

$$\nu = \frac{\nabla_{\bar g} F}{\|\nabla_{\bar g} F\|_{\bar g}}.$$

The components are actually

$$(\nabla_{\bar g} F)^\alpha \quad = \quad u_i \bar g^{i\alpha} - \bar g^{n\alpha}.$$

The norm of the gradient will not be needed explicitly, since it drops out anyway. Therefore, we can write the left hand side of (34) as follows:

$$X^\alpha \nu^\beta \bar g_{\alpha\beta} = \frac{1}{\|\nabla_{\bar g} F\|_{\bar g}} \big(X^k u_k - X^n\big). \tag{35}$$

For the right hand side, we use the Gauss formula (6) and (7). Recall, that g is the metric induced on $M = \Omega$, i.e.,

$$g_{ij} = \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \bar g_{\alpha\beta} = \bar g_{ij} + u_j\, \bar g_{in} + u_i\, \bar g_{jn} + u_i u_j \bar g_{nn}.$$

Then, multiplying (6) by $\bar g_{\alpha\delta}\nu^\delta$, we obtain

$$\frac{\partial^2 \phi}{\partial x^i \partial x^j} \bar g_{\alpha\delta} \nu^\delta + \bar\Gamma^\alpha_{\beta\gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} \bar g_{\alpha\delta} \nu^\delta = -h_{ij}. \tag{36}$$

Expansion of the terms on the left of (36) yields

$$\begin{aligned} -h_{ij} \quad &= \quad \frac{1}{\|\nabla_{\bar g} F\|_{\bar g}} \big(-u_{ij} + \bar\Gamma^k_{ij} u_k - \bar\Gamma^n_{ij} + \bar\Gamma^k_{in} u_j u_k - \bar\Gamma^n_{in} u_j + \\ &\qquad + \bar\Gamma^k_{nj} u_i u_k - \bar\Gamma^n_{nj} u_i + \bar\Gamma^k_{nn} u_i u_j u_k - \bar\Gamma^n_{nn} u_i u_j\big). \end{aligned} \tag{37}$$

Finally, using $H = g^{ij} h_{ij}$, equation (34) can be written in the form

$$\begin{aligned} X^k u_k - X^n \quad &= \quad -\, g^{ij} u_{ij} + g^{ij} \left(\bar\Gamma^k_{ij} u_k - \bar\Gamma^n_{ij}\right) + 2 g^{ij} \left(\bar\Gamma^k_{in} u_j u_k - \bar\Gamma^n_{in} u_j\right) \\ &\qquad + g^{ij} \left(\bar\Gamma^k_{nn} u_i u_j u_k - \bar\Gamma^n_{nn} u_i u_j\right). \end{aligned} \tag{38}$$

In this formula, the functions are to be evaluated in the following way: $X^i = X^i(x,u(x))$, $u_j = u_j(x)$, $\bar\Gamma^k_{ij} = \bar\Gamma^k_{ij}(x,u(x))$, and the metric g^{ij} depends on $x, u(x)$ and of $\nabla u(x)$, where x ranges in $\Omega \subset \mathbb{R}^{n-1}$.

At this point we impose boundary data

$$u = \varphi \qquad \text{on } \partial\Omega. \tag{39}$$

However, for simplicity, we will only treat the case $\varphi \equiv 0$ below. Notice, that even for homogeneous boundary conditions, problem (38)–(39) does not need to have a solution, as the example of the Grim Reaper curve shows (which, in general, cannot be represented as a graph over a segment joining two of its points). It will therefore be necessary to implement other hypotheses.

4.2. Solution by the implicit function theorem. — There are various possibilities to solve the Dirichlet problem (38)–(39). In [**17**] the Banach contraction theorem is used to construct a solution in case $N = \mathbb{R}^n$ for isometry groups of rotations. The advantage is, that the method yields a numerical algorithm at the same time. Alternatively, the Schauder fixed point Theorem can be used as well. In our quite general setting, it turns out, that the implicit function theorem still provides a rather economic proof. We are going to use the theorem with the following notation: let X, Y and Z be Banach spaces, and $W \subset X \times Y$ open and non-empty. Suppose $f \in C^1(W, Z)$ and $(x_0, y_0) \in W$ is such that
(i) $f(x_0, y_0) = 0$ and
(ii) $D_y f(x_0, y_0) \in L(Y, Z)$ is an isomorphism.

Then, there exist open neighborhoods $U(x_0), V(y_0)$ with $U(x_0) \times V(y_0) \subset W$ such that for all $x \in U(x_0)$ there exists a unique $y \in V(y_0)$ such that $f(x, y) = 0$. In particular, the mapping

$$g : U(x_0) \to V(y_0), \quad x \mapsto y$$

is well defined by $f(x, g(x)) = 0$, and $g \in C^1(U(x_0), V(y_0))$.

The aim is now to solve

$$\begin{aligned} -g^{ij}u_{ij} + g^{ij}\left(\bar\Gamma^k_{ij}u_k - \bar\Gamma^n_{ij}\right)& & \\ +2g^{ij}\left(\bar\Gamma^k_{in}u_ju_k - \bar\Gamma^n_{in}u_j\right)& & \\ +g^{ij}\left(\bar\Gamma^k_{nn}u_iu_ju_k - \bar\Gamma^n_{nn}u_iu_j\right) &= \lambda\left(X^k u_k - X^n\right) &\quad \text{on } \Omega \qquad (40)\\ u &= 0 &\quad \text{on } \partial\Omega \qquad (41) \end{aligned}$$

this way. Observe, that we have replaced the original Killing field in (38) by the Killing field λX, of course with the idea to consider sufficiently small values of λ.

Remark 4.1. — Notice, that for $\lambda = 0$, (40) is the equation of a minimal surface with the given boundary $\partial\Omega$. It is therefore natural to postulate, that the coordinates are chosen in such a way, that the graph of $u \equiv 0$ is a minimal surface. (This corresponds to the situation in [**17**] for the Euclidean case.) ♢

In order to make everything work, we choose the following spaces: $X := \mathbb{R}$, $Y := C^{2,\alpha}(\bar{\Omega}) \cap C_0(\bar{\Omega})$, $Z := C^{0,\alpha}(\bar{\Omega})$ and $W = X \times Y$. Here, $C_0(\bar{\Omega})$ is the set of continuous functions on $\bar{\Omega}$, which vanish on $\partial\Omega$. The space Y is equipped with the norm of $C^{2,\alpha}(\bar{\Omega})$. Concerning notation and definitions of Hölder spaces, we follow [**10**].

Let

$$\begin{array}{rcl} f : W & \to & Z \\ (\lambda, u) & \mapsto & -g^{ij} u_{ij} + g^{ij}\left(\bar{\Gamma}^k_{ij} u_k - \bar{\Gamma}^n_{ij}\right) + 2g^{ij}\left(\bar{\Gamma}^k_{in} u_j u_k - \bar{\Gamma}^n_{in} u_j\right) \\ & & + g^{ij}\left(\bar{\Gamma}^k_{nn} u_i u_j u_k - \bar{\Gamma}^n_{nn} u_i u_j\right) - \lambda\left(X^k u_k - X^n\right) \end{array}$$

and $(\lambda_0, u_0) = (0,0)$.

It is standard to verify that $f \in C^1\big(\mathbb{R} \times \big(C^{2,\alpha}(\bar{\Omega}) \cap C_0(\bar{\Omega})\big), C^{0,\alpha}(\bar{\Omega})\big)$ (see, e.g., [**19**] for details).

In view of Remark 4.1, property (i) in the implicit function theorem, i.e., $f(0,0) = 0$, is also satisfied.

To verify property (ii), we need to compute the derivative of f with respect to u in the point $(\lambda_0, u_0) = (0,0)$: we find

$$\begin{array}{rcl} D_u f(0,0) : C^{2,\alpha}(\bar{\Omega}) \cap C_0(\bar{\Omega}) & \to & C^{0,\alpha}(\bar{\Omega}) \\ v & \mapsto & -g^{ij}(x,0,0) v_{ij} \\ & & + g^{ij}(x,0,0)\bar{\Gamma}^k_{ij}(x,0) v_k \\ & & - D_u\left(g^{ij}(x,0,0)\bar{\Gamma}^n_{ij}(x,0)\right) v \\ & & - D_{u_k}\left(g^{ij}(x,0,0)\bar{\Gamma}^n_{ij}(x,0)\right) v_k \\ & & - 2g^{ij}(x,0,0)\bar{\Gamma}^n_{in}(x,0) v_j\,. \end{array}$$

This is a strictly elliptic operator on Ω. Let us assume, that Ω has a $C^{2,\alpha}$ boundary. Then, the condition on the term which is linear in v

$$D_u\left(g^{ij}(x,0,0)\bar{\Gamma}^n_{ij}(x,0)\right) \leq 0 \tag{42}$$

implies, by [**10**, Theorem 6.14], that $D_u f(0,0)$ is an isomorphism. Therefore, we may conclude by the inverse function theorem, that (40), (41) possesses a unique solution for all values of λ which are sufficiently small, and this solution depends continuously differentiable on λ. In order to formulate a theorem, we introduce the following manner of speaking: let N be an n-dimensional Riemannian manifold carrying a Killing vector field $X : N \to TN$. Moreover, let $N' \subset N$ be an embedded orientable minimal hypersurface, and $\psi : V \subset N \to U \subset \mathbb{R}^n$ a chart having the property that $\psi^{-1}\big((\mathbb{R}^{n-1} \times \{0\}) \cap U\big) = N'$. Such a chart (or such coordinates) will be called *minimal.* If condition (42) is satisfied on $(\mathbb{R}^{n-1} \times \{0\}) \cap U$ we call the coordinates *admissible.* If for N' admissible coordinates exist, we call N' *admissible.*

Theorem 4.2. — *Let N be an n-dimensional Riemannian manifold carrying a Killing vector field X. Suppose $N' \subset N$ is an admissible hypersurface. If γ is the boundary of a $C^{2,\alpha}$ domain $\Sigma \Subset N'$, then, for λ suitably small, there exists a soliton hypersurface with respect to the Killing field λX and having boundary γ.*

Remark 4.3. — We close the discussion at this point with a few remarks concerning the condition (42): according to (40), $g^{ij}(x,u,0)\bar{\Gamma}^n_{ij}(x,u)$ is just the mean curvature of the level set surface of $u \equiv$ constant. Therefore, if N' is a leave of a foliation of N in a neighborhood of N' consisting of minimal surfaces, then we even have $D_u\big(g^{ij}(x,0,0)\bar{\Gamma}^n_{ij}(x,0)\big) = 0$ if the coordinates are chosen such that the leaves correspond to the graphs of constant functions. Such a foliation exists in particular if X (or another Killing field on N) is transversal to N'.

If condition (42) is violated, then $D_u f(0,0)v = w$ does not need to have a solution any more. However, it is still possible to formulate a Fredholm alternative and to replace (42) by the condition that $D_u f(0,0)v = 0$ (with homogeneous boundary data) has only the trivial solution. ♢

The particular case of Theorem 4.2 when X generates a rotation on $N = \mathbb{R}^n$, and if N' is a hypersurface, was already treated in [**17**, Theorem 1].

To illustrate Theorem 4.2, we consider the case of a screw motion in $\mathbb{R}^3$. The Killing field

$$X(x) = \begin{pmatrix} -x_2\omega \\ x_1\omega \\ \lambda \end{pmatrix}$$

generates a screw motion along the x_3-axis. We want to solve the soliton equation $\langle X, \nu\rangle = -H$ locally, and represent the solution as graph of a function u over the unit disk in the (x_1,x_2)-plane. Theorem 4.2 guarantees the existence of a solution for angular velocity ω and translation velocity in x_3-direction λ both small enough. (A more precise analysis actually shows, that in this case it suffices that the transversal component, λ, be small.) For the metric, we get $g_{ij} = \delta_{ij} + u_iu_j$ and $g^{ij} = \delta^{ij} - \frac{1}{w^2}\delta^{ik}u_k\delta^{jl}u_l$ respectively, where $w^2 = 1 + \delta^{ij}u_iu_j$. The unit normal in a point of the graph of u is

$$\nu = -\frac{1}{w}(\delta^{kl}u_l\frac{\partial}{\partial x^k} - \frac{\partial}{\partial u}).$$

For the Christoffel symbols, we compute

$$\Gamma^k_{ij} = u_{ij}g^{kr}u_r\,.$$

Finally, using the Gauss equation (6), the mean curvature turns out to be

$$-H = \frac{1}{w}g^{ij}u_{ij}.$$

Therefore, the soliton equation reads as follows:

$$w^2(\omega(u_1x_2 - u_2x_1) + \lambda) = u_{11}(1 + u_2^2) + u_{22}(1 + u_1^2) - 2u_1u_2u_{12}.$$

Using the idea in [**17**, Section 3], this equation can easily be written as a fixed point problem for a contraction operator. This yields a numerical iteration scheme: in each iteration step, a linear elliptic equation has to be solved, which is achieved, e.g., by the Gauss-Seidel algorithm. The combination with a successive grid refinement yields a reasonably fast converging algorithm. The solution, for a certain choice of boundary data, is displayed below in Figure 14. Observe, that the "belly" in the central region is responsible for the translation, and the four slightly asymmetric "noses" at the boundary yield the rotation.

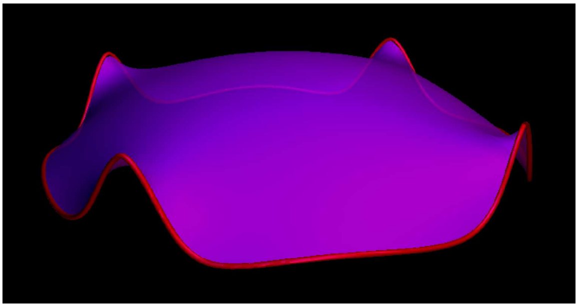

FIGURE 14. A screwing soliton

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