Norbert Hungerbühler

m-harmonic flow

m-Harmonic Flow

NORBERT HUNGERBÜHLER

Abstract

We prove that the *m*-harmonic flow of maps from a Riemannian manifold M of dimension m into a compact Riemannian manifold N has for arbitrary initial data of finite *m*-energy a global weak solution which is partially regular, i.e. up to finitely many singular times $t_1, \ldots, t_k$ the gradient is $C^{0,\alpha}$ in space-time. The number k of singular times is a priori bounded in terms of the initial energy and the geometry. Two solutions with identical initial data and bounded gradient coincide.

1. – Introduction

Let M and N be smooth compact Riemannian manifolds without boundary and with metrics γ and g respectively. Let m and n denote the dimensions of M and N. For a C^1-map $f : M \to N$ the p-energy density is defined by

$$(1) \qquad e(f)(x) := \frac{1}{p} |df_x|^p$$

and the p-energy by

$$(2) \qquad E(f) := \int_M e(f) \, d\mu \, .$$

Here, p denotes a real number in $[2, \infty[$, $|df_x|$ is the Hilbert-Schmidt norm with respect to γ and g of the differential $df_x \in T_x^*(M) \otimes T_{f(x)}(N)$ and μ is the measure on M which is induced by the metric.

The motivation to consider this class of energies is twofold: One motivation is the physics of liquid crystals if we think of a liquid crystal as consisting of bar-shaped particles described by a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}P^2$ or $f : \Omega \subset \mathbb{R}^3 \to S^2$ if the bars have distinguishable ends. Critical points of the p-energy then correspond (in this model) to stationary states of the liquid crystal

and the energy flow below suggests some aspects of its dynamical behavior (see e.g. [18]). The mathematical motivation for the p-harmonic flow is the general interest in geometric evolution problems of which the p-harmonic flow is a highly nonlinear and degenerate example and shares some features with harmonic maps and harmonic flow ($p = 2$). The main source of difficulty is that the equations exhibit a supercritical growth and since several years there has been an increasing interest in how the additional information given by the geometric structure of the problem can be used to establish compactness and existence results (see e.g. [43], [57], [52], [23], [38], [62], [39], [40], [41]).

For concrete calculations we will need $E(f)$ in local coordinates:

$$E_U(f) = \frac{1}{p} \int_\Omega \left(\gamma^{\alpha\beta} (g_{ij} \circ f) f_\alpha^i f_\beta^j \right)^{\frac{p}{2}} \sqrt{\gamma} \, dx \, .$$

Here, $U \subset M$ and $\Omega \subset \mathbb{R}^m$ denote the domain and the range of the coordinates on M and it is assumed that $f(U)$ is contained in the domain of the coordinates chosen on N. Upper indices denote components, whereas f_α denotes the derivative of f with respect to the indexed variable x^α. We use the usual summation convention, and $\sqrt{\gamma}$ always means $\sqrt{|\det \gamma|}$.

If p coincides with the dimension m of the manifold M then the energy E is conformally invariant.

Variation of the energy-functional yields the Euler-Lagrange equations of the p-energy which are

$$(3) \qquad \Delta_p f = - \left(\gamma^{\alpha\beta} g_{ij} f_\alpha^i f_\beta^j \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \Gamma_{ij}^l f_\alpha^i f_\beta^j$$

in local coordinates. The operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{ij} f_\alpha^i f_\beta^j \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} f_\alpha^l \right)$$

is called p-Laplace operator (for $p = 2$ this is just the Laplace-Beltrami operator). On the right hand side of (3) the Γ_{ij}^l denote the Christoffel-symbols related to the manifold N. According to Nash's embedding theorem we can think of N as being isometrically embedded in some Euclidean space $\mathbb{R}^k$ since N is compact. Then, if we denote by F the function f regarded as a function into $N \subset \mathbb{R}^k$, equation (3) admits a geometric interpretation, namely

$$\Delta_p F \perp T_F N$$

with Δ_p being the p-Laplace operator with respect to the manifolds M and $\mathbb{R}^k$.

For $p > 2$ the p-Laplace operator is degenerate elliptic. (Weak) solutions of (3) are called (weakly) p-harmonic maps.

The regularity theory of weakly p-harmonic maps involves an extensive part of the theory of nonlinear partial differential equations. We mention the

contributions of Hardt and Lin, Giaquinta and Modica, Fusco and Hutchinson, Luckhaus, Coron and Gulliver, Fuchs, Duzaar, DiBenedetto, Friedman, Choe, and refer to their work listed in the bibliography. One of the most important recent results is due to Hélein [38], who proved regularity of weakly harmonic maps on surfaces.

One possibility to produce p-harmonic maps is to investigate the heat flow related to the p-energy, i.e. to look at the flow-equation

$$(4) \qquad\qquad\qquad \partial_t f - \Delta_p f \perp T_f N$$

$$(5) \qquad\qquad\qquad f|_{t=0} = f_0$$

or explicitly for (4)

$$(6) \qquad\qquad \partial_t f - \Delta_p f = (p\, e(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)$$

where $A(f)(\,\cdot\,,\,\cdot\,)$ is the second fundamental form on N. For $p = 2$ Eells and Sampson showed in their famous work [17] of 1964, that there exist global solutions of (4) provided N has non-positive sectional curvature and that the flow tends for suitable $t_k \to \infty$ to a harmonic map. We see that under the mentioned geometric condition the heat flow also solves the homotopy problem, i.e. to find a harmonic map homotopic to a given map. Surprisingly, also a topological condition on the target may suffice to solve the homotopy problem. Lemaire [45] (and independently also Sacks-Uhlenbeck [54]) obtained this result under the assumption that $m = 2$ and $\pi_2(N) = 0$. For negative sectional curvature and arbitrary $p > 2$ the homotopy problem was solved by Duzaar and Fuchs in [15] by using different methods than the heat flow.

Struwe proved in [58] existence and uniqueness of partially regular weak solutions of the harmonic flow on Riemannian surfaces. Recently, Freire extended the uniqueness result to the class of weak solutions in $W^{1,2}(M \times [0, T])$ with non-increasing energy (see [20], [21], [22]). In the higher dimensional case Y. Chen [4] (and independently Keller, Rubinstein and Sternberg [42] as well as Shatah [56]) showed existence of global weak solutions of the harmonic flow into spheres by using a penalizing technique. This technique together with Struwe's monotonicity formula (see [59]) was ,used in the corresponding proof for arbitrary target-manifold N (Chen-Struwe [6]). The existence of weak solutions of the p-harmonic flow into spheres was shown by Chen, Hong and the author in [5]. This result has been extended to homogeneous spaces as target manifolds by the author in [40].

Parallel to this development the theory of degenerate parabolic systems with controllable growth has been developed by DiBenedetto, Friedman, Choe and other authors (see bibliography).

Δ_2 is a linear elliptic diagonal matrix operator in divergence form. For $p > 2$ $(0 < p < 2)$ the operator is degenerate (singular) at $\nabla f = 0$. The right hand side of (6) is for $p = 2$ a quadratic form in the first derivatives of f

with coefficients depending on f. These strong nonlinearities are caused by the non-Euclidean structure of the target manifold N and cannot be removed by special choices of coordinates on N unless N is locally isometric to the Euclidean space $\mathbb{R}^n$. But even in this case the space of mappings from M to N does not possess a natural linear structure unless N itself is a linear space. In general, the right hand side of (6) is of the order of the p-th power in the gradient of f (non-controllable or natural growth).

Our main existence result for the m-harmonic flow is Theorem 10 in Section 3.7. The uniqueness results are located in Section 4.

2. – A priori estimates

We start with the definition of the energy space:

DEFINITION 1. The space of mappings of the class $W^{1,p}$ from M to N is

$$W^{1,p}(M, N) := \{f \in W^{1,p}(M, \mathbb{R}^k); \ f(x) \in N \text{ for } \mu\text{-almost all } x \in M\}$$

equipped with the topology inherited from the topology of the linear Sobolev space $W^{1,p}(M, \mathbb{R}^k)$.

The nonlinear space $W^{1,p}(M, N)$ defined above depends on the embedding of N in $\mathbb{R}^k$. This fact does not cause any problems if M and N are compact since then different embeddings give rise to homeomorphic spaces $W^{1,p}(M, N)$.

The space $H^{1,p}(M, N)$ defined as the closure of the class of smooth functions from M to N in the $W^{1,p}$-norm is contained in $W^{1,p}(M, N)$ but does not coincide with the latter space in general (this fact gives rise to the so called "gap phenomenon" of Hardt-Lin [37]). This important observation was first made by Schoen and Uhlenbeck: see Eells and Lemaire [16] as a main reference. However, we have $H^{1,p}(M, N) = W^{1,p}(M, N)$ if $\dim(M) = p$ (see Schoen and Uhlenbeck [55], Bethuel [1] or Bethuel and Zheng [2]).

The regularity of minimizing p-harmonic mappings between two compact smooth Riemannian manifolds has been widely discussed, see Hardt and Lin [37] Giaquinta and Modica [32], Fusco and Hutchinson [31], Luckhaus [47], Coron and Gulliver [10] and Fuchs in [25] who also discussed obstacle problems for minimizing p-harmonic mappings (see [24] and [27]-[30]). The results of these investigations may be summarized briefly as follows:

Suppose $1 < p < \infty$ and let M and N be two smooth compact Riemannian manifolds, M possibly having a boundary ∂M. Consider mappings $f : M \to N$ minimizing the p-energy and having fixed trace on ∂M. Such a minimizer f is locally Hölder continuous on $M \setminus Z$ for some compact subset Z of $M \setminus \partial M$ which has Hausdorff dimension at most $\dim(M) - [p] - 1$. Moreover Z is a finite set in case $\dim(M) = [p] + 1$ and empty in the case $\dim(M) < [p] + 1$. On $(M \setminus Z) \setminus \partial M$, the gradient of f is also locally Hölder continuous.

The set Z is defined as the set of points a in M for which the normalized p-energy on the ball $B_r(a)$,

$$r^{p-\dim(M)} \int_{M \cap B_r(a)} |df|^p d\mu,$$

fails to approach zero as $r \to 0$. The technique to prove the above assertions essentially is to show that, near points $a \in (M \setminus Z) \setminus \partial M$, this normalized integral decays for $r \to 0$ like a positive power of r. Then, local Hölder continuity of f on $(M \setminus Z) \setminus \partial M$ follows by Morrey's lemma. The proof of the Hölder continuity of the gradient of f is much more difficult.

Instead of looking for minimizers of the p-energy, Uhlenbeck investigated weak solutions of systems of the form

$$\text{div}\left(\rho(|\nabla f|^2)\nabla f\right) = 0$$

where ρ satisfies some ellipticity and growth condition (see Uhlenbeck [63]). Her work prompted an extensive study of quasilinear elliptic *scalar* equations having a lack of ellipticity: Evans in [19] and Lewis in [46] showed the $C^{1,\alpha}$-regularity for rather special equations. Later DiBenedetto [11] and Tolksdorf in [60], [61] proved $C^{1,\alpha}$-regularity of the solutions of rather general quasilinear equations which are allowed to have such a lack of ellipticity. It is remarkable that Ural'ceva in [64] obtained Evan's result already in 1968.

For such equations and systems, $C^{1,\alpha}$-regularity is optimal. Tolksdorf gave in [60] an example of a scalar function minimizing the p-energy and which does not belong to $C^{1,\alpha}$, if $\alpha \in]0, 1[$ is chosen sufficiently close to one.

In contrast to equations, everywhere-regularity cannot be obtained for *general* elliptic quasilinear systems. The counterexample of Giusti and Miranda in [33] shows that it is generally impossible to obtain C^α-everywhere regularity for homogeneous quasilinear systems with analytic coefficients satisfying the usual ellipticity and growth conditions. Nevertheless almost-everywhere-regularity has been obtained for rather general classes of quasilinear elliptic systems: see Morrey [49] or Giusti-Miranda [33].

As far as the regularity of the heat flow of p-harmonic maps for $p > 2$ is concerned only results in the Euclidean case are known: In [13] DiBenedetto and Friedman investigated weak solutions $f : \Omega \times (0, T] \to \mathbb{R}^m$ of the parabolic system

$$f_t - \text{div}(|\nabla f|^{p-2}\nabla f) = 0,$$

where Ω is an open set in $\mathbb{R}^n$. The main result of DiBenedetto and Friedman is that for $\max\left\{1, \frac{2n}{n+2}\right\} < p < \infty$ weak solutions of this problem are regular in the sense that ∇f is continuous on $\Omega \times (0, T]$ with $|\nabla f(x, t) - \nabla f(\bar{x}, \bar{t})| \le \omega(|x - \bar{x}| + |t - \bar{t}|^{\frac{1}{2}})$ in any compact subset $\tilde{\Omega}$ of Ω, where ω depends only on $\tilde{\Omega}$ and the norms of f in the spaces $L^\infty(0, T; L^2(\Omega))$ and $L^p(0, T; W^{1,p}(\Omega))$. In [14] the same authors obtained for $p > \max\{1, \frac{2n}{n+2}\}$ Hölder regularity of ∇f in $\Omega \times (0, T]$ using a combination of Moser iteration and De Giorgi iteration. H. Choe investigated in [7] weak solutions of the system

$$(7) \qquad f_t - \operatorname{div}(|\nabla f|^{p-2}\nabla f) + b(x, t, f, \nabla f) = 0$$

where b respects the growth condition

$$(8) \quad |b_x(x, t, f, Q)| + |b_f(x, t, f, Q)\|Q| + |b_{Q_\alpha^i}(x, t, f, Q)Q_\alpha^i| \le c(1 + |Q|^{p-1})$$

and where $f \in C^0(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$. In this case, Choe proves $f \in C_{\text{loc}}^{0,\alpha}(\Omega \times (0, T])$ for some $\alpha > 0$ provided $f \in L_{\text{loc}}^{r_0}(\Omega \times (0, T])$ for some $r_0 > \frac{n(2-p)}{p}$. Notice that the p-harmonic flow does not satisfy condition (8): there the growth is of order $|Q|^p$. This will be one of the main difficulties in the study of the p-harmonic flow.

2.1. – Energy estimate

Now we establish an energy estimate for strong solutions of the p-harmonic flow. Notice that this lemma is true without the conformality assumption $\dim(M) = p$.

LEMMA 2. *Let $f \in C^2(M \times [0, T], N)$ be a solution to the p-harmonic flow (6). Then the following energy equality holds for all t_1, t_2 with $0 \le t_1 < t_2 \le T$:*

$$(9) \qquad \int_{t_1}^{t_2} \int_M |\partial_t f|^2 d\mu\, dt + E(f(t_2)) = E(f(t_1)).$$

PROOF.

STEP 1. We multiply (6) by $\partial_t f$. Since $\partial_t f \in T_f N$ the right hand side vanishes.

STEP 2. On the left hand side we use

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\gamma^{\alpha\beta} f_\alpha^i f_\beta^i\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta} f_\alpha^i \partial_t f^i\right) = \Delta_p f \cdot \partial_t f + \frac{d}{dt}e(f).$$

The divergence term vanishes if we integrate over M. Integrating over the time interval $[t_1, t_2]$ we get

$$(10) \qquad -\int_{t_1}^{t_2} \int_M \Delta_p f f_t\, d\mu\, dt = E(f(t_2)) - E(f(t_1))$$

and hence the desired result.   □

2.2. – L^{2p}-estimate for ∇f

We start with a variant of the Gagliardo-Nirenberg estimate. Notice that from now on we assume $p = \dim(M)$.

LEMMA 3. *Let M be a Riemannian manifold of dimension $m = p$ and ι the injectivity radius of M. Then there exist constants $c > 0$ and $R_0 \in]0, \iota]$ only depending on M, N, such that for any measurable function $f : M \times [0, T] \to N$ ($T > 0$ arbitrary), any $B_R(x) \subset M$ with $R \in]0, R_0]$ and any function $\varphi \in L^\infty(B_R(x))$ depending only on the distance from x, i.e. $\varphi(y) \equiv \varphi(|x - y|)$, and non-increasing as a function of this distance, the estimate*

$$
(11) \quad
\begin{aligned}
\int_0^T \int_M |f|^{2p} \varphi \, d\mu \, dt &\leq c \operatorname*{ess\,sup}_{0 \leq t \leq T} \left(\int_{B_R(x)} |f|^p d\mu \right)^{\frac{2}{p}} \int_0^T \int_M \left| \nabla |f|^{p-1} \right|^2 \varphi \, d\mu \, dt \\
&+ \frac{c}{\mu(B_R(x))} \operatorname*{ess\,sup}_{0 \leq t \leq T} \int_{B_R(x)} |f|^p d\mu \cdot \int_0^T \int_M |f|^p \varphi \, d\mu \, dt
\end{aligned}
$$

holds, provided $\varphi \equiv 1$ on $B_{R/2}(x)$.

REMARK. Here, $B_R(x)$ denotes the geodesic ball in M around x with radius R, i.e. $B_R(x) = \{y \in M : \operatorname{dist}_M(x, y) < R\}$, where $\operatorname{dist}_M(x, y)$ means the geodesic distance of $x, y \in M$ with respect to the given metric γ on the manifold M.

PROOF. (i) Suppose first that $\varphi \equiv 1$ and assume that the right hand side of (11) is finite. Let $g \in H^{1,2}(B_R(x))$ be a function with vanishing mean value $\int_{B_R(x)} g = 0$. Then we infer from ([44] Chap. 2, § 2, Theorem 2.2)

$$
(12) \quad \|g\|_{L^{\frac{2p}{p-1}}(B_R(x))} \leq \beta \|\nabla g\|_{L^2(B_R(x))}^{\frac{p-1}{p}} \|g\|_{L^{\frac{p}{p-1}}(B_R(x))}^{\frac{1}{p}} .
$$

We apply (12) to the function $g = |f|^{p-1} - \lambda$ with

$$
\lambda = \frac{1}{\mu(B_R(x))} \int_{B_R(x)} |f|^{p-1} d\mu .
$$

We get

$$
(13) \quad
\begin{aligned}
&\int_0^T \int_{B_R(x)} |f|^{2p} d\mu \, dt \\
&\leq c \int_0^T \int_{B_R(x)} \left| |f|^{p-1} - \lambda \right|^{\frac{2p}{p-1}} d\mu \, dt + c \int_0^T \int_{B_R(x)} \lambda^{\frac{2p}{p-1}} d\mu \, dt \\
&\leq c \operatorname*{ess\,sup}_{0 \leq t \leq T} \left(\int_{B_R(x)} \left| |f|^{p-1} - \lambda \right|^{\frac{p}{p-1}} d\mu \right)^{\frac{2}{p}} \int_0^T \int_{B_R(x)} \left| \nabla |f|^{p-1} \right|^2 d\mu \, dt \\
&+ c \int_0^T \frac{1}{\mu(B_R(x))^{\frac{2p}{p-1} - 1}} \left(\int_{B_R(x)} |f|^{p-1} d\mu \right)^{\frac{2p}{p-1}} dt .
\end{aligned}
$$

The various terms on the right hand side of (13) are estimated in the following way:

$$(14) \quad \int_{B_R(x)} \left| |f|^{p-1} - \lambda \right|^{\frac{p}{p-1}} d\mu \leq 2^{\frac{p}{p-1}} \left(\int_{B_R(x)} |f|^p d\mu + \int_{B_R(x)} \lambda^{\frac{p}{p-1}} d\mu \right)$$

$$(15) \quad \begin{aligned} \int_{B_R(x)} \lambda^{\frac{p}{p-1}} d\mu &= \mu(B_R(x))^{\frac{-1}{p-1}} \left(\int_{B_R(x)} |f|^{p-1} d\mu \right)^{\frac{p}{p-1}} \\ &\leq \int_{B_R(x)} |f|^p d\mu \end{aligned}$$

$$\left(\int_{B_R(x)} |f|^{p-1} d\mu \right)^{\frac{2p}{p-1}} \leq \left(\int_{B_R(x)} |f|^p d\mu \right)^2 \cdot \mu(B_R(x))^{\frac{2}{p-1}}$$

$$(16) \quad \leq \mu(B_R(x))^{\frac{2}{p-1}} \operatorname*{ess\,sup}_{0 \leq t \leq T} \int_{B_R(x)} |f|^p d\mu \cdot \int_{B_R(x)} |f|^p d\mu.$$

Plugging in the estimates (14)-(16) into (13) the assertion for $\varphi \equiv 1$ follows. (ii) By linearity and (i) the assertion remains true for step functions φ which are non-increasing in radial distance and which satisfy $\varphi \equiv 1$ on $B_{R/2}(x)$. Finally, the general case follows by density of the step functions in $L^\infty(B_R(x))$ in measure. $\qquad \square$

Now we try to carry over Choe's results for p-Laplace-systems in [8] to the case of natural growth in the inhomogeneity. To do this we need to control the local energy:

DEFINITION 4. We denote by

$$E(f(t), B_R(x)) = \int_{B_R(x)} e(f(\cdot, t)) \, d\mu$$

the local energy of the function $f(\cdot, t) : M \to N$ in the geodesic ball $B_R(x) \subset M$.

Given some uniform control of the local energy, we can estimate the $2p$-norm of the gradient and higher derivatives.

LEMMA 5. *Let ι denote the injectivity radius of the manifold M. If $m = \dim(M) = p$ then there exists $\varepsilon_1 > 0$ which only depends on M and N with the following property:*

If $f \in C^2(B_{3R}(y) \times [0, T[; N)$ with $E(f(t)) \leq E_0$ is a solution of the m-harmonic flow (6) on $B_{3R}(y) \times [0, T[$ for some $R \in]0, \frac{\iota}{3}[$ and if

$$\sup \{E(f(t), B_R(x)); \, 0 \leq t \leq T, x \in B_{2R}(y)\} < \varepsilon_1$$

then we have for every $x \in B_R(y)$

(17)
$$\int_0^T \int_{B_R(x)} |\nabla^2 f|^2 |\nabla f|^{2m-4} d\mu \, dt < c \, E_0 \left(1 + \frac{T}{R^m}\right)$$

and

(18)
$$\int_0^T \int_{B_R(x)} |\nabla f|^{2m} d\mu \, dt < c \, E_0 \left(1 + \frac{T}{R^m}\right)$$

for some constant c which only depends on the manifolds M and N.

PROOF. For simplicity we consider the case of a flat torus $M = \mathbb{R}^m / \mathbb{Z}^m$. (For a general manifold terms involving the metric of M and its derivative occur.) Let $\varphi \in C_0^\infty(B_{2R}(y))$ be a cutoff function satisfying $0 \leq \varphi \leq 1$, $\varphi|_{B_R(y)} \equiv 1$ and $|\nabla\varphi| < \frac{2}{R}$. The equation of the p-harmonic flow which takes the form

(19)
$$f_t - \nabla(\nabla f |\nabla f|^{p-2}) \perp T_f N$$

is now tested by the function $\nabla(\nabla f |\nabla f|^{p-2})\varphi^p$. Using the explicit form of the right hand side in (6) we get

(20)
$$|f_t \Delta_p f \varphi^p - (\Delta_p f)^2 \varphi^p| \leq c|\nabla f|^p |\Delta_p f| \varphi^p$$

with a constant c only depending on N. For brevity let $Q = B_{2R}(y) \times [0, T[$. Integrating over Q we obtain

(21)
$$\int_Q \left(\frac{1}{p}\frac{d}{dt}|\nabla f|^p \varphi^p + |\Delta_p f|^2 \varphi^p\right) dx \, dt$$
$$= \int_Q \left(-\nabla(\nabla f |\nabla f|^{p-2}\varphi^p) f_t + |\Delta_p f|^2 \varphi^p\right) dx \, dt$$
$$= \int_Q \left(-\Delta_p f \varphi^p f_t - p\nabla f |\nabla f|^{p-2}\varphi^{p-1}\nabla\varphi f_t + |\Delta_p f|^2 \varphi^p\right) dx \, dt$$
$$\leq \int_Q \left(p|\Delta_p f||\nabla f||\nabla f|^{p-2}\varphi^{p-1}|\nabla\varphi| + c|\nabla f|^p |\Delta_p f| \varphi^p\right) dx \, dt \, .$$

In the last step we used (20) and then equation (6) to substitute f_t by $\Delta_p f$. By Young's inequality we can estimate the last line in (21) by

$$(22) \qquad \int_Q \left(\frac{1}{4} |\Delta_p f|^2 \varphi^p + c |\nabla f|^{2p} \varphi^p + c |\nabla f|^{2p-2} |\nabla \varphi|^2 \varphi^{p-2} \right) dx\, dt\,.$$

By integrating by parts twice, exchanging derivatives and rearranging the resulting terms we find that for arbitrary functions $f \in C^2(Q; N)$ there holds

$$(23) \qquad \begin{aligned} \int_Q |\Delta_p f|^2 \varphi^p dx\, dt &\geq \tfrac{1}{2} \int_Q |\nabla^2 f|^2 |\nabla f|^{2p-4} \varphi^p dx\, dt \\ &\quad - c \int_Q |\nabla \varphi|^2 |\nabla f|^{2p-2} \varphi^{p-2} dx\, dt \end{aligned}$$

for a constant c. Putting (21)–(23) together we obtain

$$(24) \qquad \begin{aligned} \int_Q |\nabla^2 f|^2 &|\nabla f|^{2p-4} \varphi^p dx\, dt \\ &\leq c \int_{B_{2R}(y)} |\nabla f(\cdot, 0)|^p dx \\ &\quad + c \int_Q \left(|\nabla \varphi|^2 |\nabla f|^{2p-2} \varphi^{p-2} + |\nabla f|^{2p} \varphi^p \right) dx\, dt\,. \end{aligned}$$

The second term on the right hand side of (24) may be estimated separately by Hölder's and Young's inequality:

$$(25) \qquad \int_Q |\nabla \varphi|^2 |\nabla f|^{2p-2} \varphi^{p-2} dx\, dt \leq c \int_Q \left(|\nabla \varphi|^p |\nabla f|^p + |\nabla f|^{2p} \varphi^p \right) dx\, dt\,.$$

Hence, from (24) and (25) it follows

$$(26) \qquad \begin{aligned} \int_Q |\nabla^2 f|^2 &|\nabla f|^{2p-4} \varphi^p dx\, dt \\ &\leq c \int_{B_{2R}(y)} |\nabla f(\cdot, 0)|^p dx \\ &\quad + c \int_Q \left(|\nabla f|^{2p} \varphi^p + |\nabla \varphi|^p |\nabla f|^p \right) dx\, dt\,. \end{aligned}$$

From Lemma 3 we infer

$$(27) \qquad \begin{aligned} \int_Q |\nabla f|^{2p} &\varphi^p dx\, dt \\ &\leq c \sup_{0 \leq t \leq T, x \in B_{2R}(y)} \left(\int_{B_R(x)} |\nabla f(\cdot, t)|^p dx \right)^{\frac{2}{p}} \int_Q |\nabla^2 f|^2 |\nabla f|^{2p-4} \varphi^p dx\, dt \\ &\quad + \frac{c}{R^m} \sup_{0 \leq t \leq T, x \in B_{2R}(y)} \int_{B_R(x)} |\nabla f(\cdot, t)|^p dx \int_Q |\nabla f|^p dx\, dt\,. \end{aligned}$$

Hence, for $\varepsilon_1 > 0$ small enough, the condition

$$\sup_{0 \le t \le T, x \in B_{2R}(y)} \int_{B_R(x)} |\nabla f(\cdot, t)|^p d\mu < \varepsilon_1$$

used in (27) and in (26) implies the estimate (17) and by applying Lemma 3 once again, we get (18). □

2.3. – Higher integrability

In order to describe how much the energy is concentrated we will use the following quantity:

DEFINITION 6. For a function $f: M \times [t_1, t_2] \to N$, $f \in L^\infty(t_1, t_2; W^{1,p}(M, N))$, $\varepsilon > 0$ and $\Omega \subset M \times [t_1, t_2]$ let

$$R^*(\varepsilon, f, \Omega) = \operatorname{ess\,sup} \left\{ R \in [0, \iota] : \operatorname*{ess\,sup}_{(x,t) \in \Omega} (E(f(t)), B_R(x)) < \varepsilon \right\}$$

where ι denotes the injectivity radius of M.

LEMMA 7. *Let* $q \in]2p, \infty[$ *be a given constant. If* $\dim(M) = p$ *then there exists a constant* $\varepsilon_2 > 0$ *only depending on* M, N *and* q *with the following property:*

For any solution $f \in C^2(M \times [0, T[; N)$ *of the m-harmonic flow* (6) *with* $R^* = R^*(\varepsilon_2, f, M \times [0, T[) > 0$ *and any open set* $\Omega \subset\subset M \times]0, T[$ *there exists a constant* C *which only depends on* p, q, M, N, T, E_0, R^* *and* $\operatorname{dist}(\Omega, M \times \{0\})$ *such that*

$$\int_\Omega |\nabla f|^q dx < C.$$

E_0 *denotes the initial energy* $E_0 = E(f(\cdot, 0))$.

REMARK. It seems very inconvenient that ε_2 may not be chosen independently of the level q of integrability we want to reach. We will obtain a better result below which uses the assertion of this lemma in a technical way.

PROOF. For simplicity we consider the case $M = \mathbb{R}^m/\mathbb{Z}^m$. Again we write (6) in the form

$$(28) \qquad \partial_t f - \nabla(\nabla f |\nabla f|^{p-2}) = |\nabla f|^{p-2} A(f)(\nabla f, \nabla f)$$

Let us first fix a few notations: For $(x_0, t_0) \in M \times]0, T[$ let

$$B_R = B_R(x_0) = \{|x_0 - x| < R\}$$

$$Q_R = Q_R(x_0, t_0) = B_R \times]t_0 - R^p, t_0[$$

$$Q_R(\sigma_1, \sigma_2) = B_{R-\sigma_1 R} \times]t_0 - (1 - \sigma_2)R^p, t_0[\text{ for } \sigma_i \in]0, 1[.$$

For R small enough such that $\overline{Q}_R \subset M \times]0, T[$ we take a cutoff function ζ with

$$\zeta = 1 \quad \text{on } Q_R(\sigma_1, \sigma_2)$$

$$\zeta = 0 \quad \text{in a neighborhood of the parabolic boundary of } Q_R$$

and with

(29) $0 \le \zeta \le 1, \quad |\nabla \zeta| \le \dfrac{2}{\sigma_1 R}, \quad |\partial_t \zeta| \le \dfrac{2}{\sigma_2 R^p}$.

We now use $-\nabla(\nabla f \, v^\alpha \zeta^2)$, with $v = |\nabla f|^2$ and α to be chosen later, as a testfunction in equation (28). This gives rise to the following calculations:

(i) The first term on the left gives

$$\int_{Q_R} f_t(-\nabla(\nabla f v^\alpha \zeta^2)) \, dx \, dt = \frac{1}{2(\alpha+1)} \int_{Q_R} \zeta^2 \partial_t v^{\alpha+1} dx \, dt$$

$$= \frac{1}{2(\alpha+1)} \int_{B_R} v^{\alpha+1} \zeta^2(\cdot, t_0) \, dx$$

$$- \frac{1}{2(\alpha+1)} \int_{Q_R} v^{\alpha+1} 2\zeta \zeta_t \, dx \, dt .$$

(ii) For the second term on the left we observe that

$$\int_{Q_R} (f_k^i |\nabla f|^{p-2})_k (f_j^i v^\alpha \zeta^2)_j \, dx \, dt = \int_{Q_R} (f_k^i |\nabla f|^{p-2})_j (f_j^i v^\alpha \zeta^2)_k \, dx \, dt .$$

In order not to lose control on the various terms we proceed here in two steps.

(a) First we get

$$\int_{Q_R} |\nabla f|^{p-2} f_{jk}^i \left(f_{jk}^i v^\alpha \zeta^2 + \alpha f_j^i v^{\alpha-1} v_k \zeta^2 + 2 f_j^i v^\alpha \zeta \zeta_k \right) dx \, dt$$

$$= \int_{Q_R} \left(\zeta^2 \sum_{ijk} (f_{jk}^i)^2 v^{\frac{2\alpha+p-2}{2}} + \frac{\alpha}{2} \zeta^2 v^{\frac{2\alpha+p-4}{2}} |\nabla v|^2 + v^{\frac{2\alpha+p-2}{2}} \nabla v \zeta \nabla \zeta \right) dx \, dt.$$

(b) Second we find

$$\int_{Q_R} \left(|\nabla f|^{p-2} \right)_j f_k^i \left(f_{jk}^i v^\alpha \zeta^2 + \alpha f_j^i v^{\alpha-1} v_k \zeta^2 + 2 f_j^i v^\alpha \zeta \zeta_k \right) dx \, dt$$

$$= \int_{Q_R} \left(\frac{p-2}{4} |\nabla f|^{p-4} |\nabla v|^2 v^\alpha \zeta^2 + \frac{\alpha(p-2)}{2} |\nabla f|^{p-4} v^{\alpha-1} \zeta^2 \sum_i (\nabla f^i \nabla v)^2 \right.$$

$$\left. + (p-2)|\nabla f|^{p-4} \zeta v^\alpha \sum_i (\nabla f^i \nabla v)(\nabla f^i \nabla \zeta) \right) dx \, dt .$$

(iii) On the right hand side we finally get

(30)
$$\int_{Q_R} (|\nabla f|^{p-2} A(f)(\nabla f, \nabla f))_j f_j v^\alpha \zeta^2 dx\, dt$$
$$= \int_{Q_R} |\nabla f|^{p-2} (\nabla A(f))(\nabla f, \nabla f) \nabla f v^\alpha \zeta^2 dx\, dt.$$

In the calculation (iii) of the right hand side of (30) we used the fact that $f_j v^\alpha \zeta^2 \in T_f N$. Putting all the terms (i)-(iii) together and taking the supremum over the time interval $]t_0 - (1 - \sigma_2)R^p, t_0[$ we get by using the fact that the positive terms are non-decreasing in t

(31)
$$\frac{1}{4(\alpha + 1)} \operatorname*{ess\,sup}_{t_0 - (1-\sigma_2)R^p < t < t_0} \int_{B_R} v^{\alpha+1} \zeta^2(\cdot, t)\, dx$$
$$+ \frac{1}{2} \int_{Q_R} \zeta^2 v^{\frac{p+2\alpha-2}{2}} \sum_{ijk} (f^i_{jk})^2 dx\, dt$$
$$+ \frac{\alpha(p-2)}{4} \int_{Q_R} \zeta^2 v^{\frac{p+2\alpha-6}{2}} \sum_i (\nabla f^i \nabla v)^2 dx\, dt$$
$$+ \frac{\alpha + p - 2}{4} \int_{Q_R} v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \zeta^2 dx\, dt$$
$$\le \frac{1}{\alpha + 1} \int_{Q_R} v^{\alpha+1} \zeta |\zeta_t| dx\, dt + \int_{Q_R} v^{\frac{p+2\alpha-2}{2}} |\nabla v| \zeta |\nabla \zeta| dx\, dt$$
$$+ (p-2) \int_{Q_R} v^{\frac{p+2\alpha-6}{2}} \sum_i |\nabla f^i \nabla v| v^{\frac{3}{2}} \zeta |\nabla \zeta| dx\, dt$$
$$+ c \int_{Q_R} \zeta^2 v^{\frac{p+2\alpha+2}{2}} dx\, dt.$$

Two terms on the right hand side of the inequality (31) need to be interpolated by the binomic inequality

(32)
$$\int_{Q_R} v^{\frac{p+2\alpha-2}{2}} |\nabla v| \zeta |\nabla \zeta| dx\, dt$$
$$\le \frac{1}{2\varepsilon} \int_{Q_R} v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \zeta^2 dx\, dt + \frac{\varepsilon}{2} \int_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 dx\, dt,$$

(33)
$$(p-2) \int_{Q_R} v^{\frac{p+2\alpha-6}{2}} \sum_i |\nabla f^i \nabla v| v^{\frac{3}{2}} \zeta |\nabla \zeta| dx\, dt$$
$$\le \frac{p-2}{2\delta} \int_{Q_R} v^{\frac{p+2\alpha-6}{2}} \zeta^2 \sum_i |\nabla f^i \nabla v|^2 dx\, dt$$
$$+ \frac{\delta(p-2)}{2} \int_{Q_R} v^{\frac{p+2\alpha-6}{2}} v^3 |\nabla \zeta|^2 dx\, dt.$$

Thus, choosing the constants ε and δ in (32) and (33) appropriately, e.g. $\varepsilon = \frac{4}{\alpha+p-2}$ and $\delta = \frac{4}{\alpha}$, and absorbing the resulting terms we obtain from (31)

(34)
$$
\frac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p < t < t_0} \int_{B_R} v^{\alpha+1}\zeta^2(\cdot\,,t)\,dx
$$
$$
+ \frac{\alpha+p-2}{8} \int_{Q_R} \zeta^2 v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2 dx\,dt
$$
$$
\leq \frac{1}{\alpha+1} \int_{Q_R} v^{\alpha+1}\zeta|\zeta_t|dx\,dt + 2\left(\frac{1}{\alpha+p-2}+\frac{p-2}{\alpha}\right)
$$
$$
\cdot \int_{Q_R} v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2 dx\,dt + c \int_{Q_R} v^{\frac{p+2+2\alpha}{2}}\zeta^2 dx\,dt\,.
$$

Since we trivially have

$$
\int_{Q_R} \zeta^2 v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2 dx\,dt \geq \frac{8}{(p+2\alpha)^2} \int_{Q_R}\left|\nabla\left(v^{\frac{p+2\alpha}{4}}\zeta\right)\right|^2 dx\,dt
$$
$$
- \frac{16}{(p+2\alpha)^2}\int_{Q_R}|\nabla\zeta|^2 v^{\frac{p+2\alpha}{2}} dx\,dt
$$

it follows from (34)

(35)
$$
\frac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p < t < t_0} \int_{B_R} v^{\alpha+1}\zeta^2(\cdot\,,t)\,dx
$$
$$
+ \frac{\alpha+p-2}{(p+2\alpha)^2} \int_{Q_R}\left|\nabla\left(v^{\frac{p+2\alpha}{4}}\zeta\right)\right|^2 dx\,dt
$$
$$
\leq \frac{1}{\alpha+1} \int_{Q_R} v^{\alpha+1}\zeta|\zeta_t|dx\,dt + 2\left(\frac{1}{\alpha+p-2}+\frac{p-2}{\alpha}+\frac{\alpha+p-2}{(p+2\alpha)^2}\right)
$$
$$
\cdot \int_{Q_R} v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2 dx\,dt + c \int_{Q_R} v^{\frac{p+2+2\alpha}{2}}\zeta^2 dx\,dt\,.
$$

Now, for every fixed time t Hölder's inequality implies

$$
\|v^{\frac{p+2\alpha+2}{4}}\zeta\|^2_{L^2(B_R)} \leq \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)}\|v\|_{L^{\frac{p}{2}}(B_R)}
$$

where $2^* = \frac{2p}{p-2}$. So, we may split off a suitable factor in the last term on the right hand side of (35)

$$\frac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\,dx$$

$$+ \frac{\alpha+p-2}{(p+2\alpha)^2} \int_{t_0-R^p}^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{H_0^{1,2}(B_R)}\,dt$$

(36)
$$\leq \frac{1}{\alpha+1} \int_{Q_R} v^{\alpha+1}\zeta\,|\zeta_t|\,dx\,dt$$

$$+ 2\left(\frac{1}{\alpha+p-2} + \frac{p-2}{\alpha} + \frac{\alpha+p-2}{(p+2\alpha)^2}\right) \int_{Q_R} v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2\,dx\,dt$$

$$+ c \operatorname*{ess\,sup}_{t_0-R^p<t<t_0} \left(\int_{B_R} v^{\frac{p}{2}}(\,\cdot\,,t)\,dx\right)^{\frac{2}{p}} \int_{t_0-R^p}^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)}\,dt\,.$$

Using the Poincaré-Sobolev inequality $\|v^{\frac{p+2\alpha}{4}}\zeta\|_{H_0^{1,2}(B_R)} \geq k\|v^{\frac{p+2\alpha}{4}}\zeta\|_{L^{2^*}(B_R)}$ to estimate the last term on the left of inequality (36) we see that we may absorb the last term of the right hand side provided the quantity

$$\sup\{E(f(t),B_R(x))\,;\ 0 \leq t < T, x \in M\}$$

is smaller than a suitable constant $\varepsilon_2 > 0$ which only depends on absolute data and the level $q < \infty$ of integrability that we want to reach (it is enough to choose ε_2 such that $c\varepsilon_2^{2/p} \leq \frac{k^2(p-2)}{(q+2)^4}$, and $\varepsilon_2 \leq \varepsilon_1$ which allows to use the result of Lemma 5 later). In this way we get from (36)

$$\frac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\,dx$$

$$+ \frac{\alpha k^2}{(p+2\alpha)^2} \int_{t_0-R^p}^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)}\,dt$$

(37)
$$\leq \frac{1}{\alpha+1} \int_{Q_R} v^{\alpha+1}\zeta\,|\zeta_t|\,dx\,dt$$

$$+ 2\left(\frac{1}{\alpha+p-2} + \frac{p-2}{\alpha} + \frac{\alpha+p-2}{(p+2\alpha)^2}\right) \int_{Q_R} v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2\,dx\,dt\,.$$

Now, by Hölder's inequality and a further elementary inequality we observe

that for every $\lambda, \gamma > 0$ there holds

$$\lambda \int_{t_1}^{t_2} \int_{B_\rho} v^{\frac{p+2\alpha}{2}+\frac{2}{p}(\alpha+1)} dx\, dt \leq \lambda \operatorname*{ess\,sup}_{t \in]t_1, t_2[} \left(\int_{B_\rho} v^{\alpha+1} dx \right)^{\frac{2}{p}} \int_{t_1}^{t_2} \|v^{\frac{p+2\alpha}{4}}\|_{L^{2^*}(B_\rho)}^2 dt$$

(38)

$$\leq \lambda \left(\gamma^{\frac{p}{2}} \operatorname*{ess\,sup}_{t \in]t_1, t_2[} \int_{B_\rho} v^{\alpha+1} dx + \frac{1}{\gamma} \int_{t_1}^{t_2} \|v^{\frac{p+2\alpha}{4}}\|_{L^{2^*}(B_\rho)}^2 dt \right)^{\frac{p+2}{p}}$$

Choosing $t_1 = t_0 - (1 - \sigma_1)R^p$, $t_2 = t_0$, $\rho = (1 - \sigma_1)R$ and the constants λ and γ such that

$$\lambda^{\frac{p}{p+2}} \gamma^{\frac{p}{2}} = \frac{1}{4(\alpha+1)} \quad \text{and} \quad \frac{\lambda^{\frac{p}{p+2}}}{\gamma} = \frac{\alpha k^2}{(p+2\alpha)^2}$$

we get from (37) together with (38) and (29) that

$$(39) \qquad \int_{Q_R(\sigma_1, \sigma_2)} v^{\frac{p+2\alpha}{2}+\frac{2}{p}(\alpha+1)} dx\, dt \leq C \left(\int_{Q_R} v^{\frac{p+2\alpha}{2}} dx\, dt + 1 \right)^{1+\frac{2}{p}}$$

where the constant C in (39) may be expressed by means of the constants α, σ_1, σ_2, R, p, k. Now, the assertion follows by iteration and a covering argument. The iteration starts e.g. with $\alpha = p/2$ and the a priori estimate of Lemma 5 (we should not start with $\alpha = 0$ since we used $\alpha > 0$ in the calculations of the proof). We stop the iteration as soon as $\frac{p+2\alpha}{2} + \frac{2}{p}(\alpha+1) \geq \frac{q}{2}$. It is easy to check that our choice of ε_2 remains valid during the iteration process. □

2.4. – L^∞-estimate for ∇f

The next step is to find local a priori estimates for $\|\nabla f\|_\infty$ by a Moser iteration technique (see [50] and [51]). H. Choe used similar arguments in [8] to handle the case of systems of type (7). We start once again from the estimate (35) but this time the iteration is arranged in quite a different way.

LEMMA 8. *Let $a_0 > p$ be an arbitrary constant, $\dim(M) = p$, $f \in C^2(M \times [0, T[; N)$ a solution of the m-harmonic flow (6) and $Q_R \subset M \times]0, T[$. Then there exists a constant C which only depends on p, a_0, R, M and N with the property that*

$$\|\nabla f\|_{L^\infty(Q_R(\frac{1}{2}, \frac{1}{2}))} < C \left(1 + \int_{Q_R} |\nabla f|^{2a_0} dx\, dt \right)^{\frac{1}{a_0 - p}} .$$

PROOF. By the Hölder and the Sobolev inequality we have

$$\int_{Q_R(\sigma_1,\sigma_2)} v^{(\alpha+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p})}\,dx\,dt$$

$$(40) \qquad \leq c \underset{t_0-(1-\sigma_2)R^p<t<t_0}{\text{ess sup}} \left(\int_{B_R} v^{\alpha+1}\zeta^2 dx \right)^{\frac{2}{p}} \int_{Q_R} |\nabla(v^{\frac{2\alpha+p}{4}}\zeta)|^2 dx\,dt$$

$$\leq c \left(\underset{t_0-(1-\sigma_2)R^p<t<t_0}{\text{ess sup}} \int_{B_R} v^{\alpha+1}\zeta^2 dx + \int_{Q_R} |\nabla(v^{\frac{2\alpha+p}{4}}\zeta)|^2 dx\,dt \right)^{\frac{p+2}{p}}$$

Using (35) we conclude

$$\int_{Q_R(\sigma_1,\sigma_2)} v^{(\alpha+\frac{p}{2})\left(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p}\right)}\,dx\,dt$$

$$(41)$$

$$\leq c \left(\int_{Q_R} v^{\alpha+}\zeta|\zeta_t|dx\,dt + \int_{Q_R} v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2 dx\,dt + \alpha\int_{Q_R} v^{\frac{p+2\alpha+2}{2}}\zeta^2 dx\,dt \right)^{1+\frac{2}{p}}$$

for a new constant c which does not depend on α (we assume that $\alpha \geq \frac{p-2}{2}$). Observing (29) estimate (41) simplifies to

$$\int_{Q_R(\sigma_1,\sigma_2)} v^{(\alpha+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p})}\,dx\,dt$$

$$(42) \qquad \leq c \left(\frac{2}{\sigma_2 R^p} \int_{Q_R} v^{\alpha+1}dx\,dt + \frac{4}{\sigma_1^2 R^2} \int_{Q_R} v^{\frac{p+2\alpha}{2}}dx\,dt \right.$$

$$\left. + \alpha \int_{Q_R} v^{\frac{p+2\alpha+2}{2}}dx\,dt \right)^{1+\frac{2}{p}}.$$

Now, for every $\nu \in \mathbb{N}$ we put

$$R_\nu = \frac{R_0}{2}\left(1+\frac{1}{2^\nu}\right) \qquad \sigma_1 R = \frac{R_0}{2^{\nu+2}}$$

$$Q_R = Q_\nu \qquad\qquad \sigma_2 R^p = \frac{R_0^p}{2^{\nu+2}}$$

$$Q_R(\sigma_1,\sigma_2) = Q_{\nu+1} \qquad \alpha = \alpha_\nu$$

in (42) and get after some elementary manipulations

(43)
$$\int_{Q_{\nu+1}} v^{\left(\alpha_\nu+\frac{p}{2}\right)\left(1+\frac{2}{p}\frac{2\alpha_\nu+2}{2\alpha_\nu+p}\right)} dx\,dt$$
$$\leq \tilde{c}\left(\frac{4^\nu}{R_0^p}|Q_\nu| + \left(\alpha_\nu + \frac{4^\nu}{R_0^p}\right)\int_{Q_\nu} v^{\frac{p+2\alpha_\nu+2}{2}} dx\,dt\right)^{1+\frac{2}{p}}.$$

In order to iterate (43) we define

$$a_{\nu+1} := \left(\alpha_\nu + \frac{p}{2}\right)\left(1 + \frac{2}{p}\cdot\frac{2\alpha_\nu+2}{2\alpha_\nu+p}\right)$$

$$a_\nu := \frac{p+2\alpha_\nu+2}{2}.$$

We see that with $\mu := 1 + \frac{2}{p}$

$$a_{\nu+1} = \mu a_\nu - 2$$

and hence that

$$a_\nu = \mu^\nu(a_0 - p) + p.$$

Thus, $a_\nu \to \infty$ as $\nu \to \infty$—provided $a_0 > p$. Moreover we have $\alpha_\nu = \mu^\nu(a_0 - p) + \frac{p}{2} - 1$ such that we obtain from (43)

(44)
$$\int_{Q_{\nu+1}} v^{a_{\nu+1}} dx\,dt \leq K4^{\nu\mu}\left(1 + \int_{Q_\nu} v^{a_\nu} dx\,dt\right)^\mu$$

with $K = \tilde{c}(|M| + a_0 + \frac{1}{R_0^p})^\mu$. Now we use once more the fact that $\mu \leq 2$ and hence that $(1+x)^\mu \leq 4(1+x^\mu)$ for all $x \geq 0$. Defining

$$I_\nu = \int_{Q_\nu} v^{a_\nu} dx\,dt$$

(44) thus implies for $\nu \in \mathbb{N}_0$

(45)
$$I_{\nu+1} \leq 4\cdot 16^\nu K(1 + I_\nu^\mu).$$

Using (45) we can easily prove by induction that

$$I_\nu \leq L^{b_\nu}(1 + I_0^{\mu^\nu})$$

with $L = 4K + 16$ and a sequence b_ν satisfying

$$b_{\nu+1} = \mu b_\nu + 2\mu + \nu, \quad b_0 = 0.$$

For b_ν we find explicitly:

$$b_\nu = \mu^\nu \left(2 + b_0 + p + \frac{p^2}{4} \right) - \left(2 + p + \frac{p^2}{4} + \frac{\nu p}{2} \right).$$

Now we see that for $\nu \to \infty$

$$\frac{\mu^\nu}{a_\nu} \to A = \frac{1}{a_0 - p} \quad \text{and} \quad \frac{b_\nu}{a_\nu} \to B = \frac{2 + p + \frac{p^2}{4}}{a_0 - p}.$$

This implies

$$\|\nabla f\|^2_{L^\infty(Q_{R_0}(\frac{1}{2},\frac{1}{2}))} = \sup_{\nu \in \mathbb{N}} I_\nu^{1/a_\nu} \le L^B (1 + I_0^A). \qquad \square$$

As a corollary of the Lemmas 7 and 8 we have

LEMMA 9. *If* $\dim(M) = p$ *then there exists a constant* $\varepsilon_3 > 0$ *only depending on M and N with the following property:*

For any solution $f \in C^2(M \times [0, T[; N)$ *of the m-harmonic flow* (6) *with* $R^* = R^*(\varepsilon_3, f, M \times [0, T[) > 0$ *and any open set* $\Omega \subset\subset M \times]0, T[$ *there exists a constant C which only depends on p, E_0, M, N, R^* and* $\operatorname{dist}(\Omega, M \times \{0\})$ *such that*

$$\|\nabla f\|_{L^\infty(\Omega)} < C.$$

$E_0 = E(f(\cdot, 0))$ *again denotes the initial energy.*

2.5. – Energy concentration

In the theory of the harmonic flow an energy concentration theorem plays a fundamental role (see Struwe [58]). A modified form of this theorem also holds in the case $p > 2$. It shows that the local energy cannot concentrate too fast.

THEOREM 1. *If* $\dim(M) = p$ *and* $f \in C^2(M \times [0, \infty[; N)$ *is a solution of the m-harmonic flow* (6) *then there exist constants* $c, \varepsilon_0 > 0$ *which only depend on the geometry of the manifolds M and N, and there exists a time* $T_0 > 0$ *which depends in addition on E_0 and $R^*(\varepsilon_0, f, M \times \{0\})$, with the following properties: If the initial local energy satisfies*

$$\sup_{x \in M} E(f(0), B_{2R}(x)) < \varepsilon_0$$

then it follows

$$(46) \qquad E(f(t), B_R(x)) \le E(f(0), B_{2R}(x)) + c\, E_0^{1 - \frac{1}{p}} \frac{t}{R^m}$$

for all $(x, t) \in M \times [0, T_0]$. *Here E_0 denotes the initial energy* $E_0 = E(f(\cdot, 0))$.

PROOF. We make the same assumptions on M as in the proof of Lemma 7. We choose a testfunction $\varphi \in C_0^\infty(B_{2R}(x))$ which satisfies

$$0 \le \varphi \le 1, \quad \varphi \equiv 1 \text{ on } B_R(x), \quad |\nabla\varphi| \le \frac{2}{R}$$

where x is an arbitrary point in M and $4R \le \iota$ (ι the injectivity radius of M). Then we test

$$f_t - \nabla(\nabla f |\nabla f|^{p-2}) \perp T_f N$$

by the testfunction $f_t \varphi^2$ and obtain

$$0 = \int_0^T \int_{B_{2R}(x)} \left(f_t^2 \varphi^2 - \nabla(\nabla f |\nabla f|^{p-2}) f_t \varphi^2 \right) dx\, dt$$

(47)

$$= \int_0^T \int_{B_{2R}(x)} \left(f_t^2 \varphi^2 + \nabla f |\nabla f|^{p-2}(\nabla f_t \varphi^2 + 2\varphi \nabla\varphi f_t) \right) dx\, dt .$$

Thus,

$$\int_0^T \int_{B_{2R}(x)} \left(f_t^2 + \frac{1}{p}\frac{d}{dt}|\nabla f|^p \right) \varphi^2 dx\, dt = -2 \int_0^T \int_{B_{2R}(x)} \nabla f |\nabla f|^{p-2} \varphi \nabla\varphi f_t\, dx\, dt .$$

This implies

$$E(f(T), B_R(x)) - E(f(0), B_{2R}(x))$$

$$\le \frac{1}{p} \int_{B_{2R}(x)} |\nabla f|^p \varphi^2 dx \Big|_0^T$$

$$= - \int_0^T \int_{B_{2R}(x)} f_t^2 \varphi^2 dx\, dt - 2 \int_0^T \int_{B_{2R}(x)} \nabla f |\nabla f|^{p-2} \varphi \nabla\varphi f_t\, dx\, dt .$$

The second term on the right hand side of (48) may be estimated by Young's inequality by the first term and

$$\int_0^T \int_{B_{2R}(x)} |\nabla f|^{2p-2} |\nabla\varphi|^2 dx\, dt .$$

In this way we get from (48)

$$E(f(T), B_R(x)) - E(f(0), B_{2R}(x)) \le \frac{c}{R^2} \int_0^T \int_{B_{2R}(x)} |\nabla f|^{2p-2} dx\, dt$$

(49)

$$\le c\, \frac{T^{\frac{1}{p}}}{R} \left(\int_0^T \int_{B_{2R}(x)} |\nabla f|^{2p} dx\, dt \right)^{1-\frac{1}{p}}$$

using Hölder's inequality in the last step. Now, there exists a number L only depending on the geometry of M but not on R such that every ball $B_{2R}(x)$ may be covered by at most L balls $B_R(x_i)$. Now remember that in Lemma 5 we found a constant $\varepsilon_1 > 0$ with the property that

(50)
$$\sup_{t \in [0,T], y \in B_{4R}(x)} \int_{B_{2R}(y)} |\nabla f|^p dx < \varepsilon_1$$

$$\implies \int_0^T \int_{B_{2R}(x)} |\nabla f|^{2p} dx \, dt < cE_0 \left(1 + \frac{T}{(2R)^p}\right).$$

Now we choose $\varepsilon_0 = \frac{\varepsilon_1}{4L}$ and suppose that $R_0 < \frac{1}{4}$ is such that $\sup_{x \in M} E(f(0), B_{2R_0}(x)) < \varepsilon_0$. Then we choose T_0 as

$$T_0 = \min\left\{ \left(\frac{\varepsilon_1 R_0}{4Lc(2cE_0)^{1-\frac{1}{p}}}\right)^p, 1\right\}.$$

Now we claim: For all $(x,t) \in M \times [0, T_0]$ and all $R \le R_0$ there hold

(i)
$$\int_0^t \int_{B_{2R}(x)} |\nabla f|^{2p} dx \, dt \le cE_0 \left(1 + \frac{t}{(2R)^p}\right)$$

and

(ii)
$$\sup_{0 \le \tau \le t} \int_{B_{2R}(x)} |\nabla f(\cdot, \tau)|^p dx \le \varepsilon_1.$$

To see this let $T < T_0$ such that for $t \in [0, T]$ (i) and (ii) hold. Then it follows from (49) and (50) together with (ii) that

$$\sup_{(x,t) \in M \times [0,T]} E(f(t), B_{2R_0}(x))$$

$$\le L \sup_{(x,t) \in M \times [0,T]} E(f(t), B_{R_0}(x))$$

$$\le L \left(\sup_{x \in M} E(f(0), B_{2R_0}(x)) + c \frac{T^{\frac{1}{p}}}{R_0} \left(cE_0(1 + \frac{T}{(2R_0)^p}) \right)^{1-\frac{1}{p}} \right)$$

$$\le \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4} = \frac{\varepsilon_1}{2}$$

due to the special choice of ε_0, R_0 and T_0. Thus, (ii) and consequently (i) hold on some larger interval $[0, T + \delta]$. On the other hand the interval where (i) and (ii) hold is closed and nonempty. Hence, (i) and (ii) hold on $[0, T_0]$. Using (i) in the formula (49) the assertion follows after a short calculation with a new constant c. $\qquad\square$

3. – Existence results

In order to prove existence results for problem (6) we encounter two main difficulties:

(1) We have to assure that the image of f remains contained in N for all time: $f(M) \subset N \subset \mathbb{R}^k$ $\forall t \geq 0$.

(2) The p-Laplace operator is degenerate for $p > 2$.

In [5] existence for the p-harmonic flow is established for $N = S^n$. Due to the special geometry of the sphere the first difficulty may be handled by the "penalty trick". In case $p = 2$ the same technique has been applied successfully for general N (see Chen-Struwe [6]). For $p > 2$ this does not seem to work any more. Thus, we will try to solve the problem with Hamilton's technique of a totally geodesic embedding of N in $\mathbb{R}^k$ (see [35]).

The second difficulty will be attacked by regularising the p-energy (see Section 3.2 below). We will then apply the theory of analytic semigroups to the corresponding regularised operator. Due to the a priori estimates of Section 2 it will be possible to pass to the limit $\varepsilon \to 0$.

3.1. – Totally geodesic embedding of N in $\mathbb{R}^k$

In a first step we will work with a special embedding of N in $(\mathbb{R}^k, h)$: We equip $\mathbb{R}^k$ with a metric h such that

1. N is embedded isometrically, i.e. the metric g on N equals the metric induced by h.

2. The metric h equals the Euclidean metric outside a large ball B.

3. There exists an involutive isometry $\iota : T \to T$ on a tubular neighborhood T of N corresponding to multiplication by -1 in the orthonormal fibers of N and having precisely N for its fixed point set.

Such an embedding is called totally geodesic: The h-geodesic curve γ connecting $x, y \in N$ (x, y close enough) will always be contained in N. This follows from the (local) uniqueness of geodesics and the fact that with γ the curve $\iota \circ \gamma$ is another geodesic joining x and y.

A totally geodesic embedding can be accomplished as follows: We start with the standard Nash-embedding of $N \subset \mathbb{R}^k$ and choose a tubular neighborhood T of N: $T = T_{2\delta} = \{x \in \mathbb{R}^k : \text{dist}(x, N) < 2\delta\}$ (δ small enough and dist the Euclidean distance). Then we choose locally in $N \times] - 2\delta, 2\delta[^{k-n}$ the metric $\tilde{h}_{ij} = g_{ij} \otimes \delta_{ij}$ (or like Hamilton in [35] we just take the average of any extension of g under the action of ι). Then we smooth out $\tilde{h}$ by taking a positive C^∞ function ψ with support in $T_{2\delta}$ and $\psi \equiv 1$ on T_δ and by defining $h_{ij} = \psi \tilde{h}_{ij} + (1 - \psi)\delta_{ij}$.

3.2. – The regularised p-energy

DEFINITION 10. For $\varepsilon > 0$ the regularised p-energy density of a C^1-mapping $f : M \to N$ is

$$e_\varepsilon(f)(x) := \frac{1}{p}\,(\varepsilon + |df_x|^2)^{\frac{p}{2}}$$

and the regularised p-energy of f is

$$E_\varepsilon(f) := \int_M e_\varepsilon(f)\,d\mu$$

where the norm $|\cdot|$ and the measure μ are associated with the given Riemannian metrics on M and N.

For the heat flow of E_ε we find the equation

$$(51) \qquad\qquad\qquad f_t - \Delta_p^\varepsilon f = 0$$

where in local coordinates

$$
(52)\qquad
\begin{aligned}
\Delta_p^\varepsilon f &= \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\varepsilon + \gamma^{\alpha\beta}g_{ij}f_\alpha^i f_\beta^j\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}f_\alpha^l\right)\\
&\quad + \left(\varepsilon + \gamma^{\alpha\beta}g_{ij}f_\alpha^i f_\beta^j\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\Gamma_{ij}^l f_\alpha^i f_\beta^j
\end{aligned}
$$

(g and γ are the metrics of N and M respectively, Γ_{ij}^l denote the Christoffel symbols related to g). It will be necessary below to attach the related target-manifold in the notation; we will write ${}^g\Delta_p^\varepsilon$ or ${}^N\Delta_p^\varepsilon$ for that.

The extrinsic form of this equation which we use if N is isometrically embedded in the Euclidean space $\mathbb{R}^k$ is

$$(53) \qquad f_t - \Delta_p^\varepsilon f = (p\,e_\varepsilon(f))^{1-\frac{2}{p}}A(f)(\nabla f, \nabla f)_M$$

where in local coordinates

$$\Delta_p^\varepsilon f = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\varepsilon + \gamma^{\alpha\beta}f_\alpha^j f_\beta^j\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}f_\alpha\right)\,.$$

3.3. – The flow of the regularised p-energy

Let us first state a theorem of A. Lunardi (see [48]). We will use a version which was formulated by V. Vespri in [65].

THEOREM 2. *Consider the following Cauchy problem in a Banach space X*

(54)
$$f_t = \psi(f(t)) \quad for \ t \geq 0$$
$$f(0) = f_0$$

where ψ is a C^2 function from Y to X, Y is a continuously embedded subspace of X and $f_0 \in Y$. Assume that the linear operator $D\psi(f_0) : Y \to X$ generates an analytic semigroup in X and $\psi(f_0) \in \overline{Y}$ then there exists a strict solution f (in the sense of Lunardi [48]) of (54) on a time interval $[0, \tau]$, $\tau > 0$, and $f \in C^1([0, \tau], X) \cap C^0([0, \tau], Y)$. Moreover f is unique.

Now we will apply Lunardi's theorem to the regularised flow. To do this, consider a totally geodesic embedding of N in $(\mathbb{R}^k, h)$ and the regularised p-Laplace operator ${}^h\Delta_p^\varepsilon : Y \to X$ with $X = C^{0,\alpha}(M, \mathbb{R}^k)$ and $Y = C^{2,\alpha}(M, \mathbb{R}^k)$ for some $\alpha > 0$ (the fact that ${}^h\Delta_p^\varepsilon$ maps Y in X is checked directly in the definition (52)). Y is continuously embedded in X. Now, by expanding ${}^h\Delta_p^\varepsilon(f_0 + k)$ we find the first derivative of the regularised p-Laplace operator:

$$D({}^h\Delta_p^\varepsilon)(f_0) : Y \to X$$

$$k \mapsto \frac{1}{\sqrt{\gamma}}\Big(\sqrt{\gamma}(\tfrac{p}{2} - 1)(p\, e_\varepsilon(f_0))^{1-\frac{4}{p}} \gamma^{\sigma\rho}(h_{ij,s}k^s f_{0,\sigma}^i f_{0,\rho}^j$$

(55)
$$+ 2h_{ij} f_{0,\sigma}^i k_{,\rho}^j)\gamma^{\alpha\beta} f_{0,\alpha}^l + \sqrt{\gamma}(p\, e_\varepsilon(f_0))^{1-\frac{2}{p}} \gamma^{\alpha\beta} k_{,\alpha}^l\Big)_{,\beta}$$

$$+ (\tfrac{p}{2} - 1)(p\, e_\varepsilon(f_0))^{1-\frac{4}{p}} \gamma^{\alpha\beta} \Gamma_{ij}^l f_{0,\alpha}^i f_{0,\beta}^j \gamma^{\sigma\rho}(h_{ij,s}k^s f_{0,\sigma}^i f_{0,\rho}^j$$

$$+ 2h_{ij} f_{0,\sigma}^i k_{,\rho}^j) + (p\, e_\varepsilon(f_0))^{1-\frac{2}{p}} \gamma^{\sigma\rho}(\Gamma_{ij,s}^l k^s f_{0,\sigma}^i f_{0,\rho}^j + 2\Gamma_{ij}^l f_{0,\sigma}^i k_{,\rho}^j)$$

Here upper indices denote components whereas $_{,\alpha}$ means $\frac{\partial}{\partial x_\alpha}$. It is not difficult to check that for $f_n \to f$ in Y we have $D({}^h\Delta_p^\varepsilon)(f_n) \to D({}^h\Delta_p^\varepsilon)(f)$ in $L(Y, X)$ and hence that the mapping $f \mapsto D({}^h\Delta_p^\varepsilon)(f)$ is continuous. Analogously we find that ${}^h\Delta_p^\varepsilon$ has a continuous second derivative. The important facts about the operator $D({}^h\Delta_p^\varepsilon)(f_0)$ for $f_0 \in Y$ are

(1) The coefficients of $D({}^h\Delta_p^\varepsilon)(f_0)$ are of class $C^{0,\alpha}(M)$.

(2) For $\varepsilon > 0$ the operator $D({}^h\Delta_p^\varepsilon)(f_0)$ is elliptic in the sense that the main part satisfies a uniform strong Legendre-condition.

(1) is obvious. (2) can be checked in case of the Euclidean metric $h_{ij}(z_0) = \delta_{ij}$ directly from the definition: Then the main part of $D({}^h\Delta_p^\varepsilon)(f_0)$ in x_0 is

$$A_{il}^{\rho\beta} k_{,\rho\beta}^i = (p\, e_\varepsilon(f_0))^{1-\frac{4}{p}}\left((p - 2)\gamma^{\sigma\rho} f_{0,\sigma}^i \gamma^{\alpha\beta} f_{0,\alpha}^l + (p\, e_\varepsilon(f_0))^{\frac{2}{p}} \gamma^{\alpha\beta} \delta_\alpha^\rho \delta_i^l\right) k_{,\rho\beta}^i$$

such that we get for an arbitrary vector ξ_α^i and a uniform constant $\nu > 0$ depending on the geometry of M

$$A_{il}^{\rho\beta} \xi_\rho^i \xi_\beta^l = (p\, e_\varepsilon(f_0))^{1-\frac{4}{p}} \left((p-2)(\text{trace}(\gamma\xi f_0))^2 + (p\, e_\varepsilon(f_0))^{\frac{2}{p}} \langle \xi, \xi \rangle \right) \geq \nu |\xi|^2$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product induced by γ.

For a general metric h we use the observation that the main part of $D(^h\Delta_p^\varepsilon)(f_0)$ depends only on the metric h but not on its derivatives. For the Euclidean metric $h_{ij} = \delta_{ij}$ the required ellipticity is treated above. Then ellipticity follows at least for metrics h which are close enough to the Euclidean metric. But we *can* choose h as close as we want to the Euclidean metric by choosing the δ-neighborhood of N in the construction of the totally geodesic embedding small (compare Section 3.1).

As Vespri has shown in [65] the conditions (1) and (2) guarantee that $D(^h\Delta_p^\varepsilon)(f_0) : Y \to X$ generates an analytic semigroup in X. In fact, Vespri showed that under these conditions there exist constants $C, \omega > 0$ such that for all $\varphi \in X$ and every complex number λ satisfying $\text{Re}(\lambda) > \omega$ there exists a solution $u \in Y$ of

$$\left(\lambda - D(^h\Delta_p^\varepsilon)(f_0) \right) u = \varphi$$

with

$$\|u\|_X \leq \frac{C}{|\lambda|} \|\varphi\|_X .$$

Now, Lunardi's theorem yields existence of a unique local solution $f \in C^0([0, \tau], C^{2,\alpha}(M)) \cap C^1([0, \tau], C^{0,\alpha}(M))$ of (51) with initial data $f_0 \in Y$. Furthermore we have

THEOREM 3. *If* $\text{Im}(f_0) \subset N$, *then* $\text{Im}(f(\cdot, t)) \subset N$ *for all* $t \in [0, \tau]$.

PROOF. Let ι still denote the involutive isometry on the tubular neighborhood T of $N \subset \mathbb{R}^k$ defined in Section 3.1. We proceed by contradiction. If the image of f does not always remain in N, we can restrict ourselves to a smaller interval $M \times [0, \tau']$, $\tau' \leq \tau$, such that the image of f does not always remain in N but in the tubular neighborhood T of N. Since $\iota : T \to T$ is an isometry, the composition $\iota \circ f$ is another solution of (51). Since ι is the identity on N this solution has the same initial value as f, so by the uniqueness of the solution we have $\iota \circ f = f$. This shows that the image of f must remain in the fixed point set N of ι. □

THEOREM 4. *If* (N, g) *is a totally geodesic embedded submanifold of* $(\mathbb{R}^k, h)$ *and* $f : M \to N \subset \mathbb{R}^k$ *then* $^g\Delta_p^\varepsilon f = {}^h\Delta_p^\varepsilon f$.

PROOF. We refer to Hamilton [35], Section 4.5, page 108. The proof there is given for $p = 2$ and $\varepsilon = 0$, but it carries over to our situation. □

Thus, according to Theorem 3 we find that for initial data $f_0 : M \to N$, $f_0 \in Y$, the solution f of (51) we found above satisfies $f(M) \subset N$ for all $t \in [0, \tau]$ and is, by applying Theorem 4, a solution of (53) with initial data f_0.

Notice that since we have constructed the local solution for *fixed* ε the existence interval $[0, \tau(\varepsilon)]$ might depend on ε. So, we will have to show that $\tau(\varepsilon) \not\to 0$ as $\varepsilon \to 0$.

3.4. – An ε-independent existence interval

First we formulate the a priori estimates of Section 2 for solutions of the heat flow of the energy E_ε.

THEOREM 5. *Let* $f : M \times [0, \tau] \to N \subset \mathbb{R}^k$, $f \in C^0([0, \tau], C^{2,\alpha}(M)) \cap C^1([0, \tau], C^{0,\alpha}(M))$, *be a solution of* (53) *with initial value* f_0. *We assume that* $\varepsilon \leq 1$. *Then the following is true if* $p = m = \dim M$:

(i) $\displaystyle \int_0^t \int_M |\partial_t f|^2 d\mu \, dt + E_\varepsilon(f(t)) = E_\varepsilon(f_0) \leq E_1(f_0)$ *(energy inequality)*

(ii) *There exist constants* C, $\varepsilon_0 > 0$, *only depending on* M *and* N *(but not on* ε *and* f_0*), and* $T_0 > 0$ *depending in addition on* $E_1(f_0)$ *and* $R^*(\varepsilon_0, f, M \times \{0\})$, *such that the condition*

$$\sup_{x \in M} E_1(f_0, B_{2R}(x)) < \varepsilon_0$$

implies

$$E_\varepsilon(f(t), B_R(x)) \leq E_1(f_0, B_{2R}(x)) + C E_1(f_0)^{1 - \frac{1}{p}} \frac{t}{R^p}$$

for all $(x, t) \in M \times [0, \min\{\tau, T_0\}]$.

(iii) *There exists a constant* $\varepsilon_1 > 0$ *only depending on* M *and* N *(but not on* ε *and* f_0*) such that*

$$R^* = R^*(\varepsilon_1, f, M \times [0, \tau]) > 0$$

implies for every $\Omega \subset M \times [0, \tau]$ *with* $\mathrm{dist}(\Omega, M \times \{0\}) = \mu > 0$

$$\|\nabla f\|_{L^\infty(\Omega)} \leq C$$

where C *is a constant that depends on* p, $E_1(f_0)$, M, N, R^* *and* μ.

(iv) *For the same constant* ε_1 *as in* (iii) *we have that*

$$R^* = R^*(\varepsilon_1, f, M \times [0, \tau]) > 0$$

implies

$$\|\nabla f\|_{L^\infty(M \times [0, \tau])} \leq C$$

where C *is a constant that depends on* p, M, N, R^* *and the* L^∞-*norm of the initial value* ∇f_0.

PROOF.

(i)-(iii): We obtain these assertions by repeating the corresponding proofs of Section 2. Notice however that the regularised energy E_ε is not conformally invariant, so every argument based on this fact would break down.

(iv) Here we repeat the three steps of Section 2.2-2.4: We obtain the L^{2p}-estimate for ∇f as in Section 2.2 by testing the equation by $-\varphi^p \Delta_p^\varepsilon f$.

Using the L^{2p}-estimate for ∇f we get the L^q-estimates for $q < \infty$ as in Section 2.3 by an iteration argument: In order to obtain the estimate up to $t = 0$ we have to choose the cutoff function ζ independent of t, i.e. $\sigma_2 = 0$ (in the notation of Section 2.3). Using the testfunction

$$-\nabla(\nabla f(\varepsilon + v)^\alpha \zeta^2)$$

(with $v = |\nabla f|^2$) in (53) we find after some calculation the analogue of (37)

$$\frac{1}{4(\alpha + 1)} \operatorname*{ess\,sup}_{0 < t < t_0} \int_{B_R} v^{\alpha+1} \zeta^2(\cdot, t)\, dx$$

$$+ \frac{\alpha k^2}{(p + 2\alpha)^2} \int_0^{t_0} \| v^{\frac{p+2\alpha}{4}} \zeta \|_{L^{2*}(B_R)}^2 dt$$

$$\leq \frac{1}{\alpha + 1} \int_{B_R} v^{\alpha+1}(\cdot, 0) \zeta^2 dx$$

$$+ 2\left(\frac{1}{\alpha + p - 2} + \frac{p - 2}{\alpha} + \frac{\alpha + p - 2}{(p + 2\alpha)^2} \right) \int_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 dx\, dt\,.$$

The iteration yields

$$\int_{Q_R} v^q dx\, dt \leq C$$

where C now depends on the initial datum $\int_{B_R} v^{\alpha_0+1}(\cdot, 0) dx$ (α_0 denotes the initial value of the iteration).

Finally as in Section 2.4 we use a Moser-iteration to get the L^∞-bound for ∇f. Using the same testfunction as in the last step, we obtain the analogue of (41)

$$\int_{Q_R(\sigma_1)} v^{\left(\alpha + \frac{p}{2}\right)\left(1 + \frac{2}{p}\frac{2\alpha+2}{2\alpha+p}\right)} dx\, dt$$

$$\leq c \left(\int_{B_R} v^{\alpha+1}(\cdot, 0) \zeta^2 dx\, dt + \int_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 dx\, dt + \alpha \int_{Q_R} v^{\frac{p+2\alpha+2}{2}} \zeta^2 dx\, dt \right)^{1 + \frac{2}{p}}$$

Iteration yields

$$\| \nabla f \|_{L^\infty(Q_R(\frac{1}{2}))} \leq C$$

where C now depends on the initial datum $\| \nabla f_0 \|_{L^\infty(M)}$.   □

Now we combine (ii) and (iv): Let $R = R^*(\frac{\min\{\varepsilon_0, \varepsilon_1\}}{2}, f_0, M \times \{0\})/3$. Hence, R is a radius with the property

$$\sup_{x \in M} E_1(f_0, B_{2R}(x)) < \frac{\min\{\varepsilon_0, \varepsilon_1\}}{2}.$$

Then (ii) implies that for all $t \leq T^*$,

$$T^* = T^*(f_0) = \frac{\varepsilon_1 R^p}{2CE_1(f_0)^{1-\frac{1}{p}}},$$

the hypothesis of (iv) and hence the conclusion holds:

(56) $\|\nabla f\|_{L^\infty(M \times [0,t])} \leq C$

for $t \leq \min\{\tau, T^*\}$ with a constant depending on M, N, p, and R.

Now, for the solution $f : M \times [0, \tau] \to N$ of (53) constructed in Section 3.3 we define $B : M \times [0, \tau] \to \mathbb{R}^k$, $B = (p \, e_\varepsilon(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)_M$. Then f is a solution of

$$f_t - \Delta_p^\varepsilon f = B$$

and the results of DiBenedetto [12] apply: We have that $\|B\|_{L^\infty(M \times [0,\tau])} \leq C(f_0)$ and the operators $\{\Delta_p^\varepsilon\}_{\varepsilon \in [0,1]}$ satisfy the structural conditions of [12], Section 4.1 (ii) (notice that $B(x, t)$ does not depend on ∇f). Then we infer from [12], Section 9.1, that ∇f is Hölder continuous: there is a Hölder exponent $\tilde{\alpha} > 0$, independent of ε and τ, with

(57) $\|\nabla f\|_{C^{0,\tilde{\alpha}}(M \times [0,\tau])} \leq C(f_0)$

(see also [12], Section 8.1, p. 216).

Combining (57) with the classical results in [44] we obtain for $t \leq \min\{\tau, T^*\}$

$$\|\nabla^2 f(\cdot, t)\|_{C^{0,\alpha}(M)} \leq C$$

where C depends on f_0 and also on the modulus of ellipticity ε.

Now we can prove that $[0, T^*(f_0)]$ is the existence interval for the solution of the heat flow of the energy E_ε independent of ε by "continuous induction": Let

$$I = \{t \in [0, T^*] : \exists \tilde{\alpha} > 0, \exists f \text{ a solution of (53) on } [0, t] \text{ with}$$

$$f \in C^0([0, t], C^{2,\alpha}(M)) \cap C^1([0, t], C^{0,\alpha}(M)) \text{ and initial value } f_0\}.$$

Then the interval I is not empty (according to Section 3.3). But I is also open: If $t \in I$, then we can extend the solution beyond t by solving the flow with initial value $f(t)$ which is possible as we have seen in Section 3.3. On the other hand by our time independent bounds for the quantities $\|\nabla^2 f\|_{C^{0,\alpha}(M)}$, $\|\nabla f\|_{C^{0,\tilde{\alpha}}(M)}$ and $\|f_t\|_{L^2(M \times [0,t])}$ on $[0, T^*]$ the interval I is also closed and hence $I = [0, T^*]$.

Thus, we have

THEOREM 6. *For $p = m = \dim M$ there exists a constant $\varepsilon_2 > 0$ depending on M and N with the following property:*

For arbitrary $f_0 : M \to N \subset \mathbb{R}^k$, $f_0 \in C^{2,\alpha}(M)$ there exists a time $T^* > 0$ only depending on $E(f_0)$, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and the geometry of M and N such that for every $\varepsilon \in]0, 1]$ there exists a solution $f \in C^0([0, T^*], C^{2,\tilde{\alpha}}(M)) \cap C^1([0, T^*], C^{0,\alpha}(M))$ of (53) with initial value f_0. Moreover there exist ε-independent bounds for the following quantities:

$$\|f_t\|_{L^2(M \times [0,T^*])} \le C(E_1(f_0))$$

$$\|\nabla f\|_{L^\infty(M \times [0,T^*])} \le C(\|\nabla f_0\|_{L^\infty(M)})$$

$$\|\nabla f\|_{C^{0,\tilde{\alpha}}(M \times [0,T^*])} \le C(\|\nabla f_0\|_{C^{0,\alpha}(M)}) .$$

Of course the constants C also depend on p, M and N. The constant $\tilde{\alpha}$ depends on α and $\|\nabla f_0\|_{L^\infty(M)}$.

Combining Theorem 5 (iii) with DiBenedetto's result in [12], Theorem 1.1', Chapter IX, we obtain also

THEOREM 7. *For the solution f from Theorem 6 we have for every open $\Omega \subset M \times [0, T^*]$ with $\mathrm{dist}(\Omega, M \times \{0\}) = \mu > 0$*

$$\|\nabla f\|_{C^{0,\beta}(\Omega)} \le C$$

for some constants C (depending on p, $E_1(f_0)$, M, N, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and μ) and $\beta \in]0, 1[$ (depending on p, M and N).

3.5. – The limit $\varepsilon \to 0$

Let f_ε denote the solution of the heat flow of the energy E_ε (with initial value f_0) on the time interval $[0, T^*]$, that we have constructed in the previous section. The aim is to pass in the distributional form of (53) on $[0, T^*]$ to the limit. Due to the ε-independent bounds given in Theorem 6 we know at least that $\{f_\varepsilon\}_{\varepsilon \in]0,1]}$ is bounded in $W^{1,2}(M \times [0, T^*], N)$. Thus, we can choose a sequence $\varepsilon_k \to 0$ such that

$$f_{\varepsilon_k} \rightharpoonup f \quad \text{weakly in } W^{1,2}(M \times [0, T^*], N) .$$

But by the bound for the $C^{0,\tilde{\alpha}}$-norm of ∇f_ε we can pass to a subsequence if necessary, and obtain (observing that $C^{0,\tilde{\alpha}}(M \times [0, T^*]) \subset C^{0,\tilde{\alpha}/2}(M \times [0, T^*])$ compactly) that

$$\nabla f_\varepsilon \to \nabla f \quad \text{strongly in } C^{0,\tilde{\alpha}/2}(M \times [0, T^*]) .$$

Now, for a C_0^∞ testfunction φ we can pass to the limit $\varepsilon_k \to 0$ in

$$\int_0^{T^*} \int_M \partial_t f_\varepsilon \varphi \, d\mu \, dt + \int_0^{T^*} \int_M \frac{1}{\sqrt{\gamma}}$$

$$\cdot \left(\sqrt{\gamma} \left(\varepsilon + \gamma^{\alpha\beta} \partial_{x_\alpha} f_\varepsilon^j \partial_{x_\beta} f_\varepsilon^j \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \partial_{x_\alpha} f_\varepsilon \right) \partial_{x_\beta} \varphi \, d\mu \, dt$$

$$= \int_0^{T^*} \int_M \varphi \, (p \, e_\varepsilon(f_\varepsilon))^{1-\frac{2}{p}} A(f_\varepsilon)(\nabla f_\varepsilon, \nabla f_\varepsilon).$$

Thus, we have the following

THEOREM 8. *For $p = m = \dim M$ there exits a constant $\varepsilon_2 > 0$ depending on M and N with the following property:*

For arbitrary $f_0 : M \to N \subset \mathbb{R}^k$, $f_0 \in C^{2,\alpha}(M)$ there exists a time $T^* > 0$ only depending on $E(f_0)$, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and the geometry of M and N, and a local weak solution $f : M \times [0, T^*] \to N$ of

$$f_t - \Delta_p f \perp T_f N$$

$$f(\cdot, 0) = f_0.$$

f satisfies the energy inequality. Furthermore $\|\nabla f\|_{C^{0,\tilde{\alpha}}(M \times [0,T^*])} \le C (\|\nabla f_0\|_{C^{0,\alpha}(M)})$ and $\|\nabla f\|_{L^\infty(M \times [0,T^*])} \le C(\|\nabla f_0\|_{L^\infty(M)})$. The constants C also depend on p, M and N. Locally, for every open $\Omega \subset M \times [0, T^*]$ with $\mathrm{dist}(\Omega, M \times \{0\}) = \mu > 0$, there holds $\|\nabla f\|_{C^{0,\beta}(\Omega)} \le C$ for some constants C (depending on p, $E(f_0)$, M, N, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and μ) and $\beta \in]0, 1[$ (depending on p, M and N).

For small initial data, i.e. if $\|\nabla f_0\|_{L^p(M)} \le \varepsilon_1$, the existence is global (ε_1 is the constant from Theorem 5).

PROOF. We have already seen that f is a weak solution of the flow on $[0, T^*]$. The energy inequality follows in the limit $\varepsilon \to 0$ from Theorem 5 (i) and the bounds for $\|\nabla f\|_{C^{0,\tilde{\alpha}}(M \times [0,T^*])}$ and $\|\nabla f\|_{L^\infty(M \times [0,T^*])}$ from Theorem 6. The local bound for $\|\nabla f\|_{C^{0,\beta}(\Omega)}$ follows from Theorem 7.   □

3.6. – Short Time existence for non-smooth initial data

We can now prove short time existence for a wider class of initial values:

THEOREM 9. *For $p = m = \dim M$ there exits a constant $\varepsilon_2 > 0$ depending on M and N with the following property:*

For given initial value $f_0 \in W^{1,p}(M, N)$ there exists a time $T^* > 0$ only depending on $E(f_0)$, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and the geometry of M and N, and a weak solution $f : M \times [0, T^*] \to N$ of

$$f_t - \Delta_p f \perp T_f N$$

$$f(\cdot, 0) = f_0.$$

f satisfies the energy inequality. Locally, for every open $\Omega \subset M \times [0, T^*]$ *with* dist$(\Omega, M \times \{0\}) = \mu > 0$, *there holds* $\|\nabla f\|_{C^{0,\beta}(\Omega)} \leq C$ *for some constants C (depending on p, $E(f_0)$, M, N, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and μ) and $\beta \in]0, 1[$ (depending on p, M and N).*

For small initial data, i.e. if $\|\nabla f_0\|_{L^p(M)} \leq \varepsilon_1$, *the existence is global.*

PROOF. From Bethuel-Zheng [2] we infer that $C^\infty(M, N)$ is dense in $W^{1,p}(M, N)$. Hence, we can approximate the given f_0 by smooth functions: there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $C^\infty(M, N)$ such that $\varphi_n \to f_0$ in $W^{1,p}(M, N)$. Let f_n denote the solution of

$$\partial_t f - \Delta_p f \perp T_f N$$

$$f(\cdot, 0) = \varphi_n .$$

Notice that for every $\varepsilon > 0$ there exists a radius $R > 0$ such that $\sup_{x \in M} E(\varphi_n, B_R(x)) \leq \varepsilon$ for all $n \in \mathbb{N}$. According to the construction of the time T^* in Section 3.4 there exists an existence interval $[0, T^*]$, $T^* > 0$, valid for every solution f_n. Using the energy inequality of Theorem 8 for the solutions f_n we find a subsequence (still denoted by f_n) such that

$$f_n \rightharpoonup f \text{ weakly* in } L^\infty\left(0, T^*; W^{1,p}(M, N)\right)$$

and

$$\partial_t f_n \rightharpoonup \partial_t f \text{ weakly in } L^2\left(0, T^*; L^2(M)\right) .$$

The local estimate on $\|\nabla f_n\|_{C^{0,\beta}(M \times [t, T^*])}$ for $t > 0$ also from Theorem 8 implies (after passing to another subsequence) that

$$\nabla f_n \to \nabla f \text{ strongly in } C^{0,\beta/2}(M \times [t, T^*]) .$$

Choosing $t = \frac{T^*}{n}$ we obtain, by iterated extraction of subsequences and passing to the diagonal sequence, that

$$\nabla f_n \to \nabla f \text{ strongly in } C^{0,\beta/2}(M \times [t, T^*])$$

for all $t > 0$. This allows to go to the limit $n \to \infty$ in the weak form of the equation for f_n.

The energy inequality and the local estimate for $\|\nabla f\|_{C^{0,\beta}(\Omega)} \leq C$ also follow in the limit.   □

3.7. – Global existence and partial regularity

Once we have established local existence for initial data in $W^{1,m}(M, N)$, we can try to extend the local solution beyond an occurring singularity. It will be possible to find an a priori bound for the number of singular times which enables us to obtain global existence by repeating the extension finitely many times.

THEOREM 10. *For given initial value $f_0 \in W^{1,m}(M, N)$ there exists a weak solution $f : M \times [0, \infty[\to N$ of the m-harmonic flow*

$$f_t - \Delta_m f \perp T_f N$$

$$f(\cdot, 0) = f_0.$$

f satisfies the energy inequality and is in $W^{1,m}(M)$ weakly continuous in time. There exists a set $\Sigma = \cup_{k=1}^{K} \Sigma_k \times \{T_k\}$, $\Sigma_k \subset M$, $0 < T_k \leq \infty$, such that on every open set $\Omega \subset M \times [0, \infty[$ with dist$(\Omega, (M \times \{0\}) \cup \Sigma) = \mu > 0$ there holds $\|\nabla f\|_{C^{0,\beta}(\Omega)} \leq C$ for some constants C (depending on m, $E(f_0)$, M, N and μ) and $\beta \in]0, 1[$ (depending on m, M and N). The number K of singular times is a priori bounded by $K \leq \varepsilon_1^{-1} E(f_0)$ and the singular points (x, T_k) are characterized by the condition $\limsup_{t \nearrow T_k} E(f(t), B_R(x)) \geq \varepsilon_1$ for any $R > 0$. At every singular time T_k the decrease of the m-energy is at least ε_1:

$$E(f(T_k)) \leq \liminf_{t \nearrow T_k} E(f(t)) - \varepsilon_1.$$

PROOF. We can extend the local solution of the last section, which is defined on $[0, T^*]$, to a maximal interval $[0, T_1[$ where T_1 is characterized by

(i) $\nabla f \in C_{\text{loc}}^{0,\beta}(M \times]0, T_1[)$,
(ii) there exists $x \in M$ such that $\limsup_{t \nearrow T_1} E(f(t), B_R(x)) \geq \varepsilon_1$ for any $R > 0$.

In fact, if f is a solution on $I = [0, t[$ or $I = [0, t]$ such that $\nabla f \in C_{\text{loc}}^{0,\beta}(M \times I)$ and for all $x \in M$ there exists $R > 0$ with $\limsup_{\tau \nearrow t} E(f(\tau), B_R(x)) \leq \varepsilon_1$, then there holds $R^* = R^*(\varepsilon_1, f, M \times I) > 0$. According to the construction in Section 3.4 we set

$$T^* = \frac{\varepsilon_1 (R^*)^p}{4CE(f_0)^{1-\frac{1}{p}}}$$

and can then find a solution of the m-harmonic flow with initial value $f(\cdot, t - \frac{T^*}{2})$ on the time interval $[t - \frac{T^*}{2}, t + \frac{T^*}{2}]$. (This extension of the solution is unique as we will see in the next Chapter.)

Now, the energy inequality implies that

(58) $f(\cdot, t) \rightharpoonup f(\cdot, T_1)$ weakly in $W^{1,p}(M)$.

In fact, since $f_t \in L^2(M \times [0, T_1[)$ we have that $f_t(x, \cdot) \in L^2(0, T_1[)$ for a.e. $x \in M$, and hence we can write $f(x, T_1) = f(x, 0) + \int_0^{T_1} f_t(x, s) ds$ for almost all $x \in M$ and $f(x, t) \to f(x, T_1)$ a.e. $x \in M$ as $t \nearrow T_1$. On the other hand, since $\|f(\cdot, t)\|_{W^{1,p}(M)}$ is bounded on $[0, T_1[$ we have (58) at least for a sequence $t_k \nearrow T_1$. To prove that we have convergence for an arbitrary sequence

$t_k \nearrow T_1$ we argue as follows: Let $\phi, \psi \in L^{p'}(M)$ be fixed. Then we have for $\widetilde{\phi}, \widetilde{\psi} \in C^\infty(M)$

$$\left| \int_M ((f(x,t) - f(x,T_1))\phi(x) + \nabla(f(x,t) - f(x,T_1))\psi(x)) \, d\mu \right|$$

$$\leq \int_M |f(x,t) - f(x,T_1)| \, |\widetilde{\phi}(x)| d\mu + \int_M |f(x,t) - f(x,T_1)| \, |\phi(x) - \widetilde{\phi}(x)| d\mu$$

$$+ \left| \int_M \nabla(f(x,t) - f(x,T_1))\widetilde{\psi}(x) d\mu \right|$$

$$+ \int_M |\nabla(f(x,t) - f(x,T_1))(\psi(x) - \widetilde{\psi}(x)| d\mu$$

$$\leq \max_{x \in M} |\widetilde{\phi}(x)| \int_M |f(x,t) - f(x,T_1)| d\mu$$

$$+ \left(\int_M |f(x,t) - f(x,T_1)|^p d\mu \right)^{\frac{1}{p}} \left(\int_M |\phi(x) - \widetilde{\phi}(x)|^{p'} d\mu \right)^{\frac{1}{p'}}$$

$$+ \max_{x \in M} |\nabla \widetilde{\psi}(x)| \int_M |f(x,t) - f(x,T_1)| d\mu$$

$$+ \left(\int_M |\nabla(f(x,t) - f(x,T_1))|^p d\mu \right)^{\frac{1}{p}} \left(\int_M |\psi(x) - \widetilde{\psi}(x)|^{p'} d\mu \right)^{\frac{1}{p'}} .$$

Let some $\varepsilon > 0$ be given. Since $f(\cdot, t)$ is bounded in $W^{1,p}(M)$ we can make the second and the fourth term each smaller than $\frac{\varepsilon}{4}$ by choosing $\widetilde{\phi}$ close to ϕ and $\widetilde{\psi}$ close to ψ in $L^{p'}(M)$. Then, by choosing t close to T_1 the first and the third term become smaller than $\frac{\varepsilon}{4}$ (this follows by Lebesgue's theorem).

On the other hand we have seen that at time T_1 there exist points $x \in M$ such that

(59) $$\limsup_{t \nearrow T_1} E(f(\cdot, t), B_R(x)) \geq \varepsilon_1$$

for any $R > 0$. Now for such a point x satisfying (59) and for $R > 0$ let $M_R = M \setminus B_R(x)$ and $E_0 = E(f(0))$. Then we have

(60)
$$E_0 - \varepsilon_1 \geq E_0 - \limsup_{t \nearrow T_1} E(f(t), B_R(x)) \geq$$
$$\geq \liminf_{t \nearrow T_1} (E(f(t)) - E(f(t), B_R(x))) =$$
$$= \liminf_{t \nearrow T_1} E(f(t), M_R) \geq$$
$$\geq E(f(T_1), M_R) \xrightarrow{R \searrow 0} E(f(T_1), M) \geq 0 .$$

From (60) we conclude that we have

$$E(f(T_1)) \leq E_0 - \varepsilon_1$$

for the energy at time T_1.

Now, for any $\Omega \subset\subset M \times]0, T_1] \setminus \Sigma_1 \times \{T_1\}$ (Σ_1 the set of singular points, i.e. the set of points satisfying 59)), there exists a radius $R > 0$ depending on Ω such that

$$\sup_{(x,t) \in \Omega} E(f(t), B_R(x)) < \varepsilon_1.$$

Thus, our solution f on $[0, T_1[$ extends to a solution on $M \times [0, T_1] \setminus \Sigma_1 \times \{T_1\}$ with $\nabla f \in C^{0,\beta}(\Omega)$.

In view of (58) we can use $f(\cdot, T_1)$ as new initial value and iterate this process. Piecing all the resulting solutions together, we obtain a global solution as asserted. Applying (60) at every occurring singular time T_k we conclude that we are a priori given an upper bound for the number K of singular times $T_1, \ldots, T_K$, namely

$$K \leq \frac{E(f_0)}{\varepsilon_1}. \qquad \qquad \square$$

4. – Uniqueness in the class $L^\infty(0, T; W^{1,\infty}(M))$

In the case $p = 2$ Struwe proved uniqueness of the harmonic flow on Riemannian surfaces within the class in which he obtained existence (see Struwe [58]). In the non-conformal case uniqueness fails to be true as counterexamples of Coron (for $p = 2$) and of the author (for $p > 2$) show (see [9] and [39]). In this section we show that if two solutions of the p-harmonic flow coincide at time $t = 0$ they coincide on the time interval $[0, T[$ provided the L^∞-norm of the gradients remains bounded during that time. Let us start with a technical lemma.

LEMMA 11. *Let $p \geq 2$. Then there holds for all $a, b \in \mathbb{R}^k$*

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq c|a - b|^2 \big||a| + |b|\big|^{p-2}.$$

with a constant $c > 0$ which only depends on the inner product.

PROOF. By a suitable rotation and dilatation, the problem reduces to two dimensions where the verification is elementary. In the case of the standard inner product the best possible constant is $c = \frac{1}{2^{p-2}}$. $\qquad \square$

THEOREM 11. *Let f_1, f_2 be weak solutions of*

$$\begin{cases} \frac{\partial f}{\partial t} - \Delta_p f & \perp T_f N \\ f|_{t=0} & = f_0 \end{cases}$$

and suppose that for $t \in [0, T[$ there holds $|\nabla f_1| + |\nabla f_2| \leq C < \infty$. Then $f_1 \equiv f_2$ on $M \times [0, T]$.

PROOF. We test the difference of the equations for f_1 and for f_2 with the testfunction $v = f_1 - f_2$ and get

$$\begin{aligned}
(61) \quad & \frac{1}{2} \int_M |v(\cdot, t)|^2 d\mu \\
& + \int_0^t \int_M \left(\left(\gamma^{\alpha\beta} \frac{\partial f_1}{\partial x^\alpha} \frac{\partial f_1}{\partial x^\beta} \right)^{\frac{p}{2}-1} \frac{\partial f_1}{\partial x^\sigma} \right. \\
& \qquad\qquad \left. - \left(\gamma^{\alpha\beta} \frac{\partial f_2}{\partial x^\alpha} \frac{\partial f_2}{\partial x^\beta} \right)^{\frac{p}{2}-1} \frac{\partial f_2}{\partial x^\sigma} \right) \\
& \qquad \cdot \gamma^{\sigma\rho} \left(\frac{\partial f_1}{\partial x^\rho} - \frac{\partial f_2}{\partial x^\rho} \right) d\mu \, dt \\
& \leq c \int_0^t \int_M \left(|v| \, |\nabla v| |\nabla F|^{p-1} + v^2 |\nabla F|^p \right) d\mu \, dt
\end{aligned}$$

with the shorthand notation $|\nabla F| := |\nabla f_1| + |\nabla f_2|$.

The second term II on the left hand side of (61) is estimated by

$$(62) \qquad II \geq c \int_0^t \int_M |\nabla v|^2 |\nabla F|^{p-2} d\mu \, dt \, .$$

This inequality follows from Lemma 11.

Now we interpolate the term on the right hand side of (62) by using Young's inequality for the first term on the right of (61):

$$\begin{aligned}
(63) \quad & \int_0^t \int_M |v| \|\nabla v\| |\nabla F|^{p-1} \, d\mu \, dt \\
& \leq \frac{\varepsilon}{2} \int_0^t \int_M |\nabla v|^2 |\nabla F|^{p-2} \, d\mu \, dt \\
& \quad + \frac{1}{2\varepsilon} \int_0^t \int_M v^2 |\nabla F|^p \, d\mu \, dt \, .
\end{aligned}$$

Putting all the above inequalities (61)-(63) together, we get

$$(64) \quad \int_M |v(\cdot, t)|^2 d\mu + \int_0^t \int_M |\nabla v|^2 |\nabla F|^{p-2} \, d\mu \, dt \leq c \int_0^t \int_M |v|^2 |\nabla F|^p \, d\mu \, dt \, .$$

Using the assumption $|\nabla F| < C$ on $M \times [0, T[$ we get from (64)

$$(65) \qquad \int_M |v(\cdot, t)|^2 d\mu \le c \int_0^t \int_M v^2 d\mu \, dt$$

with a new constant c. The right hand side of (65) is increasing in t and hence we get

$$\sup_{t' \in [0,t]} \int_M |v(\cdot, t')|^2 d\mu \le c\,t \sup_{t' \in [0,t]} \int_M |v(\cdot, t')|^2 d\mu.$$

Thus, for $t < \frac{1}{c}$ we get $v \equiv 0$ for $t' \in [0, t]$. Iteration of the argument proves the assertion (notice that the constant c remains the same during the iteration process). $\qquad\qquad\square$

In the conformal case, we get the following corollary:

COROLLARY 1. *Let f_1 and f_2 be weak solutions of*

$$\begin{cases} \frac{\partial f}{\partial t} - \Delta_p f & \perp T_f N \\ f|_{t=0} & = f_0 \end{cases}$$

for initial data $f_0 \in W^{1,\infty}(M)$ with $p = \dim M$. Then $f_1 = f_2$ on a time-interval $[0, T]$ where T depends on $\|\nabla f_0\|_{L^\infty(M)}$. Furthermore, there exists $\varepsilon_1 > 0$ such that

$$\int_M |\nabla f_0|^p d\mu < \varepsilon_1$$

implies $f_1 = f_2$ for all $t \ge 0$.

REMARK. It is an open question whether the m-harmonic flow ($m \ge 3$) develops singularities in finite time (numerical calculation lend some support to the conjecture that this is the case).

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