Young measures and nonlinear PDEs

Norbert Hungerbühler

Birmingham, October 1999

Contents

Introduction

In these notes, we deal with nonlinear (or more precisely with quasilinear) elliptic and parabolic systems of partial differential equations in divergence form. The ellipticity (and parabolicity) of such systems can be phrased by imposing certain monotonicity conditions on the operator. The classical assumptions in this context have been formulated by Leray and Lions in [LeLi-65]. These conditions guarantee solvability of the corresponding elliptic and parabolic equations. The aim of this text is to prove analogous existence results under relaxed monotonicity assumptions. The main technical tool we advocate and use throughout the proofs are Young measures.

This text is set up in the following way: Chapter 1 states a refined version of Ball's fundamental theorem on Young measures (see Section 1.1 and 1.2): For a sequence $u_j : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ generating the Young measure ν_x, $x \in \Omega$, Ball's Theorem asserts, that a tightness condition, preventing mass in the target space from escaping to infinity, implies that ν_x is a probability measure and that $f(u_k) \rightharpoonup \langle \nu_x, f \rangle$ in L^1 provided the sequence is equiintegrable. We show that Ball's tightness condition is necessary for the conclusions to hold and that in fact all three, the tightness condition, the assertion $\|\nu_x\|_{\mathscr{M}} = 1$, and the convergence conclusion, are equivalent. This theorem is then used to prove several technical lemmas (see Section 1.3) which build the basic tools in the Chapters 3 and 4. Most of Chapter 1 follows [Hu-97]. We also recall in this chapter some relevant facts about gradient Young measures (see Section 1.4) which will be used throughout the text.

Chapter 2 gives a brief overview on monotone operator theory and its applications to nonlinear partial differential equations. The aim of this chapter is to explain the connection of the known theory to the new results in the Chapters 3 and 4. In particular, we recall the result on Leray-Lions operators (see Theorem 2.7).

In Chapter 3 we prove the Leray-Lions result in the elliptic case but with considerably relaxed monotonicity assumptions. In particular, we can drop the strict monotonicity condition and replace it by monotonicity (together with a mild regularity condition, which can as well be dropped if the problem is variational in the gradient variable). We also deal with an integrated form of monotonicity (called quasimonotonicity). Its definition is phrased in terms of gradient Young measures (see Definition 3.1). The main result is stated in Theorem 3.2. The main technical point is a div-curl inequality, which allows to pass to the limit in the approximate equations and hence to prove existence of a solution.

In Chapter 3 we treat the parabolic problem which corresponds to Chapter 3. That is, we consider a time dependent Leray-Lions operator and relax its monotonicity assumption to those stated in the stationary case. Again, the use of Young measures allows to prove existence of a solution.

Each chapter (with the exception of Chapter 2) starts with a short abstract which summarizes the contents and which should allow the reader to navigate easily through the text.

Chapter 1

A refinement of Ball's Theorem on Young measures

ABSTRACT: For a sequence $u_j : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ generating the Young measure ν_x, $x \in \Omega$, Ball's Theorem on Young measures asserts, that a tightness condition, preventing mass in the target space from escaping to infinity, implies that ν_x is a probability measure and that $f(u_k) \rightharpoonup \langle \nu_x, f \rangle$ in L^1 provided the sequence is equiintegrable. We show that Ball's tightness condition is necessary for the conclusions to hold and that in fact all three, the tightness condition, the assertion $\|\nu_x\|_{\mathscr{M}} = 1$, and the convergence conclusion, are equivalent. We give some simple applications of this observation which are useful in the theory of nonlinear partial differential equations, in particular in the Chapters 3 and 4. We also recall some basic facts about gradient Young measures which will be used throughout this text.

1.1 The fundamental theorem on Young measures

Young measures have in recent years become an increasingly indispensable tool in the calculus of variations and in the theory of nonlinear partial differential equations (see, e.g., [Ta-79], [Ta-82], or [DoHuMü-97]. For a list of references for general Young measure theory see, e.g., [Val-94]). In [Ball-89] Ball stated the following version of the fundamental theorem of Young measures which is tailored for applications in these fields:

Mathematics Subject Classification: 46E27, 28A33, 28A20

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $u_j \colon \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_j \to K$ in measure as $j \to \infty$, i.e. given any open neighborhood U of K in $\mathbb{R}^m$*

$$\lim_{j\to\infty} |\{x \in \Omega \colon u_j(x) \notin U\}| = 0.$$

Then there exists a subsequence u_k of u_j and a family (ν_x), $x \in \Omega$, of positive measures on $\mathbb{R}^m$, depending measurably on x, such that

(i) $\|\nu_x\|_{\mathscr{M}} := \int_{\mathbb{R}^m} d\nu_x \leqslant 1$ *for a.e.* $x \in \Omega$,

(ii) $\operatorname{spt} \nu_x \subset K$ *for a.e.* $x \in \Omega$, *and*

(iii) $f(u_k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda)$ *in* $L^\infty(\Omega)$ *for each continuous function* $f \colon \mathbb{R}^m \to \mathbb{R}$ *satisfying* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

Suppose further that $\{u_k\}$ satisfies the boundedness condition

$$\forall R > 0 \colon \lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R \colon |u_k(x)| \geqslant L\}| = 0, \tag{1.1}$$

where $B_R = B_R(0)$. Then

$$\|\nu_x\|_{\mathscr{M}} = 1 \qquad \text{for a.e. } x \in \Omega \tag{1.2}$$

(i.e. ν_x is a probability measure), and there holds:

$$\begin{cases} \text{For any measurable } A \subset \Omega \text{ and any continuous function } f \colon \mathbb{R}^m \to \\ \mathbb{R} \text{ such that } \{f(u_k)\} \text{ is sequentially weakly relatively compact in} \\ L^1(A) \text{ we have } f(u_k) \rightharpoonup \langle \nu_x, f\rangle \text{ in } L^1(A). \end{cases} \tag{1.3}$$

Improved versions of this theorem can be found, e.g., in [Kr-94]. The main result of this chapter is to prove that (1.1) is necessary for (1.2) and (1.3) to hold, and that in fact (1.1), (1.2) and (1.3) are equivalent. We will give some simple consequences of this fact which will be useful in the subsequent chapters on nonlinear elliptic and parabolic systems of partial differential equations.

Theorem 1.2 *Let Ω, u_j and ν_x be as in Theorem 1.1. Then (1.1), (1.2) and (1.3) are equivalent.*

Remarks:

(a) It was proved in [Ball-89] that (1.1) is equivalent to the following tightness condition: Given any $R > 0$ there exists a continuous nondecreasing function $g_R\colon [0,\infty) \to \mathbb{R}$, with $\lim_{t\to\infty} g_R(t) = \infty$, such that

$$\sup_{k\in\mathbb{N}} \int_{\Omega\cap B_R} g_R(|u_k(x)|)dx < \infty.$$

(b) In [Ball-89] it is also shown, that under hypothesis (1.1) for any measurable $A \subset \Omega$

$$f(\cdot, u_k) \rightharpoonup \langle \nu_x, f(x,\cdot)\rangle \quad \text{in } L^1(A)$$

for every Carathéodory function $f\colon A \times \mathbb{R}^m \to \mathbb{R}$ such that $\{f(\cdot, u_k)\}$ is sequentially weakly relative compact in $L^1(A)$. Hence, this fact is also equivalent to (1.1), (1.2) and (1.3).

(c) Ball also shows in [Ball-89], that if u_k generates the Young measure ν_x, then for $\psi \in L^1(\Omega; C_0(\mathbb{R}^m))$

$$\lim_{k\to\infty} \int_\Omega \psi(x, u_k(x))dx = \int_\Omega \langle \nu_x, \psi(x,\cdot)\rangle dx.$$

Here, $C_0(\mathbb{R}^m)$ denotes the Banach space of continuous functions $f\colon \mathbb{R}^m \to \mathbb{R}$ satisfying $\lim_{|\lambda|\to\infty} f(\lambda) \to 0$ equipped with the L^∞-norm.

The proof of Theorem 1.2 which we give in the following section follows [Hu-97].

1.2 Proof of the refined theorem

First we prove (1.2) $\Longrightarrow$ (1.1). We assume by contradiction that (1.2) holds and that there exists $R > 0$ and $\varepsilon > 0$ with the following property: There exists a sequence $L_i \to \infty$ and integers k_i such that $|\{x \in \Omega \cap B_R\colon |u_{k_i}(x)| \geqslant L_i\}| > \varepsilon$ for all $i \in \mathbb{N}$. For $\rho > 0$ consider the function

$$\alpha_\rho(t) := \begin{cases} 1 & \text{if } t \leqslant \rho \\ 0 & \text{if } t \geqslant \rho + 1 \\ \rho + 1 - t & \text{if } \rho < t < \rho + 1. \end{cases}$$

Then $\varphi_\rho\colon \mathbb{R}^m \to \mathbb{R}, x \mapsto \alpha_\rho(|x|)$, is in $C_0(\mathbb{R}^m)$. Hence, applying the first part of Theorem 1.1, we have that

$$\lim_{k\to\infty} \int_\Omega \varphi_\rho(u_k)\chi_{B_R} dx = \int_\Omega \int_{\mathbb{R}^m} \varphi_\rho(\lambda) d\nu_x(\lambda) \chi_{B_R} dx. \tag{1.4}$$

Notice that $k_i \to \infty$ for $i \to \infty$ since the functions u_j are finite for a.e. $x \in \Omega$. Hence, u_{k_i} is a subsequence of u_k and for i large enough, we find

$$|\Omega \cap B_R| - \varepsilon \geqslant \int_\Omega \varphi_\rho(u_{k_i})\chi_{B_R} dx$$

such that (1.4) implies

$$|\Omega \cap B_R| - \varepsilon \geqslant \int_\Omega \int_{\mathbb{R}^m} \varphi_\rho(\lambda) d\nu_x(\lambda)\chi_{B_R} dx. \tag{1.5}$$

On the other hand, by the monotone convergence theorem, we conclude that the right hand side of (1.5) converges for $\rho \to \infty$ to

$$\int_\Omega \int_{\mathbb{R}^m} d\nu_x(\lambda)\chi_{B_R} dx = \int_\Omega \|\nu_x\|_{\mathscr{M}} \chi_{B_R} dx = |\Omega \cap B_R|$$

by (1.2) and this contradicts (1.5).

Second we prove that (1.3) $\Longrightarrow$ (1.2). Let $R > 0$ be fixed and let f denote the function constant 1 on $\mathbb{R}^m$. Then $f(u_j)$ is sequentially weakly relative compact on $\Omega \cap B_R$ and (1.3) implies

$$|\Omega \cap B_R| = \int_{\Omega\cap B_R} f(u_k)\chi_{B_R} dx \to$$
$$\to \int_{\Omega\cap B_R} \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda)\chi_{B_R} dx = \int_{\Omega\cap B_R} \|\nu_x\|_{\mathscr{M}} dx. \tag{1.6}$$

Since $\|\nu_x\|_{\mathscr{M}} \leqslant 1$ by (i) in Theorem 1.1, we conclude that $\|\nu_x\|_{\mathscr{M}} = 1$ for a.e. $x \in \Omega \cap B_R$. Since R was arbitrary, the claim follows. $\square$

1.3 Applications

In this section we make available some tools which will be used in Chapter 3 and 4. The following propositions are certainly well known to people working in the field, but we want to show that the sharp version of Ball's theorem which we now have at our disposal, comes in very handy in the proofs.

Proposition 1.3 *If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence u_j then there holds*

$$u_j \to u \text{ in measure} \iff \nu_x = \delta_{u(x)} \text{ for a. e. } x \in \Omega.$$

A weak form of this proposition can be found e.g. in [Ev-90] (see also [Kr-94]).

Proof. Let us first assume that $u_j \to u$ in measure, i.e. for all $\varepsilon > 0$ we have

$$\lim_{j\to\infty} |\{|u_j - u| > \varepsilon\}| = 0. \tag{1.7}$$

For $\varphi \in C_c^\infty(\mathbb{R}^m)$ and $\zeta \in L^1(\Omega)$ there holds

$$| \int_\Omega \zeta\,(\varphi(u_j) - \varphi(u))dx| \leqslant$$
$$| \int_{|u_j-u|>\varepsilon} \zeta\,(\varphi(u_j) - \varphi(u))dx| + | \int_{|u_j-u|\leqslant\varepsilon} \zeta\,(\varphi(u_j) - \varphi(u))dx| =: I + II.$$

By choosing ε appropriately, we can make II as small as we want, since we observe that $II \leqslant \varepsilon \|D\varphi\|_{L^\infty}\|\zeta\|_{L^1}$. For I we then have

$$I \leqslant 2\|\varphi\|_{L^\infty} \int_{|u_j-u|>\varepsilon} |\zeta| dx$$

which converges to 0 as j tends to ∞ by absolute continuity of the integral and (1.7). Since C_c^∞ is dense in C_0 we conclude that for all $\varphi \in C_0$

$$\varphi(u_j) \overset{*}{\rightharpoonup} \langle \delta_{u(x)}, \varphi\rangle \quad \text{in } L^\infty(\Omega)$$

and hence $\nu_x = \delta_{u(x)}$.

Now, for the opposite implication, we assume $\nu_x = \delta_{u(x)}$, hence (1.2) is fulfilled.

First step: we consider the case that u_j is bounded in L^∞. Then by (1.3) we conclude that for $\varphi(x) := |x|^2$

$$\|u_j\|_{L^2}^2 = \int_\Omega \varphi(u_j)dx \to \int_\Omega \varphi(u)dx = \|u\|_{L^2}^2 \tag{1.8}$$

for $j \to \infty$. On the other hand choosing $\varphi = \mathrm{id}$ we similarly find that $u_j \rightharpoonup u$ weakly in $L^2(\Omega)$, which in combination with (1.8) gives that $u_j \to u$ in $L^1(\Omega)$. Thus for all $\alpha > 0$ we have

$$\alpha|\{|u_j - u| \geqslant \alpha\}| \leqslant \int_\Omega |u_j - u| dx \to 0$$

as $j \to \infty$, and hence $u_j \to u$ in measure.

Second step: We show that if u_j generates the Young measure $\delta_{u(x)}$ then $T_R(u_j) \to T_R(u)$ in measure, if T_R denotes the truncation $T_R(x) := x\min\{1, \frac{R}{|x|}\}$, $R > 0$ fixed. In fact, for $f \in C_0(\mathbb{R}^m)$ we have that $f \circ T_R$ is continuous and $f(T_R(u_j))$

is equiintegrable (and hence by the Dunford-Pettis theorem sequentially weakly precompact in $L^1(\Omega)$). Since (1.2) is fulfilled, we conclude by (1.3) that for $\zeta \in L^\infty(\Omega)$

$$\int_\Omega \zeta\, f(T_R(u_j))dx \to \int_\Omega \zeta\, f(T_R(u))dx.$$

This implies that $T_R(u_j)$ generates the Young measure $\delta_{T_R(u(x))}$ and by the first step, the claim follows.

Third step: We show, that $u_j \to u$ in measure. Let $\varepsilon > 0$ be given. Then we have:

$$\begin{aligned}|\{|u_j - u| > \varepsilon\}| \leqslant |\{|u_j - u| > \varepsilon, |u| \leqslant R, |u_j| \leqslant R\}| + \\ +|\{|u| > R\}| + |\{|u_j| > R\}| =: I + II + III.\end{aligned}$$

II can be made arbitrarily small by choosing $R > 0$ large enough. By (1.2) we have (1.1) which implies that III is, again for R large enough, uniformly in j as small as we want. Finally by the second step, $I \to 0$ for $j \to \infty$. □

Our second application is the following proposition:

Proposition 1.4 *Let $|\Omega| < \infty$. If the sequences $u_j\colon \Omega \to \mathbb{R}^m$ and $v_j\colon \Omega \to \mathbb{R}^k$ generate the Young measures $\delta_{u(x)}$ and ν_x respectively, then (u_j, v_j) generates the Young measure $\delta_{u(x)} \otimes \nu_x$.*

This result also holds for sequences μ_j, λ_j of Young measures converging in the narrow topology to μ and λ respectively: see [Val-94]. However it is false if both μ and λ are not Dirac measures. E.g. consider the Rademacher functions $u_1(x) := (-1)^{\lfloor x \rfloor}$ and $u_n(x) = u_1(nx)$. u_n and $-u_n$ generate the Young measure $\frac{1}{2}(\delta_{-1} + \delta_1)$, but (u_n, u_n) and $(-u_n, u_n)$ obviously generate different measures (consider the sets $K = \{(-1,-1),(1,1)\}$ and $K = \{(-1,1),(1,-1)\}$ respectively in Theorem 1.1).

Proof of Proposition 1.4. We have to show that for all $\varphi \in C_c^\infty(\mathbb{R}^m \times \mathbb{R}^k)$ there holds $\varphi(u_j, v_j) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^k} \varphi(u(x), \lambda) d\nu_x(\lambda)$. So, let $\zeta \in L^1(\Omega)$. We have

$$\begin{aligned}&|\int_\Omega \zeta\, (\varphi(u_j, v_j) - \int_{\mathbb{R}^k} \varphi(u, \lambda) d\nu_x(\lambda))dx| \leqslant \\ &\leqslant |\int_{|u_j - u| < \varepsilon} \zeta\, (\varphi(u_j, v_j) - \varphi(u, v_j))dx| + |\int_{|u_j - u| \geqslant \varepsilon} \zeta\, (\varphi(u_j, v_j) - \varphi(u, v_j))dx| \\ &\qquad + |\int_\Omega \zeta\, (\varphi(u, v_j) - \int_{\mathbb{R}^k} \varphi(u, \lambda) d\nu_x(\lambda))dx| =: I + II + III.\end{aligned}$$

Since $I \leqslant \varepsilon \|\zeta\|_{L^1(\Omega)} \|D\varphi\|_{L^\infty}$, the first term is small for $\varepsilon > 0$ small. For $\varepsilon > 0$ fixed, we have for $j \to \infty$

$$II \leqslant 2\|\varphi\|_{L^\infty} \int_{|u_j - u| \geqslant \varepsilon} |\zeta| dx \to 0$$

since by Proposition 1.3 the sequence u_j converges to u in measure. Since $L^\infty(\Omega)$ is dense in $L^1(\Omega)$ we may assume that $\zeta \in L^\infty(\Omega)$. Thus, the function $\zeta(x)\varphi(u(x),\cdot)$ is in $L^1(\Omega, C_0(\mathbb{R}^k))$ and hence $III \to 0$ as $j \to \infty$ by Remark (c). $\square$

The following Fatou-type lemma will also be used in the Chapters 3 and 4. The proof follows [DoHuMü-97]. In the sequel, by $\mathbb{M}^{m\times n}$ we mean the linear space of real $m \times n$ matrices.

Lemma 1.5 *Let $F\colon \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $u_k\colon \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that $u_k \to u$ in measure and such that Du_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathscr{M}} = 1$ for almost every $x \in \Omega$. Then*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k(x), Du_k(x))\, dx \geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u, \lambda)\, d\nu_x(\lambda)\, dx \tag{1.9}$$

provided that the negative part $F^-(x, u_k(x), Du_k(x))$ is equiintegrable.

More general versions of this lemma may be found in [Bald-84], [Bald-91] and [Val-94], [Val-90]. Our assumptions allow the following more elementary proof.

Proof
We may assume that the limes inferior on the left-hand side of (1.9) agrees with the limit and is not equal to $+\infty$. Consider the Carathéodory functions $F_R(x,u,p) = \min\{R, F(x,u,p)\}$ for $R > 0$. For fixed $R > 0$ the sequence $\{F_R(x, u_k(x), Du_k(x))\}_k$ is equiintegrable. We have

$$\int_\Omega F_R(x, u_k(x), Du_k(x))\, dx \leqslant \int_\Omega F(x, u_k(x), Du_k(x))\, dx \leqslant C < \infty$$

for all k and $R > 0$. Since, by assumption, $\|\nu_x\|_{\mathscr{M}} = 1$ for almost every $x \in \Omega$, we have by Remark (b) in Section 1.1 that for all $R > 0$

$$\lim_{k\to\infty} \int_\Omega F_R(x, u_k(x), Du_k(x))\, dx = \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx \leqslant C,$$

and by monotone convergence of the integrands as $R \to \infty$

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx \leqslant C < \infty\,. \tag{1.10}$$

On the other hand

$$\begin{aligned}
&\int_\Omega F(x,u_k(x),Du_k(x))\,dx - \int_\Omega\int_{\mathbb{M}^{m\times n}} F(x,u(x),\lambda)\,d\nu_x(\lambda)\,dx = \\
&= \int_\Omega F(x,u_k(x),Du_k(x))\,dx - \int_\Omega F_R(x,u_k(x),Du_k(x))\,dx + \\
&\qquad + \int_\Omega F_R(x,u_k(x),Du_k(x))\,dx - \int_\Omega\int_{\mathbb{M}^{m\times n}} F_R(x,u(x),\lambda)\,d\nu_x(\lambda)\,dx + \\
&\qquad + \int_\Omega\int_{\mathbb{M}^{m\times n}} F_R(x,u(x),\lambda)\,d\nu_x(\lambda)\,dx - \int_\Omega\int_{\mathbb{M}^{m\times n}} F(x,u(x),\lambda)\,d\nu_x(\lambda)\,dx \\
&=: I_k + II_k + III\,.
\end{aligned}$$

Now we have

$$\begin{aligned}
&I_k \geqslant 0\,, \\
&II_k \to 0 \quad \text{for any fixed } R>0 \text{ as } k\to\infty, \\
&III \to 0 \quad \text{as } R\to\infty, \text{ because of (1.10) and monotone convergence,}
\end{aligned}$$

and the claim follows. □

As a last application of the refined Young measure Theorem 1.2, we consider a criterion for the pointwise convergence of Fourier series, which is similar to Dini's test (see [Zy-77]).

Theorem 1.6 *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R})$ be a 2π periodic complex function. If $z\in\mathbb{R}$ is a point with the property that*

$$\int_{-\pi}^{\pi}\left|\frac{f(x)-f(z)}{x-z}\right|\,dx<\infty \tag{1.11}$$

then the Fourier series of f converges in z to $f(z)$.

Proof
With

$$D_N(x) := \frac{\sin(N+\frac12)x}{\sin\frac{x}{2}}$$

the Nth Fourier approximation of f is $s_N(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z-x)D_N(x)dx$. The Young measure generated by the (whole) sequence $\sin(N+\frac12)x$ is a probability measure ν_x with vanishing first moment $\langle\nu_x,\mathrm{id}\rangle=0$ (see e.g. [Val-94]). Hence, the Young measure μ_x generated by the sequence $(f(z-x)-f(z))D_N(x)$ has for a. e. x also these properties. Now, (1.11) implies that the sequence $(f(z-x)-f(z))D_N(x)$ is equiintegrable and hence (since μ_x is a probability measure for a. e. x) by equivalence of (1.2) and (1.3), we have as $N \to \infty$

$$s_N(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(z-x) - f(z)) D_N(x) dx + \frac{f(z)}{2\pi} \int_{-\pi}^{\pi} D_N(x) dx \to f(z)$$

since the first term converges to zero and the second term equals $f(z)$ for all N.

□

1.4 Gradient Young measures

In this section, we briefly summarize the relevant facts about $W^{1,p}$ gradient Young measures that we are going to use in the Chapters 3 and 4. In the sequel, we confine ourselves to the case $1 \leqslant p < \infty$. For the case $p = \infty$ see [KiPe-91a]–[KiPe-94].

For the rest of this section, Ω denotes a measurable bounded set in $\mathbb{R}^n$. Suppose, f_k is a bounded sequence in $L^p(\Omega; \mathbb{R}^N)$. Then, according to Theorem 1.2, there exists a family of probability measures (ν_x), $x \in \Omega$, and a (not relabeled) subsequence of the f_k, such that whenever

$$\psi(f_k) \rightharpoonup \bar{\psi} \qquad \text{in } L^1(\Omega), \text{ for } \psi \in C(\mathbb{R}^N),$$

then

$$\bar{\psi}(x) = \int_{\mathbb{R}^N} \psi(\lambda) d\nu_x(\lambda) \qquad \text{in } \Omega \text{ a.e.} \tag{1.12}$$

For example, from Hölder's inequality and the Dunford-Pettis theorem, the conclusion (1.12) follows if

$$|\psi(\lambda)| \leqslant C(1 + |\lambda|^q), \qquad \text{for all } \lambda \in \mathbb{R}^N \tag{1.13}$$

whenever $q < p$. In order to deal with the case $q = p$, one considers the space

$$E^p := \{\psi \in C(\mathbb{M}^{n \times N}) : \lim_{|A| \to \infty} \frac{\psi(A)}{1 + |A|^p} \text{ exists}\}.$$

E^p is isomorphic to the continuous functions on the Alexandrov one-point compactification of $\mathbb{M}^{n \times N}$ and is separable. Thus, we arrive at the definition of p-Young measures (see [KiPe-94]):

Definition 1.7 *A family (ν_x), $x \in \Omega$, is a p-Young measure, provided there is a sequence $f_k \in L^p(\Omega; \mathbb{R}^N)$, $1 \leqslant p < \infty$, and a $g \in L^1(\Omega)$ such that* *(i)* $|f_k|^p \rightharpoonup g$ *in* $L^1(\Omega)$,

(ii) $\psi(f_k) \rightharpoonup \bar{\psi}$ *in* $L^1(\Omega)$, *where*

$$\bar{\psi}(x) = \int_{\mathbb{R}^N} \psi(\lambda) d\nu_x(\lambda) \qquad \text{in } \Omega \text{ a.e. for } \psi \in E^p.$$

Alternatively, one may define biting Young measures for an arbitrary sequence: Recall that if g_k is a bounded sequence in $L^1(\Omega)$, then there is a sequence of measurable sets $E_j \subset \Omega$, $E_{j+1} \subset E_j$, $|E_j| \to 0$, and a $g \in L^1(\Omega)$ such that for a (not relabeled) subsequence of the g_k,

$$g_k \rightharpoonup g \qquad \text{in } L^1(\Omega \setminus E_j) \text{ for each } j.$$

This is the conclusion of Chacon's biting lemma (see [BrCh-80], and [BaMu-89]) and is usually written as

$$g_k \overset{b}{\rightharpoonup} g \qquad \text{in } L^1(\Omega).$$

The corresponding notion of biting Young measures is

Definition 1.8 *A family* (ν_x), $x \in \Omega$, *is a biting Young measure, provided there is a sequence* $f_k \in L^p(\Omega; \mathbb{R}^N)$, $1 \leqslant p < \infty$, *and a* $g \in L^1(\Omega)$ *such that*

(i) $|f_k|^p \overset{b}{\rightharpoonup} g$ *in* $L^1(\Omega)$,

(ii) $\psi(f_k) \overset{b}{\rightharpoonup} \bar{\psi}$ *in* $L^1(\Omega)$, *where*

$$\bar{\psi}(x) = \int_{\mathbb{R}^N} \psi(\lambda) d\nu_x(\lambda) \qquad \text{in } \Omega \text{ a.e. for } \psi \in E^p$$

The bitten sets E_j do not depend on the particular ψ. A p-Young measure is a biting Young measure.

Now, we impose the constraint that the functions f_k that generate the Young measure are gradients. The associated measures are called $H^{1,p}$ Young measures and $H^{1,p}$ biting Young measures respectively (or $W^{1,p}$ Young measures and $W^{1,p}$ biting Young measures if the spaces $H^{1,p}$ and $W^{1,p}$ coincide). The principal result in [KiPe-94] is that $H^{1,p}$ Young measures and $H^{1,p}$ biting Young measures are the same and can be characterized as follows:

Theorem 1.9 (Kinderlehrer-Pedregal) *Let* $(\nu)_x$, $x \in \Omega$, *be a family of probability measures in* $C(\mathbb{M}^{n\times N})'$. *Then,* $(\nu_x)_{x\in\Omega}$ *is an* $H^{1,p}$ *Young measure if and only if* *(i) there is a* $u \in H^{1,p}(\Omega; \mathbb{R}^n)$ *such that*

$$Du(x) = \int_{\mathbb{M}^{n\times N}} A d\nu_x(A) \qquad \text{in } \Omega \text{ a.e.},$$

(ii) Jensen's inequality

$$\varphi(Du(x)) \leqslant \int_{\mathbb{M}^{n\times N}} \varphi(A) d\nu_x(A)$$

holds for all $\varphi \in X^p$ *quasiconvex, and*

(iii) the function

$$\Psi(x) = \int_{\mathbb{M}^{n\times N}} |A|^p d\nu_x(A) \in L^1(\Omega).$$

Here, X^p denotes the (not separable) space

$$X^p := \{\psi \in C(\mathbb{M}^{n\times N}) : |\psi(A)| \leqslant C(1 + |A|^p) \text{ for all } A \in \mathbb{M}^{n\times N}\}$$

that is suggested by (1.13).

For improved versions of this characterization, see [FoMüPe-98] and [Kr-99, Theorem 8.1].

An important special case, which we will encounter in the Chapters 3 and 4 are homogeneous gradient Young measures:

Definition 1.10 *The* $W^{1,p}$ *gradient Young measure* $(\nu_x)_{x\in\Omega}$ *is called homogeneous, if it does not depend on* x*, i.e., if* $\nu_x = \nu$ *for almost all* $x \in \Omega$*.*

Chapter 2

Review on monotone operator theory

2.1 Monotone operators in Banach spaces

An operator $A : D(A) \subset X \to X^*$ on a normed vector space X is called *monotone* if

$$\langle Au - Av, u - v\rangle \geqslant 0 \qquad \text{for all } u, v \in D(A), \tag{2.1}$$

strongly monotone if

$$\langle Au - Av, u - v\rangle > 0 \qquad \text{for all } u, v \in D(A) \text{ with } u \neq v, \tag{2.2}$$

and *strongly monotone* if

$$\langle Au - Av, u - v) \geqslant \|u - v\|\gamma(\|u - v\|) \qquad \text{for all } u, v \in D(A), \tag{2.3}$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $\gamma(t) \to +\infty$ for $t \to \infty$, and with $\gamma(t) = 0$ only if $t = 0$. If in addition, $t \mapsto t\gamma(t)$ is strictly monotone increasing, then A is called *uniformly monotone*. Finally, A is *dissipative*, if $-A$ is monotone. If X is a complex space, one requires that the inequalities (2.1)–(2.3) hold for the real part. In particular, if A is linear, then A is monotone if and only if A is a positive operator. If X has an inner product structure, we may identify X with the dual space X^* in the usual way, and hence one can define $A : D(A) \subset X \to X$ to be monotone if $(Au - Av|u - v) \geqslant 0$. If, in this case, $X = D(A) = \mathbb{R}$ is one dimensional, then A is monotone if and only if A is a monotone increasing function. If, still in case $X = \mathbb{R}$, A is continuous and satisfies a coercivity condition

$$(Au|u) \geqslant \gamma(|u|)|u| \qquad \text{as } u \to \infty$$

for a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(t) \to +\infty$ as $t \to +\infty$, then Bolzano's intermediate value theorem assures that A is surjective. And strict monotonicity of A

easily implies (for general X) injectivity. The tremendous success of the theory of monotone operators is based on the fact that the stated surjectivity result for one dimension carries over to general reflexive Banach spaces X. On the other hand, the theory of monotone operators amplifies parts of the calculus of variations: Namely, if Ω is an open convex set in a real vector space and F a potential operator on Ω with potential f, then F is (strictly) monotone if and only if f is (strictly) convex.

Monotone operators have good analytic properties: A monotone operator $A : X \to X^*$ on a real reflexive Banach space X is, e.g., automatically locally bounded (see, e.g., [Ze-90]). In particular, a linear positive operator which is defined on the whole Banach space X is always continuous. Furthermore, already very weak continuity properties for monotone operators imply numerous convergence theorems like Minty's trick (see below).

We start our overview with the main theorem on monotone operators, which was proved by Browder [Bro-63] and Minty [Mi-63] (see also [Bro-63a]):

Theorem 2.1 (Browder, Minty) *Let X be a real reflexive Banach space and $A : X \to X^*$ monotone, hemicontinuous and coercive. Then A is surjective.*

Here, A is *hemicontinuous* if the map

$$t \mapsto \langle A(u + tv), w\rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in X$. And A is *coercive* if

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty.$$

A Hilbert space version of this theorem has been proved beforehand by Minty in [Mi-62]. Similar results in connection with partial differential equations appear in [Vi-61] and [Vi-63].

One possible proof of the main theorem of monotone operators uses Minty's trick for which we give here a slightly refined version:

Theorem 2.2 (Minty's Trick) *If $A : X \to X^*$ is monotone and hemicontinuous on a real Banach space X, and if $u_n \rightharpoonup u$ in X and $Au_n \overset{*}{\rightharpoonup} b$ in X^*, then the following is true:*

(a) $\liminf_{n\to\infty}\langle Au_n, u_n\rangle \leqslant \langle b, u\rangle \implies Au = b.$

(b) $\limsup_{n\to\infty}\langle Au_n, u_n\rangle \leqslant \langle b, u\rangle \implies \langle Au_n, u_n\rangle \to \langle b, u\rangle$ *and* $Au = b.$

Minty's trick follows easily from the *maximal monotonicity property*:

Proposition 2.3 *If $A : X \to X^*$ is monotone and hemicontinuous on a real Banach space X, then A is maximal monotone, i.e., for all $u \in X$ and all $b \in X^*$ the following implication holds:*

$$\langle b - Av, u - v\rangle \geqslant 0 \quad \textit{for all } v \in X \implies Au = b.$$

For the theory of maximal monotone operators (in particular time dependent problems), see [Bré-73].

2.2 Applications to nonlinear partial differential equations

Monotone operator theory goes hand in hand with progress in the theory of nonlinear partial differential equations: see, e.g., the monographs [Bro-68/76], [Li-69], [Vai-72] (in connection with variational methods and Hammerstein integral equations), [Sk-73] and [Sk-86] (on mapping degree and elliptic equations), [GGZ-74], [Lang-76] (monotone potential operators), [Bar-76] (on time dependent problems), [PS-78], [Kl-79] (variational inequalities), [De-85]. As far as applications of monotone operator theory to quasilinear elliptic differential equations is concerned (see next chapter of this text), we refer to [Li-69], [Bro-68], [Du-76], [FuFu-80], [Pe-80], [Pe-81], [Ne-83], [Sk-86].

The main theorem for monotone operators applies directly to the model problem involving the p-Laplace operator

$$-\operatorname{div}(|Du|^{p-2}Du) = f \qquad \text{on } \Omega$$

(with appropriate boundary conditions), which is of variational form. Also nonlinear problems of non-variational form are accessible, e.g.,

$$Lu + F(u) = f \qquad \text{on } \Omega \tag{2.4}$$

with boundary conditions

$$u = 0 \qquad \text{on } \partial\Omega. \tag{2.5}$$

Here,

$$Lu = -\operatorname{div}\sigma(Du)$$

and we are looking for a solution $u \in W_0^{1,p}(\Omega)$ for some $1 < p < \infty$ and a bounded open set $\Omega \in \mathbb{R}^n$. We impose the following conditions:

(i) *Monotonicity for the principle part Lu*:

$$(\sigma(\xi) - \sigma(\xi')) \cdot (\xi - \xi') \geqslant 0 \qquad \text{for all } \xi, \xi' \in \mathbb{R}^n.$$

(ii) *Monotonicity for Fu*:

$$(F(u) - F(u'))(u - u') \geqslant 0 \qquad \text{for all } u, v \in \mathbb{R}.$$

(iii) *Coerciveness for $Lu + F(u)$*: For some fixed $c > 0$

$$\sigma(\xi) \cdot \xi + F(u)u \geqslant c(|\xi|^p - 1)$$

holds for all $\xi \in \mathbb{R}^n, u \in \mathbb{R}$.

(iv) *Growth condition:* The functions σ and F are continuous and there exists a constant $d > 0$ such that

$$\begin{aligned} |F(u)| &\leqslant d(1 + |u|^{p-1}) \qquad && \text{for all } u \in \mathbb{R}, \\ |\sigma(\xi)| &\leqslant d(1 + |\xi|^{p-1}) \qquad && \text{for all } \xi \in \mathbb{R}^n. \end{aligned}$$

The solvability of problem (2.4), (2.5) under the given conditions (i)–(iv) follows from the following variant (Theorem 2.4) of the main theorem for monotone operators stated in the previous section: Again, let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $2 \leqslant p < \infty$. We are looking for a solution $u \in W_0^{1,p}(\Omega)$ of the following elliptic system of order $2m$

$$Lu = f \qquad \text{in } \Omega \tag{2.6}$$

$$D^\beta u = 0 \qquad \text{on } \partial\Omega, \text{ for all } \beta \text{ with } |\beta| < m, \tag{2.7}$$

where

$$(Lu)(x) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, Du(x))$$

with $m \geqslant 1$ and $Du = (D^\gamma u)_{|\gamma| \leqslant m}$. We think of A_α as a real function of the variables $x \in \Omega$ and $\xi \in \mathbb{R}^M$, where $\xi = (\xi^\gamma)_{|\gamma| \leqslant m}$. The following conditions are assumed:

(H1) *Carathéodory condition:* For all α with $|\alpha| \leqslant m$, the functions $A_\alpha : \Omega \times \mathbb{R}^M \to \mathbb{R}$ has the properties

$$\begin{aligned} x &\mapsto A_\alpha(x, \xi) \qquad && \text{is measurable on } \Omega \text{ for all } \xi \in \mathbb{R}^M, \\ \xi &\mapsto A_\alpha(x, \xi) \qquad && \text{is continuous on } \mathbb{R}^M \text{ for almost all } x \in \Omega. \end{aligned}$$

(H2) *Growth condition:* There exists $g \in L^{p'}(\Omega)$ and $C > 0$ such that for all $\xi \in \mathbb{R}^M$, $|\alpha| \leqslant m$, $x \in \Omega$ there holds

$$|A_\alpha(x,\xi)| \leqslant C\big(g(x) + \sum_{|\gamma|\leqslant m} |\xi^\gamma|^{p-1}\big)$$

(H3) *Monotonicity condition:* For all $\xi, \xi' \in \mathbb{R}^M$, $x \in \Omega$

$$\sum_{|\alpha|\leqslant m} (A_\alpha(x,\xi) - A_\alpha(x,\xi'))(\xi - \xi') \geqslant 0.$$

(H4) *Coerciveness condition:* There exists $c > 0$ such that for all $\xi \in \mathbb{R}^M$, $x \in \Omega$

$$\sum_{|\alpha|\leqslant m} A_\alpha(x,\xi)\xi^\alpha \geqslant c \sum_{|\gamma|=m} |\xi^\gamma|^p - h(x)$$

for a function $h \in L^1(\Omega)$.

An operator L satisfying the conditions (H1) through (H4) is called *monotone coercive quasilinear elliptic differential operator*. Now, let $X = W_0^{1,p}(\Omega)$, and let us set

$$a(u,v) = \int_\Omega \sum_{|\alpha|\leqslant m} A_\alpha(x, Du(x)) D^\alpha v(x) dx$$

and

$$b(v) = \int_\Omega f v \, dx$$

with $f \in L^{p'}(\Omega)$. Then the following holds:

Theorem 2.4 (Browder, Višik) *Assume that (H1)–(H4) is satisfied. Then there exists a unique operator $A : X \to X^*$ such that*

$$\langle Au, v\rangle = a(u,v) \qquad \textit{for all } u, v \in X.$$

The problem

$$a(u,v) = b(v) \qquad \textit{for all } v \in X$$

is equivalent to the operator equation

$$Au = b \qquad u \in X$$

and the operator $A : X \to X^*$ is monotone, coercive, continuous and bounded. Thus the main theorem for monotone operators (Theorem 2.1) applies and hence (2.6), (2.7) has a solution for arbitrary $f \in L^{p'}(\Omega)$. The set of solutions is closed and convex.

As usual, uniform monotonicity would give uniqueness of the solution.

2.3 Pseudomonotone operators

In case the nonlinear lower order term F of problem (2.4), (2.5) does not satisfy the monotonicity assumption (ii), then the theory of *proper monotone operators*, *pseudomonotone operators* and *semimonotone operators* has to be used instead. We give a brief overview on that part of the theory next.

An operator $A : X \to X^*$ on a real reflexive Banach space X is called *pseudomonotone* if

$$u_n \rightharpoonup u \qquad \text{in } X \text{ as } n \to \infty$$

and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leqslant 0$$

implies

$$\langle Au, u - w\rangle \leqslant \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle \qquad \text{for all } w \in X.$$

For example, a monotone and hemicontinuous operator is pseudomonotone. In fact, the class of pseudomonotone operators is “intermediate” between monotone, hemicontinuous and so called type M operators (see, e.g. [Sh-97]). And is is easy to see, that this class is “strictly intermediate”. Pseudomonotone operators have good properties, to mention one, the sum of two pseudomonotone operators is again pseudomonotone. Moreover, the sum of a pseudomonotone and a strongly continuous operator is pseudomonotone. (We recall that A is *strongly continuous* if $u_n \rightharpoonup u$ in X implies that $Au_n \to Au$ in Y.)

We should also mention the connection to proper maps: we recall, that a map is called *proper*, if the preimages of compact sets are again compact. Then, the following is true:

Theorem 2.5 *Let $A_1, A_2 : X \to X^*$ be operators on the real reflexive Banach space X and $A = A_1 + A_2$. If*

(i) A_1 is uniformly monotone and continuous,

(ii) A_2 is compact,

(iii) A is coercive,

then A is proper. If, moreover, A_2 is strongly continuous, then A is pseudomonotone.

For relations with semimonotone operators, we refer to [Ze-90].

The main theorem on pseudomonotone operators is due to Brézis (see [Bré-68]):

Theorem 2.6 (Brézis) *Assume, the operator $A : X \to X^*$ is pseudomonotone, bounded and coercive on the real separable and reflexive Banach space X. Then, for each $b \in X^*$ the equation*

$$Au = b$$

has a solution.

As an application, we consider the problem

$$-\operatorname{div} \sigma(Du) + g(u) = f \qquad \text{in } \Omega \tag{2.8}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2.9}$$

for a function $u : \Omega \to \mathbb{R}$ on a bounded open domain $\Omega \subset \mathbb{R}^n$. Here Du denotes (in contrast to the example before) the usual gradient of u. We make the following assumptions:

(A1) *Coerciveness for g:* The function $g : \mathbb{R} \to \mathbb{R}$ is continuous and $\inf_{u\in\mathbb{R}} g(u)u > -\infty$.

(A2) *Growth condition for g :* For all $u \in \mathbb{R}$

$$|g(u)| \leqslant C(1 + |u|^{r-1}),$$

where $1 < p, r < \infty$, $p^{-1} - n^{-1} < r^{-1}$.

(A3) *Monotonicity condition for the principle part:* For all $\xi, \xi' \in \mathbb{R}^n$

$$(\sigma(\xi) - \sigma(\xi')) \cdot (\xi - \xi') \geqslant 0.$$

(A4) *Coerciveness condition for the principle part:* There is a number $c > 0$ such that for all $\xi \in \mathbb{R}^n$

$$\sigma(\xi) \cdot \xi \geqslant c|\xi|^p.$$

(A5) *Growth condition for the principle part:* The function σ is continuous and for all $\xi \in \mathbb{R}^n$

$$|\sigma(\xi)| \leqslant c(1 + |\xi|^{p-1})$$

Then, we consider

$$\begin{aligned} a_1(u, v) &= \int_\Omega \sigma(Du) \cdot Dv dx \\ a_2(u, v) &= \int_\Omega g(u)v dx \\ b(v) &= \int_\Omega fv\, dx \end{aligned}$$

and we seek $u \in X = W_0^{1,p}(\Omega)$ such that

$$a_1(u,v) + a_2(u,v) = b(v) \qquad \text{for all } v \in X.$$

Under these assumptions it is not hard to show that $A_1 : X \to X^*$ characterized by $\langle A_1 u, v\rangle = a_1(u,v)$ is monotone, coercive continuous and bounded. Moreover, the operator $A_2 : X \to X^*$ given by $\langle A_2 u, v\rangle = a_2(u,v)$ is strongly continuous. Therefore, (2.8), (2.9) is equivalent to the operator equation

$$Au := A_1 u + A_2 u = b.$$

As mentioned before, A is the sum of a pseudomonotone and a strongly continuous operator, and is hence pseudomonotone. A is clearly also coercive, and hence the main theorem on pseudomonotone operators applies and ensures existence of a solution to (2.8), (2.9).

More generally, we recall now the classical result of Leray and Lions (see [LeLi-65]). $\Omega \subset \mathbb{R}^n$ continues to denote a bounded open set and $X = W_0^{1,p}(\Omega)$, $1 < p < \infty$. We consider the elliptic problem

$$-\operatorname{div} \sigma(x,u,Du) = f \qquad \text{in } \Omega, \tag{2.10}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{2.11}$$

We impose the following conditions

(L1) *Carathéodory condition:* $\sigma(x,u,\xi)$ is measurable in x for all $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and continuous in (u,ξ) for almost all $x \in \Omega$.

(L2) *Growth:* $|\sigma(x,u,\xi)| \leqslant c(k(x) + |u|^{p-1} + |\xi|^{p-1})$ on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ for a constant $c > 0$ and a function $k \in L^{p'}(\Omega)$.

(L3) *Monotonicity:* For all $\xi, \xi' \in \mathbb{R}^n$ with $\xi \neq \xi'$ and all $(x,u) \in \Omega \times \mathbb{R}$

$$(\sigma(x,u,\xi) - \sigma(x,u,\xi') \cdot (\xi - \xi') > 0.$$

(L4) *Coercivity:*

$$\frac{\sigma(x,u,\xi)\cdot\xi}{|\xi|^{p-1}} \to +\infty \qquad \text{as } |\xi| \to \infty$$

uniformly for u bounded, at almost every $x \in \Omega$.

Theorem 2.7 *Let V be a closed subspace, $W_0^{1,p}(\Omega) \subset V \subset W^{1,p}(\Omega)$, such that the embedding $V \subset L^p(\Omega)$ is compact. Then, the operator $A : V \to V^*$, given by*

$$\langle Au, v\rangle = \int_\Omega \sigma(x,u,Du)\cdot Dv\,dx$$

with σ satisfying (L1)–(L4), is pseudomonotone.

Of course, in combination with Brézis' existence result for pseudomonotone operators (Theorem 2.6), solvability of the corresponding elliptic equation (2.10), (2.11) follows.

Browder has recently proved in [Bro-97] that the pseudomonotonicity property of a Leray-Lions operator follows without a coercivity assumption.

Recently, perturbations of Leray-Lions operators have attracted attention, for example, lower order terms with critical growth in the gradient. We refer to the work of Landes and Mustonen listed in the bibliography.

It is a matter of experience, that results obtained by monotone operator theory for elliptic equations usually carry over to the corresponding parabolic problems. The next chapter on elliptic problems basically tries to prove the Leray-Lions result but with relaxed monotonicity assumptions compared to (L3). In particular, we will deal with an integrated form of monotonicity (called quasimonotonicity). The aim of Chapter 4 is to carry out the analogous idea for parabolic systems.

Clearly, the given historical overview is limited to the milestones of the development of monotone operator theory and to some results which are related to the following two chapters. For more detailed descriptions of the historical context, we refer to [Pe-70], [Vai-72], [Bré-73], [Du-76] and [Ze-90].

For a good overview on recent topics and trends in monotone operators and nonlinear partial differential equations, see [Sh-97].

Chapter 3

Quasilinear elliptic systems in divergence form with weak monotonicity

Abstract: We consider the Dirichlet problem for the quasilinear elliptic system

$$\begin{aligned} -\operatorname{div} \sigma(x, u(x), Du(x)) &= f && \text{on } \Omega \\ u(x) &= 0 && \text{on } \partial\Omega \end{aligned}$$

for a function $u : \Omega \to \mathbb{R}^m$, where Ω is a bounded open domain in $\mathbb{R}^n$. For arbitrary right hand side $f \in W^{-1,p'}(\Omega)$ we prove existence of a weak solution under classical regularity, growth and coercivity conditions, but with only very mild monotonicity assumptions.

3.1 Introduction

On a bounded open domain $\Omega \subset \mathbb{R}^n$ we consider the Dirichlet problem for the quasilinear elliptic system

$$-\operatorname{div} \sigma(x, u(x), Du(x)) = f \qquad \text{on } \Omega \tag{3.1}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega \tag{3.2}$$

for a function $u : \Omega \to \mathbb{R}^m$. Here, $f \in W^{-1,p'}(\Omega) := (W_0^{1,p}(\Omega))'$ for some $p \in (1, \infty)$, and σ satisfies the conditions (E0)–(E2) below. A feature of the Young measure

Mathematics Subject Classification: 35J65, 47H15

technique we are going to use is that we can treat a class of problems for which the classical monotone operator methods developed by Višik [Vi-63], Minty [Mi-62], Browder [Bro-68], Brézis [Bré-73], Lions [Li-69] and others do not apply. The reason for this is that σ does not need to satisfy the strict monotonicity condition of a typical Leray-Lions operator (see [LeLi-65]). The tool we use in order to prove the needed compactness of approximating solutions is Young measures. The methods are inspired by [DoHuMü-97] and the proofs follow [Hu-99].

To fix some notation, let $\mathbb{M}^{m\times n}$ denote the real vector space of $m \times n$ matrices equipped with the inner product $M : N = M_{ij}N_{ij}$ (with the usual summation convention).

The following notion of monotonicity will play a rôle in part of the exposition: Instead of assuming the usual pointwise monotonicity condition for σ, we will also use a weaker, integrated version of monotonicity which is called quasimonotonicity (see [DoHuMü-97]). The definition is phrased in terms of gradient Young measures (see Section 1.4). Note, however, that although quasimonotonicity is "monotonicity in integrated form", the gradient $D\eta$ of a quasiconvex function η is not necessarily quasimonotone.

Definition 3.1 *A function $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is said to be strictly p-quasimonotone, if*

$$\int_{\mathbb{M}^{m\times n}} (\eta(\lambda) - \eta(\bar{\lambda})) : (\lambda - \bar{\lambda}) d\nu(\lambda) > 0$$

for all homogeneous $W^{1,p}$ gradient Young measures ν with center of mass $\bar{\lambda} = \langle \nu, \mathrm{id}\rangle$ which are not a single Dirac mass.

A simple example is the following: Assume that η satisfies the growth condition

$$|\eta(F)| \leqslant C\, |F|^{p-1}$$

with $p > 1$ and the structure condition

$$\int_{\Omega} (\eta(F + \nabla\varphi) - \eta(F)) : \nabla\varphi\, dx \geqslant c \int_{\Omega} |\nabla\varphi|^r dx$$

for constants $c > 0$, $r > 0$, and for all $\varphi \in C_0^\infty(\Omega)$ and all $F \in \mathbb{M}^{m\times n}$. Then η is strictly p-quasimonotone. This follows easily from the definition if one uses that for every $W^{1,p}$ gradient Young measure ν there exists a sequence $\{Dv_k\}$ generating ν for which $\{|Dv_k|^p\}$ is equiintegrable (see [FoMüPe-98], [KiPe-94]).

Now, we state our main assumptions.

(E0) (Continuity) $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. $x \mapsto \sigma(x, u, F)$ is measurable for every $(u, F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(u, F) \mapsto \sigma(x, u, F)$ is continuous for for almost every $x \in \Omega$.

(E1) (Growth and coercivity) There exist $c_1 \geqslant 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega)$, $0 < \alpha < p$ and $0 < q \leqslant n\frac{p-1}{n-p}$ such that

$$\begin{aligned} |\sigma(x,u,F)| &\leqslant \lambda_1(x) + c_1(|u|^q + |F|^{p-1}) \\ \sigma(x,u,F) : F &\geqslant -\lambda_2(x) - \lambda_3(x)|u|^\alpha + c_2|F|^p \end{aligned}$$

(E2) (Monotonicity) σ satisfies one of the following conditions:

(a) For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $F \mapsto \sigma(x,u,F)$ is a C^1-function and is monotone, i.e.

$$(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) \geqslant 0$$

for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$, and $F \mapsto W(x,u,F)$ is convex and C^1.

(c) σ is strictly monotone, i.e. σ is monotone and $(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) = 0$ implies $F = G$.

(d) $\sigma(x,u,F)$ is strictly p-quasimonotone in F.

The condition (E0) ensures that $\sigma(x,u(x),U(x))$ is measurable on Ω for measurable functions $u : \Omega \to \mathbb{R}^m$ and $U : \Omega \to \mathbb{M}^{m\times n}$. (E1) states standard growth and coercivity conditions. The main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (E2) as it is usually assumed in previous work (see, e.g., [La-80] or [LaMu-80]). For example, take a potential $W(x,u,F)$, which is only convex but not strictly convex in F, and consider the corresponding elliptic problem (3.1)–(3.2) with $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$. Even such a very simple situation cannot be treated by conventional methods: The problem is that the gradients of approximating solutions do not converge pointwise where W is not strictly convex. The idea is now, that in a point where W is not strictly convex, it is locally affine, and therefore, passage to the limit should locally still be possible. Technically, this can indeed be achieved by a suitable blow-up process, or (and this seems to be much more efficient) by considering the Young measure generated by the sequence of gradients of approximating solutions.

The assumption (d) in (E2) is motivated by the study of nonlinear elastostatics by Ball (see [Ball-76/77] and [Ball-77]): For non-hyperelastic materials the static equation is not given by a potential map. Subsequently quasimonotone systems have been studied by Zhang (see [Zh-88], [Zh-92]) and by Zhang and Chabrowski (see [ChZh-92]) who investigated the existence of solutions for perturbed systems. However, a slightly different notion of quasimonotonicity is used in the mentioned papers. The regularity problems for such systems were studied by Fuchs [Fu-87].

We prove the following result:

Theorem 3.2 *If σ satisfies the conditions (E0)–(E2), then the Dirichlet problem (3.1), (3.2) has a weak solution $u \in W_0^{1,p}(\Omega)$ for every $f \in W^{-1,p}(\Omega)$.*

3.2 Galerkin approximation

Let $V_1 \subset V_2 \subset \ldots \subset W_0^{1,p}(\Omega)$ be a sequence of finite dimensional subspaces with the property that $\cup_{i\in\mathbb{N}} V_i$ is dense in $W_0^{1,p}(\Omega)$. We define the operator

$$\begin{aligned} F : W_0^{1,p}(\Omega) &\to W^{-1,p'}(\Omega) \\ u &\mapsto \Big(w \mapsto \int_\Omega \sigma(x, u(x), Du(x)) : Dw\, dx - \langle f, w\rangle\Big), \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Observe that for arbitrary $u \in W_0^{1,p}(\Omega)$, the functional $F(u)$ is well defined by the growth condition in (E1), linear, and bounded (again by the growth condition in (E1)).

By the continuity assumption (E0) and the growth condition in (E1), it is easy to check, that the restriction of F to a finite linear subspace of $W_0^{1,p}(\Omega)$ is continuous.

Let us fix some k and assume that V_k has dimension r and that $\varphi_1, \ldots, \varphi_r$ is a basis of V_k. Then we define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r, \quad \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle F(a^i\varphi_i), \varphi_1\rangle \\ \langle F(a^i\varphi_i), \varphi_2\rangle \\ \vdots \\ \langle F(a^i\varphi_i), \varphi_r\rangle \end{pmatrix}.$$

G is continuous, since F is continuous on finite dimensional subspaces. Moreover, for $a = (a_1, \ldots, a_r)^t$ and $u = a^i\varphi_i \in V_k$, we have by the coercivity assumption in (E1) that

$$G(a) \cdot a = (F(u), u) \to \infty$$

as $\|a\|_{\mathbb{R}^r} \to \infty$. Hence, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a) \cdot a > 0$ and the usual topological argument (see, e.g., [Mi-62] or [Li-69]) gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all k there exists $u_k \in V_k$ such that

$$\langle F(u_k), v\rangle = 0 \qquad \text{for all } v \in V_k. \tag{3.3}$$

3.3 The Young measure generated by the Galerkin approximation

From the coercivity assumption in (E1) it follows that there exists $R > 0$ with the property, that $\langle F(u), u\rangle > 1$ whenever $\|u\|_{W_0^{1,p}(\Omega)} > R$. Thus, for the sequence of Galerkin approximations $u_k \in V_k$ constructed above, we have a uniform bound

$$\|u_k\|_{W_0^{1,p}(\Omega)} \leqslant R \qquad \text{for all } k. \tag{3.4}$$

Thus, we may extract a subsequence (for convenience not relabeled) such that

$$u_k \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega)$$

and such that

$$u_k \rightharpoonup u \qquad \text{in measure and in } L^s(\Omega)$$

for all $s < p^*$. The sequence of gradients Du_k generates a Young measure ν_x, and since u_k converges in measure to u, we infer from Proposition 1.3 and Proposition 1.4 that the sequence (u_k, Du_k) generates the Young measure $\delta_{u(x)} \otimes \nu_x$. Moreover, for almost all $x \in \Omega$, ν_x

(i) is a probability measure,

(ii) is a homogeneous $W^{1,p}$ gradient Young measure, and

(iii) satisfies $\langle \nu_x, \mathrm{id}\rangle = Du(x)$.

The proofs for (i)–(iii) are standard. In particular, (i) follows directly from Theorem 1.2. For (ii) and (iii) see, e.g., [DoHuMü-97].

3.4 A div-curl inequality

In this section, we prove a div-curl inequality, which will be the key ingredient to pass to the limit in the approximating equations and to prove, that the weak limit u of the Galerkin approximations u_k is indeed a solution of (3.1)–(3.2).

Let us consider the sequence

$$I_k := \big(\sigma(x, u_k, Du_k) - \sigma(x, u, Du)\big) : \big(Du_k - Du\big)$$

and prove, that its negative part I_k^- is equiintegrable: To do this, we write I_k^- in the form

$$\begin{aligned} I_k &= \sigma(x, u_k, Du_k) : Du_k - \sigma(x, u_k, Du_k) : Du \\ &\quad -\sigma(x, u, Du) : Du_k + \sigma(x, u, Du) : Du =: II_k + III_k + IV_k + V_k. \end{aligned}$$

The sequences II_k^- and V_k^- are easily seen to be equiintegrable by the coercivity condition in (E1). Then, to see equiintegrability of the sequence III_k we take a measurable subset $\Omega' \subset \Omega$ and write

$$\begin{aligned}
&\int_{\Omega'} |\sigma(x, u_k, Du_k) : Du| dx \leqslant \\
&\leqslant \ \Big(\int_{\Omega'} |\sigma(x, u_k, Du_k)|^{p'} dx\Big)^{1/p'} \Big(\int_{\Omega'} |Du|^p dx\Big)^{1/p} \\
&\leqslant \ C\Big(\int_{\Omega'} (|\lambda_1(x)^{p'} + |u_k|^{qp'} + |Du_k|^p) dx\Big)^{1/p'} \Big(\int_{\Omega'} |Du|^p dx\Big)^{1/p}.
\end{aligned}$$

The first integral is uniformly bounded in k by (3.4). The second integral is arbitrarily small if the measure of Ω' is chosen small enough. A similar argument gives the equiintegrability of the sequence IV_k.

Having established the equiintegrability of I_k^-, we may use the Fatou-Lemma 1.5 of Chapter 1 which gives that

$$X := \liminf_{k\to\infty} \int_\Omega I_k \geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : (\lambda - Du) d\nu_x(\lambda) dx. \tag{3.5}$$

On the other hand, we will now see that $X \leqslant 0$. To do this, we choose a sequence v_k such that

(i) v_k belongs to the same finite dimensional space V_k as u_k,

(ii) $v_k \to u$ in $W_0^{1,p}(\Omega)$.

This allows us in particular, to use $u_k - v_k$ as a test function in (3.3). We have

$$\begin{aligned}
X \ &= \ \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Du) dx \\
&= \ \liminf_{k\to\infty} \Big(\int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Dv_k) dx + \\
&\qquad + \int_\Omega \sigma(x, u_k, Du_k) : (Dv_k - Du) dx\Big) \\
&\leqslant \ \liminf_{k\to\infty} \Big(\Big(\int_\Omega |\sigma(x, u_k, Du_k)|^{p'} dx dt\Big)^{1/p'} \|v_k - u\|_{W^{1,p}(\Omega))} + \\
&\qquad + \langle f, u_k - v_k\rangle\Big).
\end{aligned} \tag{3.6}$$

The term

$$\int_\Omega |\sigma(x, u_k, Du_k)|^{p'} dx\Big)^{1/p'}$$

is bounded uniformly in k by the growth condition in (E1) and (3.4). The second factor

$$\|v_k - u\|_{W^{1,p}(\Omega))}$$

converges to zero for $k \to \infty$ by construction of the sequence v_k. Hence, the first term on the right of (3.6) vanishes in the limit.

The second term in (3.6)

$$\langle f, u_k - v_k \rangle$$

converges to zero, since $u_k - v_k \rightharpoonup 0$ in $L^p(W^{1,p}(\Omega))$. This proves $X \leqslant 0$.

We conclude from (3.5) the following "div-curl inequality":

Lemma 3.3 *The Young measure ν_x generated by the gradients Du_k of the Galerkin approximations u_k has the property, that*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda) dx \leqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du d\nu_x(\lambda) dx. \tag{3.7}$$

Remark: The naming ("div-curl inequality") can be explained as follows: Suppose for a moment that $\operatorname{div} \sigma(x, u_k, Du_k) = 0$ for all k and that $\sigma(x, u_k, Du_k) : Du_k$ is equiintegrable. Then, the weak limit of $\sigma(x, u_k, Du_k) : Du_k$ in $L^1(\Omega)$ is given by $\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)$. On the other hand, by the usual div-curl lemma (see [Mu-78], [Ta-79], [Ta-82]) we conclude that $\int_\Omega \sigma(x, u_k, Du_k) : Du_k dx$ converges to $\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du d\nu_x(\lambda) dx$ and hence, the lemma would follow with equality.

3.5 Passage to the limit

Now, we prove Theorem 3.2 separately in the cases (a), (b), (c) and (d) of (E2). We start with the easiest case:

Case (d): Assume that ν_x is not a Dirac mass on a set $x \in M$ of positive Lebesgue measure $|M| > 0$. Then, by the strict p-quasimonotonicity of $\sigma(x, u, \cdot)$ and the fact that ν_x is a homogeneous $W^{1,p}$ gradient Young measure for almost every $x \in \Omega$ (see Section 3.3), we have for a.e. $x \in M$

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda) > \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda) : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{= Du(x)}.$$

Hence, by integrating over Ω and using Lemma 3.3, we get

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)d\nu_x(\lambda) : Du(x)dx \geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)d\nu_x(\lambda) : Du(x)dx$$

which is a contradiction. Hence, we have $\nu_x = \delta_{Du(x)}$ for almost every $x \in \Omega$. From this, it follows by Proposition 1.3 that $Du_k \to Du$ in measure for $k \to \infty$, and thus, $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ almost everywhere (after extraction of a suitable subsequence, if necessary). Since, by the growth condition in (E1), $\sigma(x,u_k,Du_k)$ is equiintegrable, it follows that $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ in $L^1(\Omega)$ by the Vitali convergence theorem. This implies that $\langle F(u),v\rangle = 0$ for all $v \in \cup_{k\in\mathbb{N}} V_k$ and hence $F(u) = 0$, which proves the theorem in this case.

To prepare the proof in the remaining cases (a)–(c), we proceed as follows: From inequality (3.7) in Lemma 3.3, we infere, that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,Du)\big) : \big(\lambda - Du\big)d\nu_x(\lambda)dx \leqslant 0. \tag{3.8}$$

On the other hand, the integrand in (3.8) is nonnegative by monotonicity. It follows that the integrand must vanish almost everywhere with respect to the product measure $d\nu_x \otimes dx$. Hence, we have that for almost all $x \in \Omega$

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \quad \text{on } \operatorname{spt}\nu_x \tag{3.9}$$

and thus

$$\operatorname{spt}\nu_x \subset \{\lambda \mid (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0\}. \tag{3.10}$$

Now, we proceed with the proof in the single cases.

Case (c): By strict monotonicity, it follows from (3.9) that $\nu_x = \delta_{Du(x)}$ for almost all $x \in \Omega$, and hence $Du_k \to Du$ in measure, again by Proposition 1.3. The reminder of the proof in this case is exactly as in case (d).

Case (b): We start by showing that for almost all $x \in \Omega$, the support of ν_x is contained in the set where W agrees with the supporting hyper-plane $L := \{(\lambda, W(x,u,Du) + \sigma(x,u,Du)(\lambda - Du))\}$ in $Du(x)$, i.e. we want to show that

$$\operatorname{spt}\nu_x \subset K_x = \{\lambda \in \mathbb{M}^{m\times n} : W(x,u,\lambda) = W(x,u,Du) + \sigma(x,u,Du) : (\lambda - Du)\}.$$

If $\lambda \in \operatorname{spt}\nu_x$ then by (3.10)

$$(1-t)(\sigma(x,u,Du) - \sigma(x,u,\lambda)) : (Du - \lambda) = 0 \quad \text{for all } t \in [0,1]. \tag{3.11}$$

On the other hand, by monotonicity, we have for $t \in [0,1]$ that

$$0 \leqslant (1-t)(\sigma(x,u,Du+t(\lambda-Du)) - \sigma(x,u,\lambda)) : (Du-\lambda). \tag{3.12}$$

Subtracting (3.11) from (3.12), we get

$$0 \leqslant (1-t)(\sigma(x,u,Du+t(\lambda-Du)) - \sigma(x,u,Du)) : (Du-\lambda) \tag{3.13}$$

for all $t \in [0,1]$. But by monotonicity, in (3.13) also the reverse inequality holds and we may conclude, that

$$(\sigma(x,u,Du+t(\lambda-Du)) - \sigma(x,u,Du)) : (\lambda-Du) = 0 \tag{3.14}$$

for all $t \in [0,1]$, whenever $\lambda \in \operatorname{spt}\nu_x$. Now, it follows from (3.14) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,Du) + \int_0^1 \sigma(x,u,Du+t(\lambda-Du)) : (\lambda-Du)dt \\ &= W(x,u,Du) + \sigma(x,u,Du) : (\lambda-Du) \end{aligned}$$

as claimed.

By the convexity of W we have $W(x,u,\lambda) \geq W(x,u,Du) + \sigma(x,u,Du) : (\lambda-Du)$ for all $\lambda \in \mathbb{M}^{m\times n}$ and thus L is a supporting hyper-plane for all $\lambda \in K_x$. Since the mapping $\lambda \mapsto W(x,u,\lambda)$ is by assumption continuously differentiable we obtain

$$\sigma(x,u,\lambda) = \sigma(x,u,Du) \quad \text{for all } \lambda \in K_x \supset \operatorname{spt}\nu_x \tag{3.15}$$

and thus

$$\bar{\sigma} := \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)\, d\nu_x(\lambda) = \sigma(x,u,Du)\,. \tag{3.16}$$

Now consider the Carathéodory function

$$g(x,u,p) = |\sigma(x,u,p) - \bar{\sigma}(x)|\,.$$

The sequence $g_k(x) = g(x,u_k(x),Du_k(x))$ is equiintegrable and thus

$$g_k \rightharpoonup \bar{g} \quad \text{weakly in } L^1(\Omega)$$

and the weak limit $\bar{g}$ is given by

$$\begin{aligned} \bar{g}(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |\sigma(x,\eta,\lambda) - \bar{\sigma}(x)|\, d\delta_{u(x)}(\eta) \otimes d\nu_x(\lambda) \\ &= \int_{\operatorname{spt}\nu_x} |\sigma(x,u(x),\lambda) - \bar{\sigma}(x)|\, d\nu_x(\lambda) = 0 \end{aligned}$$

by (3.15) and (3.16). Since $g_k \geqslant 0$ it follows that

$$g_k \to 0 \quad \text{strongly in } L^1(\Omega).$$

This again suffices to pass to the limit in the equation and the proof of the case (b) is finished.

We remark, that (3.16) already states that $\sigma(x, u, Du)$ is the weak L^1 limit of $\sigma(x, u_k, Du_k)$, which is enough to pass to the limit in the equation. However, we wanted to point out that in this case, the convergence is even strong in L^1.

Case (a): We claim that in this case for almost all $x \in \Omega$ the following identity holds for all $\mu \in \mathbb{M}^{m\times n}$ on the support of ν_x:

$$\sigma(x, u, \lambda) : \mu = \sigma(x, u, Du) : \mu + (\nabla\sigma(x, u, Du)\mu) : (Du - \lambda), \tag{3.17}$$

where ∇ is the derivative with respect to the third variable of σ. Indeed, by the monotonicity of σ we have for all $t \in \mathbb{R}$

$$(\sigma(x, u, \lambda) - \sigma(x, u, Du + t\mu)) : (\lambda - Du - t\mu) \geq 0,$$

whence, by (3.9),

$$\begin{aligned}
&-\sigma(x, u, \lambda) : (t\mu) \geqslant \\
&\quad \geq \quad -\sigma(x, u, Du) : (\lambda - Du) + \sigma(x, u, Du + t\mu) : (\lambda - Du - t\mu) \\
&\quad = \quad t\big((\nabla\sigma(x, u, Du)\mu)(\lambda - Du) - \sigma(x, u, Du) : \mu\big) + o(t).
\end{aligned}$$

The claim follows from this inequality since the sign of t is arbitrary. Since the sequence $\sigma(x, u_k, Du_k)$ is equiintegrable, its weak L^1-limit $\bar{\sigma}$ is given by

$$\begin{aligned}
\bar{\sigma} &= \int_{\operatorname{spt}\nu_x} \sigma(x, u, \lambda) d\nu_x(\lambda) \\
&= \int_{\operatorname{spt}\nu_x} \sigma(x, u, Du) d\nu_x(\lambda) + (\nabla\sigma(x, u, Du))^t \int_{\operatorname{spt}\nu_x} (Du - \lambda) d\nu_x(\lambda) \\
&= \sigma(x, u, Du),
\end{aligned}$$

where we used (3.17) in this calculation. This finishes the proof of the case (a) and hence of the theorem.

Remark. Notice, that in case (a), we only have $\sigma(x, u_k, Du_k) \rightharpoonup \sigma(x, u, Du)$ weakly in $L^1(\Omega)$, whereas in case (b), (c) and (d) we have $\sigma(x, u_k, Du_k) \to \sigma(x, u, Du)$ in $L^1(\Omega)$. In the cases (c) and (d), we even have $Du_k \to Du$ in measure as $k \to \infty$.

Chapter 4

Quasilinear parabolic systems in divergence form with weak monotonicity

ABSTRACT: We consider the initial and boundary value problem for the quasilinear parabolic system

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u(x,t),Du(x,t)) &= f && \text{on } \Omega \times (0,T) \\
u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T) \\
u(x,0) &= u_0(x) && \text{on } \Omega
\end{aligned}
$$

for a function $u : \Omega \times [0,T) \to \mathbb{R}^m$, $T > 0$. Here, $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ for some $p \in (\frac{2n}{n+2},\infty)$, and $u_0 \in L^2(\Omega;\mathbb{R}^m)$. We prove existence of a weak solution under classical regularity, growth and coercivity conditions for σ, but with only very mild monotonicity assumptions.

4.1 Introduction

On a bounded open domain $\Omega \subset \mathbb{R}^n$ we consider the initial and boundary value problem for the quasilinear parabolic system

---

Mathematics Subject Classification: 35K55

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u(x,t),Du(x,t)) = f \qquad \text{on } \Omega \times (0,T) \tag{4.1}$$
$$u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{4.2}$$
$$u(x,0) = u_0(x) \qquad \text{on } \Omega \tag{4.3}$$

for a function $u : \Omega \times [0,T) \to \mathbb{R}^m$, $T > 0$. Here, $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ for some $p \in (\frac{2n}{n+2},\infty)$, $u_0 \in L^2(\Omega;\mathbb{R}^m)$, and σ satisfies the conditions (P0)–(P2) below.

As in the previous chapter on the corresponding elliptic problem, existence of a weak solution follows by standard methods in the classical theory of monotone operators (see [Vi-62], [Mi-62], [Bro-68], [Bré-73], [Li-69]) if one is willing to impose strict monotonicity of $\sigma(x,t,u,F)$ in F or monotonicity in (u,F). However, we will only assume weaker monotonicity properties for σ (see below). The tool we use in order to prove the needed compactness of approximating solutions is (as in the previous chapter) Young measures.

As before, we denote by $\mathbb{M}^{m\times n}$ the real vector space of $m \times n$ matrices equipped with the inner product $M : N = M_{ij}N_{ij}$ (with the usual summation convention).

Now, we state our main assumptions.

(P0) (Continuity) $\sigma : \Omega\times(0,T)\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. $(x,t) \mapsto \sigma(x,t,u,F)$ is measurable for every $(u,F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(u,F) \mapsto \sigma(x,t,u,F)$ is continuous for almost every $(x,t) \in \Omega \times (0,T)$.

(P1) (Growth and coercivity) There exist $c_1 \geqslant 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega \times (0,T))$, $\lambda_2 \in L^1(\Omega \times (0,T))$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega \times (0,T))$, $0 < \alpha < p$, such that

$$\begin{aligned} |\sigma(x,t,u,F)| &\leqslant \lambda_1(x,t) + c_1(|u|^{p-1} + |F|^{p-1}) \\ \sigma(x,t,u,F) : F &\geqslant -\lambda_2(x,t) - \lambda_3(x,t)|u|^\alpha + c_2|F|^p \end{aligned}$$

(P2) (Monotonicity) σ satisfies one of the following conditions:

(a) For all $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^m$, the map $F \mapsto \sigma(x,t,u,F)$ is a C^1-function and is monotone, i.e.

$$(\sigma(x,t,u,F) - \sigma(x,t,u,G)) : (F - G) \geqslant 0$$

for all $(x,t) \in \Omega \times (0,T)$, $u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times (0,T) \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,t,u,F) = \frac{\partial W}{\partial F}(x,t,u,F)$, and $F \mapsto W(x,t,u,F)$ is convex and C^1 for all $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^m$.

(c) σ is strictly monotone, i.e. σ is monotone and $(\sigma(x,t,u,F)-\sigma(x,t,u,G)) : (F-G) = 0$ implies $F = G$.

(d) $\sigma(x,t,u,F)$ is strictly p-quasimonotone in F.

The Carathéodory condition (P0) ensures that $\sigma(x,t,u(x,t),U(x,t))$ is measurable on $\Omega \times (0,T)$ for measurable functions $u : \Omega \times (0,T) \to \mathbb{R}^m$ and $U : \Omega \times (0,T) \to \mathbb{M}^{m\times n}$ (see, e.g., [Ze-90]). (P1) states standard growth and coercivity conditions: They are used in the construction of approximate solutions by a Galerkin method and when we pass to the limit. The strict monotonicity condition (c) in (P2) ensures existence of weak solutions of the corresponding parabolic systems by standard methods. However, the main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (a) (b) or (d) as it is usually assumed in previous work (see, e.g., [BoMu-92], [BréBro-79], [La-85], [LaMu-87], [La-90], [LaMu-94] and the references therein).

We prove the following result:

Theorem 4.1 *If σ satisfies the conditions (P0)–(P2) for some $p \in (\frac{2n}{n+2}, \infty)$, then the parabolic system (4.1)–(4.3) has a weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $f \in L^{p'}(0,T;\, W^{-1,p}(\Omega))$ and every $u_0 \in L^2(\Omega)$.*

Remark: The result for the case (d) in (P2) answers in particular a question by Frehse [Fr-99].

4.2 Choice of the Galerkin base

Let $s \geqslant 1 + n(\frac{1}{2} - \frac{1}{p})$. Then, $W_0^{s,2}(\Omega) \subset W_0^{1,p}(\Omega)$. For $\zeta \in L^2(\Omega)$ we consider the linear bounded map

$$\varphi : W_0^{s,2}(\Omega) \to \mathbb{R}, \quad v \mapsto (\zeta, v)_{L^2}$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. By the Riesz Representation Theorem there exists a unique $K\zeta \in W_0^{s,2}(\Omega)$ such that

$$\varphi(v) = (\zeta, v)_{L^2} = (K\zeta, v)_{W^{s,2}} \qquad \text{for all } v \in W_0^{s,2}(\Omega).$$

The map $L^2 \to L^2$, $\zeta \mapsto K\zeta$, is linear, symmetric, bounded and (due to the compact embedding $W_0^{s,2}(\Omega) \subset L^2(\Omega)$) compact. Moreover, since

$$(\zeta, K\zeta)_{L^2} = (K\zeta, K\zeta)_{W^{s,2}} \geqslant 0$$

the operator K is (strictly) positive. Hence there exists an L^2-orthonormal base $W := \{w_1, w_2, \ldots\}$ of eigenvectors of K and positive real eigenvalues λ_i with $Kw_i =$

$\lambda_i w_i$. This in particular means that $w_i \in W_0^{s,2}(\Omega)$ for all i and that for all $v \in W_0^{s,2}(\Omega)$

$$\lambda_i(w_i, v)_{W^{s,2}} = (Kw_i, v)_{W^{s,2}} = (w_i, v)_{L^2}. \tag{4.4}$$

Notice that therefore the functions w_i are orthogonal also with respect to the inner product of $W^{s,2}(\Omega)$: In fact, for $i \neq j$, we get by choosing $v = w_j$ in (4.4)

$$0 = \frac{1}{\lambda_i}(w_i, w_j)_{L^2} = (w_i, w_j)_{W^{s,2}}.$$

Notice also that, by choosing $v = w_i$ in (4.4),

$$1 = \|w_i\|_{L^2}^2 = (w_i, w_i)_{L^2} = \lambda_i(w_i, w_i)_{W^{s,2}} = \lambda_i\|w_i\|_{W^{s,2}}^2.$$

Thus, $\widetilde{W} = \{\tilde{w}_1, \tilde{w}_2, \ldots\}$, with $\tilde{w}_i := \sqrt{\lambda_i}\, w_i$, is an orthonormal set for $W_0^{s,2}(\Omega)$. Actually, $\widetilde{W}$ is a basis for $W_0^{s,2}(\Omega)$. To see this, observe that for arbitrary $v \in W_0^{s,2}(\Omega)$, the Fourier series

$$s_n(v) := \sum_{i=1}^{n}(\tilde{w}_i, v)_{W^{s,2}}\tilde{w}_i \to \tilde{v} \qquad \text{in } W_0^{s,2}(\Omega)$$

converges to some $\tilde{v}$. On the other hand, we have

$$s_n(v) = \sum_{i=1}^{n}(w_i, v)_{L^2} w_i \to v \qquad \text{in } L^2(\Omega)$$

and by the uniqueness of the limit, $\tilde{v} = v$.

We will need below the L^2-orthonormal projector $P_k : L^2 \to L^2$ onto span$(w_1, w_2, \ldots, w_k)$, $k \in \mathbb{N}$. Of course, the operator norm $\|P_k\|_{\mathscr{L}(L^2,L^2)} = 1$. But notice that also $\|P_k\|_{\mathscr{L}(W^{s,2},W^{s,2})} = 1$ since for $u \in W^{s,2}(\Omega)$

$$P_k u = \sum_{i=1}^{k}(w_i, u)_{L^2} w_i = \sum_{i=1}^{k}(\tilde{w}_i, u)_{W^{s,2}}\tilde{w}_i.$$

4.3 Galerkin approximation

We make the following ansatz for approximating solutions of (4.1)–(4.3):

$$u_k(x,t) = \sum_{i=1}^{k} c_{ki}(t) w_i(x),$$

where $c_{ki} : [0,T) \to \mathbb{R}$ are supposed to be measurable bounded functions. Each u_k satisfies the boundary condition (4.2) by construction in the sense that $u_k \in$

$L^p(0,T;W_0^{1,p}(\Omega))$. We take care of the initial condition (4.3) by choosing the initial coefficients $c_{ki}(0) := (u_0, w_i)_{L^2}$ such that

$$u_k(\cdot,0) = \sum_{i=1}^{k} c_{ki}(0) w_i(\cdot) \to u_0 \quad \text{in } L^2(\Omega) \text{ as } k \to \infty. \tag{4.5}$$

We try to determine the coefficients $c_{ik}(t)$ in such a way, that for all $k \in \mathbb{N}$ the system of ordinary differential equations

$$(\partial_t u_k, w_j)_{L^2} + \int_\Omega \sigma(x,t,u_k,Du_k) : Dw_j dx = \langle f(t), w_j \rangle \tag{4.6}$$

(with $j \in \{1,2,\ldots,k\}$) is satisfied in the sense of distributions. In (4.6), $\langle \cdot,\cdot \rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Now, we fix $k \in \mathbb{N}$ for the moment. Let $0 < \varepsilon < T$ and $J = [0,\varepsilon]$. Moreover we choose $r > 0$ large enough, such that the set $B_r(0) \subset \mathbb{R}^k$ contains the vector $(c_{1k}(0),\ldots,c_{kk}(0))$, and we set $K = \overline{B_r(0)}$. Observe that by (P0), the function

$$\begin{aligned} F : J \times K &\to \mathbb{R}^k \\ (t,c_1,\ldots,c_k) &\mapsto \Big(\langle f(t), w_j \rangle \\ &\qquad - \int_\Omega \sigma(x,t,\sum_{i=1}^k c_i w_i, \sum_{i=1}^k c_i Dw_i) : Dw_j dx \Big)_{j=1,\ldots,k} \end{aligned}$$

is a Carathéodory function. Moreover, each component F_j may be estimated on $J \times K$ by

$$\begin{aligned} |F_j(t,c_1,\ldots,c_k)| &\leqslant \|f(t)\|_{W^{-1,p'}} \|w_j\|_{W_0^{1,p}} + \\ &+ \Big(\int_\Omega |\sigma(x,t,\sum_{i=1}^k c_i w_i, \sum_{i=1}^k c_i Dw_i)|^{p'} dx \Big)^{1/p'} \Big(\int_\Omega |Dw_j|^p dx \Big)^{1/p}. \end{aligned} \tag{4.7}$$

Using the growth condition in (P1), the right hand side of (4.7) can be estimated in such a way that

$$|F_j(t,c_1,\ldots,c_k)| \leqslant C(r,k) M(t) \tag{4.8}$$

uniformly on $J \times K$, where $C(r,k)$ is a constant which depends on r and k, and where $M(t) \in L^1(J)$ (independent of j, k and r). Thus, the Carathéodory existence result on ordinary differential equations (see, e.g., [Ka-60]) applied to the system

$$c_j'(t) = F_j(t,c_1(t),\ldots,c_k(t)) \tag{4.9}$$

$$c_j(0) = c_{kj}(0) \tag{4.10}$$

(for $j \in \{1,\ldots,k\}$) ensures existence of a distributional, continuous solution c_j (depending on k) of (4.9)–(4.10) on a time interval $[0,\varepsilon')$, where $\varepsilon' > 0$, a priori, may depend on k. Moreover, the corresponding integral equation

$$c_j(t) = c_j(0) + \int_0^t F_j(\tau, c_1(\tau), \ldots, c_k(\tau))d\tau \tag{4.11}$$

holds on $[0, \varepsilon')$. Then, $u_k := \sum_{j=1}^k c_j(t)w_j$ is the desired (short time) solution of (4.6) with initial condition (4.5).

Now, we want to show, that the local solution constructed above can be extended to the whole interval $[0, T)$ independent of k. As a word of warning we should mention, that the solution need not be unique.

The first thing we want to establish is a uniform bound on the coefficients $|c_{ki}(t)|$: Since (4.6) is linear in w_j, it is allowed to use u_k as a test function in equation (4.6) in place of w_j. This gives for an an arbitrary time τ in the existence interval

$$\underbrace{\int_0^\tau (\partial_t u_k, u_k)_{L^2} dt}_{=:I} + \underbrace{\int_0^\tau \int_\Omega \sigma(x, t, u_k, Du_k) : Du_k dxdt}_{=:II} = \underbrace{\int_0^\tau \langle f(t), u_k \rangle}_{=:III}.$$

For the first term we have

$$I = \tfrac{1}{2}\|u_k(\cdot, \tau)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u_k(\cdot, 0)\|^2_{L^2(\Omega)}.$$

Using the coercivity in (P1) for the second term, we obtain

$$\begin{aligned} II \geqslant -\|\lambda_2\|_{L^1(\Omega\times(0,T))} - \|\lambda_3\|_{L^{(p/\alpha)'}(\Omega\times(0,T))}\|u_k\|^\alpha_{L^p(\Omega\times(0,\tau))} + \\ + c_2\|u_k\|^p_{L^p(0,\tau;W^{1,p}_0(\Omega))}. \end{aligned}$$

For the third term, we finally get

$$III \leqslant \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\|u_k\|_{L^p(0,\tau;W^{1,p}_0(\Omega))}.$$

The combination of these three estimates gives

$$|(c_{ki}(\tau))_{i=1,\ldots,k}|^2_{\mathbb{R}^k} = \|u_k(\cdot, \tau)\|^2_{L^2(\Omega)} \leqslant \bar{C}$$

for a constant $\bar{C}$ which is independent of τ (and of k).

Now, let

$$\Lambda := \{t \in [0, T) : \text{there exists a weak solution of (4.9)–(4.10) on } [0, t)\}.$$

Λ is non-empty since we proved local existence above.

Moreover Λ is an open set: To see this, let $t \in \Lambda$ and $0 < \tau_1 < \tau_2 \leqslant t$. Then, by (4.11) and (4.8), we have

$$\begin{aligned} |c_{kj}(\tau_1) - c_{kj}(\tau_2)| &\leqslant \int_{\tau_1}^{\tau_2} |F_j(\tau, c_{k1}(\tau), \dots, c_{kk}(\tau))| d\tau \\ &\leqslant C(\bar{C}, k) \int_{\tau_1}^{\tau_2} |M(t)| d\tau. \end{aligned}$$

Since $M \in L^1(0,T)$, this implies that $\tau \mapsto c_{kj}(\tau)$ is uniformly continuous. Thus, we can restart to solve (4.6) at time t with initial data $\lim_{\tau \nearrow t} u_k(\tau)$ and hence get a solution of (4.9)–(4.10) on $[0, t+\varepsilon)$.

Finally, we prove that Λ is also closed. To see this, we consider a sequence $\tau_i \nearrow t$, $\tau_i \in \Lambda$. Let $c_{kj,i}$ denote the solution of (4.9)–(4.10) we constructed on $[0, \tau_i]$ and define

$$\tilde{c}_{kj,i}(\tau) := \begin{cases} c_{kj,i}(\tau) & \text{if } \tau \in [0, \tau_i] \\ c_{kj,i}(\tau_i) & \text{if } \tau \in (\tau_i, t). \end{cases}$$

The sequence $\{c_{kj,i}\}_i$ is bounded and equicontinuous on $[0,t)$, as seen above. Hence, by the Arzela-Ascoli Theorem, a subsequence (again denoted by $\tilde{c}_{kj,i}(\tau)$) converges uniformly in τ on $[0,t)$ to a continuous function $c_{kj}(\tau)$. Using Lebesgue's convergence theorem in (4.11) it is now easy to see that $c_{kj}(\tau)$ solves (4.9) on $[0,t)$. Hence $t \in \Lambda$ and thus Λ is indeed closed. And as claimed, it follows that $\Lambda = [0,T)$.

4.4 Compactness of the Galerkin approximation

By testing equation (4.6) by u_k in place of w_j we obtain, as above in Section 4.3, that the sequence $\{u_k\}_k$ is bounded in

$$L^\infty(0,T; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)).$$

Therefore, by extracting a suitable subsequence which is again denoted by u_k, we may assume

$$\begin{aligned} u_k &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T; L^2(\Omega)) \\ u_k &\rightharpoonup u \quad \text{in } L^p(0,T; W_0^{1,p}(\Omega)) \end{aligned}$$

At this point, the idea is to use Aubin's Lemma in order to prove compactness of the sequence $\{u_k\}$ in an appropriate space. Technically this is achieved by the following Lemma which is slightly more flexible than e.g. the version in [Li-69, Chap. 1, Sect. 5.2] or in [Si-87].

Lemma 4.2 *Let B, B_0 and B_1 be Banach spaces, B_0 and B_1 reflexive. Let $i : B_0 \to B$ be a compact linear map and $j : B \to B_1$ an injective bounded linear operator. For T finite and $1 < p_i < \infty$, $i = 0, 1$,*

$$W := \{v \mid v \in L^{p_0}(0,T;B_0),\ \frac{d}{dt}(j \circ i \circ v) \in L^{p_1}(0,T;B_1)\}$$

is a Banach space under the norm $\|v\|_{L^{p_0}(0,T;B_0)} + \|j \circ i \circ v\|_{L^{p_1}(0,T;B_1)}$. Then, if $V \subset W$ is bounded, the set $\{i \circ v \mid v \in V\}$ is precompact in $L^{p_0}(0,T;B)$.

The proof of Lemma 4.2 is given in Appendix I.

Now, we apply Lemma 4.2 to the following case: $B_0 := W_0^{1,p}(\Omega)$, $B := L^q(\Omega)$ (for some q with $2 < q < p^* := \frac{np}{n-p}$ if $p < n$ and $2 < p < \infty$ if $p \geqslant n$) and $B_1 := (W_0^{s,2}(\Omega))'$. Since we assumed that $p \in (\frac{2n}{n+2}, \infty)$, we have the following chain of continuous injections:

$$B_0 \overset{i}{\hookrightarrow} B \overset{i_0}{\hookrightarrow} L^2(\Omega) \overset{\gamma}{\cong} (L^2(\Omega))' \overset{i_1}{\hookrightarrow} B_1. \tag{4.12}$$

Here, $L^2(\Omega) \cong (L^2(\Omega))'$ is the canonical isomorphism γ of the Hilbert space $L^2(\Omega)$ and its dual. For $i : B_0 \to B$ we take simply the injection mapping, and for $j : B \to B_1$ we take the concatenation of injections and the canonical isomorphism given by (4.12), i.e. $j := i_1 \circ \gamma \circ i_0$.

Then, as stated at the beginning of this section, $\{u_k\}_k$ is a bounded sequence in $L^p(0,T;B_0)$. Observe that the time derivative $\frac{d}{dt}(j \circ i \circ u_k)$ is according to (4.6) given by

$$\begin{aligned} \frac{d}{dt}(j \circ i \circ u_k) : [0,T) &\to B_1 = (W_0^{s,2}(\Omega))' \\ t &\mapsto \Big(\varphi \mapsto -\int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\varphi)dx + \\ &\qquad + \langle f(t), P_k\varphi\rangle\Big). \end{aligned}$$

(We recall that the projection operators P_k are selfadjoint with respect to the L^2 inner product.) Now we claim that indeed $\{\partial_t j \circ i \circ u_k\}_k$ is a bounded sequence in $L^{p'}(0,T;(W_0^{s,2}(\Omega))')$: Namely, we have by the growth condition in (P1) that

$$\begin{aligned} &| -\int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : D(P_k\varphi)dxdt + \langle f, P_k\varphi\rangle| \leqslant \\ &\qquad \leqslant C(\|\lambda_1\|_{L^{p'}((0,T)\times\Omega)} + \|u_k\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))} + \\ &\qquad\qquad + \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))})\|P_k\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \end{aligned} \tag{4.13}$$

and the claim follows since

$$\|P_k\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant \|P_k\varphi\|_{L^p(0,T;W_0^{s,2}(\Omega))} \leqslant \|\varphi\|_{L^p(0,T;W_0^{s,2}(\Omega))}.$$

In the last inequality we used the remark at the end of Section 4.2.

Hence, from Lemma 4.2, we may conclude that there exists a subsequence, which we still denote by u_k, having the property that

$$u_k \to u \quad \text{in } L^p(0,T;L^q(\Omega)) \text{ for all } q < p^* \text{ and in measure on } \Omega \times (0,T).$$

Notice that in order to have the strong convergence simultaneously for all $q < p^*$, the usual diagonal sequence procedure applies.

For further use, we note that from (4.13) we can conclude that $\partial_t u$ (or rather $\partial_t(j \circ i \circ u)$) is an element of the space $L^{p'}(0,T;W^{-1,p'}(\Omega))$. (This follows easily from the fact, that the set $\{\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) : \exists k \in \mathbb{N} \text{ such that } P_k\varphi = \varphi\}$ is dense in $L^p(0,T;W_0^{1,p}(\Omega))$, as proved in Appendix II.) See also Appendix III.

Recall that the space

$$\{u \in L^p(0,T;W_0^{1,p}(\Omega)) : \partial_t(j \circ i \circ u) \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$$

is continuously embedded in

$$C^0([0,T];L^2(\Omega)).$$

Hence, we have that $u \in C^0([0,T];L^2(\Omega))$ after possible modification of u on a Lebesgue zero-set of $[0,T]$. This gives $u(t,\cdot) \in L^2(\Omega)$ a pointwise interpretation for all $t \in [0,T]$ and allows in particular to state that $u(t,\cdot)$ attains its initial value

$$u(\cdot,0) = u_0 \tag{4.14}$$

continuously in $L^2(\Omega)$ (see Appendix III for a proof of (4.14)).

At this point, we would like to mention, that in the case when σ depends only on t and in a strictly quasimonotonic way on Du, a quite simple proof gives the existence result. This is carried out in Appendix IV. However, to obtain the general result stated in Theorem 4.1, some more work is needed in order to pass to the limit.

4.5 The Young measure generated by the Galerkin approximation

The sequence (or at least a subsequence) of the gradients Du_k generates a Young measure $\nu_{(x,t)}$, and since u_k converges in measure to u on $\Omega \times (0,T)$, the sequence (u_k, Du_k) generates the Young measure $\delta_{u(x,t)} \otimes \nu_{(x,t)}$ (see Proposition 1.3 and 1.4). Now, we collect some facts about the Young measure ν in the following proposition:

Proposition 4.3 *The Young measure $\nu_{(x,t)}$ generated by the sequence $\{Du_k\}_k$ has the following properties:*

(i) $\nu_{(x,t)}$ *is a probability measure on* $\mathbb{M}^{m\times n}$ *for almost all* $(x,t)\in\Omega\times(0,T)$.

(ii) $\nu_{(x,t)}$ *satisfies* $Du(x,t)=\langle\nu_{(x,t)},\mathrm{id}\rangle$ *for almost every* $(x,t)\in\Omega\times(0,T)$.

(iii) $\nu_{(x,t)}$ *has finite* p*-th moment for almost all* $(x,t)\in\Omega\times(0,T)$.

(iv) $\nu_{(x,t)}$ *is a homogeneous* $W^{1,p}$ *gradient Young measure for almost all* $(x,t)\in\Omega\times(0,T)$.

Proof
(i) The first observation is simple: To see that $\nu_{(x,t)}$ is a probability measure on $\mathbb{M}^{m\times n}$ for almost all $(x,t)\in\Omega\times(0,T)$ it suffices to recall the fact that Du_k is a bounded sequence in $L^1(\Omega\times(0,T))$ and to use Theorem 1.2.

(ii) As we have stated at the beginning of Section 4.4 $\{Du_k\}_k$ is bounded in $L^p(0,T;L^p(\Omega))$ and we may assume that

$$Du_k\rightharpoonup Du\qquad\text{in } L^p(0,T;L^p(\Omega)).$$

On the other hand it follows that the sequence $\{Du_k\}_k$ is equiintegrable on $\Omega\times(0,T)$ and hence, by the Dunford-Pettis Theorem (see, e.g., [DuSc-88]), the sequence is sequentially weakly precompact in $L^1(\Omega\times(0,T))$ which implies that

$$Du_k\rightharpoonup\langle\nu_{(x,t)},\mathrm{id}\rangle\qquad\text{in } L^1(0,T;L^1(\Omega)).$$

Hence, we have $Du(x,t)=\langle\nu_{(x,t)},\mathrm{id}\rangle$ for almost every $(x,t)\in\Omega\times(0,T)$.

(iii) The next thing we have to check is, that $\nu_{(x,t)}$ has finite p-th moment for almost all $(x,t)\in\Omega\times(0,T)$. To see this, we choose a cut-off function $\eta\in C_0^\infty(B_{2\alpha}(0);\mathbb{R}^m)$ with $\eta=\mathrm{id}$ on $B_\alpha(0)$ for some $\alpha>0$. Then, the sequence

$$D(\eta\circ u_k)=(D\eta)(u_k)Du_k$$

generates a probability Young measure $\nu^\eta_{(x,t)}$ on $\Omega\times(0,T)$ with finite p-th moment, i.e.

$$\int_{\mathbb{M}^{m\times n}}|\lambda|^p d\nu^\eta_{(x,t)}(\lambda)<\infty$$

for almost all $(x,t)\in\Omega\times(0,T)$. Now, for $\varphi\in C_0^\infty(\mathbb{M}^{m\times n})$ we have

$$\varphi(D(\eta\circ u_k))\rightharpoonup\langle\nu^\eta_{(x,t)},\varphi\rangle=\int_{\mathbb{M}^{m\times n}}\varphi(\lambda)d\nu^\eta_{(x,t)}(\lambda)$$

weakly in $L^1(\Omega \times (0,T))$. Rewriting the left hand side, we have (by Proposition 1.4) also

$$\varphi((D\eta)(u_k)Du_k) \rightharpoonup \int_{\mathbb{M}^{m\times n}} \varphi(D\eta(u(x,t))\lambda)d\nu_{(x,t)}(\lambda).$$

Hence,

$$\nu^\eta_{(x,t)} = \nu_{(x,t)} \qquad \text{if } |u(x,t)| < \alpha.$$

Since α was arbitrary, it follows that indeed $\nu_{(x,t)}$ has finite p-th moment for almost all $(x,t) \in \Omega \times (0,T)$.

(iv) Finally, we have to show, that $\{\nu_{(x,t)}\}_{x\in\Omega}$ is for almost all $t \in (0,T)$ a $W^{1,p}$ gradient Young measure. To see this, we take a quasiconvex function q on $\mathbb{M}^{m\times n}$ with $q(F)/|F| \to 1$ as $F \to \infty$. Then, we fix $x \in \Omega$, $\delta \in (0,1)$ and use inequality (1.21) from [Kr-99, Lemma 1.6] with u replaced by $u_k(x,t)$, with $a := u(x,t) - Du(x,t)x$ and with $X := Du(x,t)$. Furthermore, we choose $r > 0$ such that $B_r(x) \subset \Omega$. Observe, that the singular part of the distributional gradient vanishes for u_k and, after integrating the inequality over the time interval $[t_0 - \varepsilon, t_0 + \varepsilon] \subset (0,T)$, we get

$$\begin{aligned}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)} q(Du_k(y,t))dydt+ \\ + \frac{1}{(1-\delta)r}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)\setminus B_{\delta r}(x)} |u_k(y,t) - u(x,t) - Du(x,t)(y-x)|dydt \geqslant \\ \geqslant |B_{\delta r}(x)| \int_{t_0-\varepsilon}^{t_0+\varepsilon} q(Du(x,t))dt.\end{aligned}$$

Letting k tend to infinity in the inequality above, we obtain

$$\begin{aligned}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)}\int_{\mathbb{M}^{m\times n}} q(\lambda)d\nu_{(y,t)}(\lambda)dydt+ \\ + \frac{1}{(1-\delta)r}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_r(x)\setminus B_{\delta r}(x)} |u(y,t) - u(x,t) + Du(x,t)(y-x)|dydt \geqslant \\ \geqslant |B_{\delta r}(x)| \int_{t_0-\varepsilon}^{t_0+\varepsilon} q(Du(x,t))dt.\end{aligned}$$

Now, we let $\varepsilon \to 0$ and $r \to 0$ and use the differentiability properties of Sobolev functions (see, e.g., [EvGa-92]) and obtain, that for almost all $(x,t_0) \in \Omega \times (0,T)$

$$\int_{\mathbb{M}^{m\times n}} q(\lambda)d\nu_{(x,t_0)}(\lambda) \geqslant \frac{|B_{\delta r}(x)|}{|B_r(x)|}\, q(Du(x,t_0)).$$

Since $\delta \in (0,1)$ was arbitrary, we conclude that Jensen's inequality holds true for q and the measure $\nu_{(x,t)}$ for almost all $(x,t) \in \Omega \times (0,T)$. Using the characterization of $W^{1,p}$ gradient Young measures of [KiPe-94] (e.g., in the form of [Kr-99, Theorem

8.1]), we conclude that in fact $\{\nu_{x,t}\}_{x\in\Omega}$ is a $W^{1,p}$ gradient Young measure on Ω for almost all $t \in (0,T)$. By the localization principle for gradient Young measures, we conclude then, that $\nu_{(x,t)}$ is a homogeneous $W^{1,p}$ gradient Young measure for almost all $(x,t) \in \Omega \times (0,T)$. $\square$

4.6 A parabolic div-curl inequality

In this section, we prove a parabolic version of the div-curl Lemma 3.3, which will be the key ingredient to pass to the limit in the approximating equations and to prove, that the weak limit u of the Galerkin approximations u_k is indeed a solution of (4.1)–(4.3).

Let us consider the sequence

$$I_k := \big(\sigma(x,t,u_k,Du_k) - \sigma(x,t,u,Du)\big) : \big(Du_k - Du\big)$$

and prove, that its negative part I_k^- is equiintegrable on $\Omega \times (0,T)$: To do this, we write I_k^- in the form

$$\begin{aligned} I_k &= \sigma(x,t,u_k,Du_k) : Du_k - \sigma(x,t,u_k,Du_k) : Du \\ &\quad -\sigma(x,t,u,Du) : Du_k + \sigma(x,t,u,Du) : Du =: II_k + III_k + IV_k + V_k. \end{aligned}$$

The sequences II_k^- and V_k^- are easily seen to be equiintegrable by the coercivity condition in (P1). Then, to see equiintegrability of the sequence III_k, we take a measurable subset $S \subset \Omega \times (0,T)$ and write

$$\begin{aligned} &\int_S |\sigma(x,t,u_k,Du_k) : Du|dxdt \leqslant \\ &\leqslant \Big(\int_S |\sigma(x,t,u_k,Du_k)|^{p'}dxdt\Big)^{1/p'}\Big(\int_S |Du|^p dxdt\Big)^{1/p} \\ &\leqslant C\Big(\int_S (|\lambda_1(x,t)^{p'} + |u_k|^p + |Du_k|^p)dxdt\Big)^{1/p'}\Big(\int_S |Du|^p dxdt\Big)^{1/p}. \end{aligned}$$

The first integral is uniformly bounded in k (see Section 4.4). The second integral is arbitrarily small if the measure of S is chosen small enough. A similar argument gives the equiintegrability of the sequence IV_k.

Having established the equiintegrability of I_k^-, we may use, as in the previous section, the Fatou-Lemma 1.5 which gives that

$$\begin{aligned} X &:= \liminf_{k\to\infty} \int_0^T \int_\Omega I_k dxdt \geqslant \\ &\geqslant \int_0^T \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - Du)d\nu_{(x,t)}(\lambda)dxdt. \end{aligned} \tag{4.15}$$

On the other hand, we will now see that $X \leqslant 0$: According to Mazur's Theorem (see, e.g., [Yo, Theorem 2, page 120]), there exists a sequence v_k in $L^p(0,T;W_0^{1,p}(\Omega))$ where each v_k is a convex linear combination of $\{u_1,\dots,u_k\}$ such that $v_k \to u$ in $L^p(0,T;W_0^{1,p}(\Omega))$. In particular, $v_k(t,\cdot) \in \operatorname{span}(w_1,w_2,\dots,w_k)$ for all $t \in [0,T]$.

Now, we have

$$
\begin{aligned}
X &= \liminf_{k\to\infty} \int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : (Du_k - Du)\,dxdt \\
&= \liminf_{k\to\infty} \Big(\int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : (Du_k - Dv_k)\,dxdt + \\
&\quad + \int_0^T \int_\Omega \sigma(x,t,u_k,Du_k) : (Dv_k - Du)\,dxdt \Big) \\
&\leqslant \liminf_{k\to\infty} \Big(\big(\int_0^T \int_\Omega |\sigma(x,t,u_k,Du_k)|^{p'} dxdt \big)^{1/p'} \|v_k - u\|_{L^p(0,T;W^{1,p}(\Omega))} + \\
&\quad + \langle f, u_k - v_k\rangle - \int_0^T \int_\Omega (u_k - v_k)\partial_t u_k\,dxdt \Big). \qquad (4.16)
\end{aligned}
$$

Observe that $u_k - v_k \in \operatorname{span}(w_1,w_2,\dots,w_k)$ which allowed to use (4.6) in the inequality above. The first factor in the first term in (4.16)

$$
\int_0^T \int_\Omega |\sigma(x,t,u_k,Du_k)|^{p'} dxdt \big)^{1/p'}
$$

is uniformly bounded in k by the growth condition in (P1) and the bound for u_k in $L^p(0,T;W^{1,p}(\Omega))$ (see Section 4.4). The second factor

$$
\|v_k - u\|_{L^p(0,T;W^{1,p}(\Omega))}
$$

converges to zero for $k \to \infty$ by construction of the sequence v_k. Hence, the first term in (4.16) vanishes in the limit.

The second term in (4.16)

$$
\langle f, u_k - v_k\rangle
$$

converges to zero, since $u_k - v_k \rightharpoonup 0$ in $L^p(0,T;W^{1,p}(\Omega))$.

Finally, for the last term in (4.16) , we have

$$
\begin{aligned}
&-\int_0^T \int_\Omega (u_k - v_k)\partial_t u_k\,dxdt = \\
&\qquad = -\int_0^T \int_\Omega \tfrac{1}{2}\partial_t u_k^2\,dxdt + \int_0^T \int_\Omega v_k \partial_t u_k\,dxdt \\
&\qquad = -\tfrac{1}{2}\|u_k(\cdot,T)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|u_k(\cdot,0)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega v_k \partial_t u_k\,dxdt. \qquad (4.17)
\end{aligned}
$$

Concerning the last term in (4.17) we claim that for $k \to \infty$ we have

$$\int_0^T \int_\Omega v_k \partial_t u_k dxdt \to \int_0^T \int_\Omega u \partial_t u dxdt = \tfrac{1}{2}\|u(\cdot,T)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{4.18}$$

To see this, let $\varepsilon > 0$ be given. Then, there exists M such that for all $l \geqslant m \geqslant M$ we have

(i) $|\int_0^T \int_\Omega (u - v_m)\partial_t u dxdt| \leqslant \varepsilon$. This is possible, since $\partial_t(j \circ i \circ u) \in L^{p'}(0,T; W^{-1,p'}(\Omega))$ and $v_m \to u$ in $L^p(0,T; W_0^{1,p}(\Omega))$.

(ii) $|\int_0^T \int_\Omega (v_l - v_m)\partial_t u_l dxdt| \leqslant \varepsilon$. This is possible by (4.13) since $v_l - v_m \in \operatorname{span}(w_1, \dots, w_l)$ for all fixed $t \in (0,T)$.

Now, we fix $m \geqslant M$ and choose $m_0 \geqslant m$ such that for all $l \geqslant m_0$

$$|\int_0^T \int_\Omega v_m(\partial_t u - \partial_t u_l) dxdt| \leqslant \varepsilon.$$

This is possible, since $\partial_t u_l \overset{*}{\rightharpoonup} \partial_t u$ in $L^{p'}(0,T; (W_0^{s,2}(\Omega))')$. Combination yields for all $l = l(\varepsilon) \geqslant m_0(\varepsilon)$

$$\begin{aligned} |\int_0^T \int_\Omega v_l \partial_t u_l dxdt - \int_0^T \int_\Omega u \partial_t u dxdt| \leqslant \\ |\int_0^T \int_\Omega (v_l - v_m)\partial_t u_l dxdt| + |\int_0^T \int_\Omega v_m(\partial_t u_l - \partial_t u) dxdt| + \\ |\int_0^T \int_\Omega (v_m - u)\partial_t u dxdt| \leqslant 3\varepsilon. \end{aligned}$$

This establishes (4.18). On the other hand, since $\{u_k\}_k$ is bounded in $L^\infty(0,T; L^2(\Omega))$, we have (after extraction of a further subsequence if necessary) that $u_k(\cdot,T) \rightharpoonup u(\cdot,T)$ in $L^2(\Omega)$ (see Appendix III for a proof). Hence,

$$\liminf_{k\to\infty} \|u_k(\cdot,T)\|_{L^2(\Omega)} \geqslant \|u(\cdot,T)\|_{L^2(\Omega)}. \tag{4.19}$$

By construction of u_k we also have

$$\lim_{k\to\infty} \|u_k(\cdot,0)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}. \tag{4.20}$$

Using (4.19), (4.20) and (4.18) in (4.17), we conclude

$$\liminf_{k\to\infty} -\int_0^T \int_\Omega (u_k - v_k)\partial_t u_k dxdt \leqslant 0$$

This establishes $X \leqslant 0$, and we infer from (4.15), that the following "div-curl inequality" holds:

Lemma 4.4 *The Young measure $\nu_{(x,t)}$ generated by the gradients Du_k of the Galerkin approximations u_k has the property, that*

$$\int_0^T \int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : \big(\lambda - Du\big) d\nu_{(x,t)}(\lambda) dx dt \leqslant 0. \quad (4.21)$$

4.7 Passage to the limit

We start with the easiest case:

Case (d): Suppose that $\nu_{(x,t)}$ is not a Dirac mass on a set $(x,t) \in M \subset \Omega \times (0,T)$ of positive Lebesgue measure $|M| > 0$. Then, by the strict p-quasimonotonicity of $\sigma(x,t,u,\cdot)$, and the fact that $\nu_{(x,t)}$ is a homogeneous $W^{1,p}$ gradient Young measure (see Section 4.5) for almost all $(x,t) \in \Omega \times (0,T)$, we have for a.e. $(x,t) \in M$

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) > \\ > \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)}_{= Du(x,t)} .$$

Hence, by integrating over $\Omega \times (0,T)$, we get together with Lemma 4.4

$$\int_0^T \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) : Du(x,t) dx dt \geqslant \\ \geqslant \int_0^T \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dx dt > \\ > \int_0^T \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) : Du(x,t) dx dt$$

which is a contradiction. Hence, we have $\nu_{(x,t)} = \delta_{Du(x,t)}$ for almost every $(x,t) \in \Omega \times (0,T)$. From this, it follows by Proposition 1.3 that $Du_k \to Du$ on $\Omega \times (0,T)$ in measure for $k \to \infty$, and thus, $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ almost everywhere on $\Omega \times (0,T)$ (up to extraction of a further subsequence). Since, by the growth condition in (P1), $\sigma(x,t,u_k,Du_k)$ is equiintegrable, it follows that $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ in $L^1(\Omega \times (0,T))$ by the Vitali convergence theorem. Now, we take a test function $w \in \cup_{i\in\mathbb{N}} \operatorname{span}(w_1,\dots,w_i)$ and $\varphi \in C_0^\infty([0,T])$ in (4.6) and integrate over the interval $(0,T)$ and pass to the limit $k \to \infty$. The resulting equation is

$$\int_0^T \int_\Omega \partial_t u(x)\varphi(t)w(x) dx dt + \int_0^T \int_\Omega \sigma(x,t,u,Du) : Dw(x)\varphi(t) dx dt = \langle f, \varphi w \rangle,$$

for arbitrary $w \in \cup_{i \in \mathbb{N}} \operatorname{span}(w_1, \dots, w_i)$ and $\varphi \in C^\infty([0,T])$. By density of the linear span of these functions in $L^p(0,T;W^{1,p}(\Omega))$, this proves, that u is in fact a weak solution. Hence the Theorem follows in case (d).

Now, we prepare the proof of Theorem 4.1 in the remaining cases as follows: Observe that the integrand in (4.21) is nonnegative by monotonicity. Thus, it follows from Lemma 4.4 that the integrand must vanish almost everywhere with respect to the product measure $d\nu_{(x,t)} \otimes dx \otimes dt$. Hence, we have that for almost all $(x,t) \in \Omega \times (0,T)$

$$(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)) : (\lambda - Du) = 0 \quad \text{on } \operatorname{spt} \nu_{(x,t)} \tag{4.22}$$

and thus

$$\operatorname{spt} \nu_{(x,t)} \subset \{\lambda \mid (\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)) : (\lambda - Du) = 0\}. \tag{4.23}$$

Now, we proceed with the proof in the single cases (a), (b) and (c) of (P2). We start with the simplest case (c):

Case (c): By strict monotonicity, it follows from (4.22) or (4.23) that $\nu_{(x,t)} = \delta_{Du(x,t)}$ for almost all $(x,t) \in \Omega \times (0,T)$, and hence $Du_k \to Du$ in measure on $\Omega \times (0,T)$. The rest of the proof is identical to the proof for case (d).

Case (b): We start by showing that for almost all $(x,t) \in \Omega \times (0,T)$, the support of $\nu_{(x,t)}$ is contained in the set where W agrees with the supporting hyper-plane $L := \{(\lambda, W(x,t,u,Du) + \sigma(x,t,u,Du)(\lambda - Du))\}$ in $Du(x,t)$, i.e. we want to show that

$$\begin{aligned} &\operatorname{spt} \nu_{(x,t)} \subset K_{(x,t)} = \\ &\quad = \{\lambda \in \mathbb{M}^{m \times n} : W(x,t.u,\lambda) = W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du)\}. \end{aligned}$$

If $\lambda \in \operatorname{spt} \nu_{(x,t)}$ then by (4.23)

$$(1-\tau)(\sigma(x,t,u,Du) - \sigma(x,t,u,\lambda)) : (Du - \lambda) = 0 \quad \text{for all } \tau \in [0,1]. \tag{4.24}$$

On the other hand, by monotonicity, we have for $\tau \in [0,1]$ that

$$0 \leqslant (1-\tau)(\sigma(x,t,u,Du + \tau(\lambda - Du)) - \sigma(x,t,u,\lambda)) : (Du - \lambda). \tag{4.25}$$

Subtracting (4.24) from (4.25), we get

$$0 \leqslant (1-\tau)(\sigma(x,t,u,Du + \tau(\lambda - Du)) - \sigma(x,t,u,Du)) : (Du - \lambda) \tag{4.26}$$

for all $\tau \in [0,1]$. But by monotonicity, in (4.26) also the reverse inequality holds and we may conclude, that

$$(\sigma(x,t,u,Du + \tau(\lambda - Du)) - \sigma(x,t,u,Du)) : (\lambda - Du) = 0 \tag{4.27}$$

for all $\tau \in [0,1]$, whenever $\lambda \in \operatorname{spt}\nu_{(x,t)}$. Now, it follows from (4.27) that

$$\begin{aligned} W(x,t,u,\lambda) &= W(x,t,u,Du) + \int_0^1 \sigma(x,t,u,Du+\tau(\lambda - Du)) : (\lambda - Du) d\tau \\ &= W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du) \end{aligned}$$

as claimed.

By the convexity of W we have $W(x,t,u,\lambda) \geqslant W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du)$ for all $\lambda \in \mathbb{M}^{m\times n}$ and thus L is a supporting hyper-plane for all $\lambda \in K_{(x,t)}$. Since the mapping $\lambda \mapsto W(x,t,u,\lambda)$ is by assumption continuously differentiable we obtain

$$\sigma(x,t,u,\lambda) = \sigma(x,t,u,Du) \quad \text{for all } \lambda \in K_{(x,t)} \supset \operatorname{spt}\nu_{(x,t)} \tag{4.28}$$

and thus

$$\bar\sigma := \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda)\, d\nu_{(x,t)}(\lambda) = \sigma(x,t,u,Du)\,. \tag{4.29}$$

Now consider the Carathéodory function

$$g(x,t,u,p) = |\sigma(x,t,u,p) - \bar\sigma(x,t)|\,.$$

The sequence $g_k(x,t) = g(x,t,u_k(x,t),Du_k(x,t))$ is equiintegrable and thus

$$g_k \rightharpoonup \bar g \quad \text{weakly in } L^1(\Omega\times(0,T))$$

and the weak limit $\bar g$ is given by

$$\begin{aligned} \bar g(x,t) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |\sigma(x,t,\eta,\lambda) - \bar\sigma(x,t)|\, d\delta_{u(x,t)}(\eta)\otimes d\nu_{(x,t)}(\lambda) \\ &= \int_{\operatorname{spt}\nu_{(x,t)}} |\sigma(x,t,u(x,t),\lambda) - \bar\sigma(x,t)|\, d\nu_{(x,t)}(\lambda) = 0 \end{aligned}$$

by (4.28) and (4.29). Since $g_k \geqslant 0$ it follows that

$$g_k \to 0 \quad \text{strongly in } L^1(\Omega\times(0,T)).$$

This again suffices to pass to the limit in the equation and the proof of the case (b) is finished.

Case (a): We claim that in this case for almost all $(x,t) \in \Omega\times(0,T)$ the following identity holds for all $\mu \in \mathbb{M}^{m\times n}$ on the support of $\nu_{(x,t)}$:

$$\sigma(x,t,u,\lambda) : \mu = \sigma(x,t,u,Du) : \mu + (\nabla\sigma(x,t,u,Du)\mu) : (Du - \lambda), \tag{4.30}$$

where ∇ is the derivative with respect to the third variable of σ. Indeed, by the monotonicity of σ we have for all $\tau \in \mathbb{R}$

$$(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du+\tau\mu)) : (\lambda - Du - \tau\mu) \geqslant 0,$$

whence, by (4.22),

$$\begin{aligned} -\sigma(x,t,u,\lambda) : (\tau\mu) &\geqslant \\ &\geqslant -\sigma(x,t,u,Du) : (\lambda - Du) + \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu) \\ &= \tau\big((\nabla\sigma(x,t,u,Du)\mu)(\lambda - Du) - \sigma(x,t,u,Du) : \mu\big) + o(\tau). \end{aligned}$$

The claim follows from this inequality since the sign of τ is arbitrary. Since the sequence $\sigma(x,t,u_k,Du_k)$ is equiintegrable, its weak L^1-limit $\bar\sigma$ is given by

$$\begin{aligned} \bar\sigma &= \int_{\operatorname{spt}\nu_{(x,t)}} \sigma(x,t,u,\lambda)\,d\nu_{(x,t)}(\lambda) \\ &= \int_{\operatorname{spt}\nu_{(x,t)}} \sigma(x,t,u,Du)\,d\nu_{(x,t)}(\lambda) + \\ &\quad +(\nabla\sigma(x,t,u,Du))^t \int_{\operatorname{spt}\nu_{(x,t)}} (Du-\lambda)\,d\nu_{(x,t)}(\lambda) \\ &= \sigma(x,t,u,Du), \end{aligned}$$

where we used (4.30) in this calculation. This finishes the proof of the case (a) and hence of the theorem.

Remark: Notice, that in case (a) we have $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$, in case (b) we have $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ in $L^1(\Omega \times (0,T))$, and in case (c), we even have $Du_k \to Du$ in measure on $\Omega \times (0,T)$ as $k \to \infty$.

Appendix I

Here we give the proof of the modified Lemma of Aubin 4.2.

Let $\tilde B_0 := j(i(B_0)) \subset B_1$ be the Banach space equipped with the norm

$$\|\tilde x\|_{\tilde B_0} := \inf_{\substack{x \in B_0 \\ j\circ i(x) = \tilde x}} \|x\|_{B_0}$$

and $\tilde B := j(B) \subset B_1$ be the Banach space equipped with the norm

$$\|x\|_{\tilde B} := \|j^{-1}(x)\|_B$$

(we recall that j is supposed to be injective). Now, we consider a bounded sequence $\{v_n\}_n$ in W. Let $\tilde{v}_n := j \circ i \circ v_n$. Then, $\{\tilde{v}_n\}_n$ is bounded in

$$\tilde{W} := \{\tilde{v} \mid \tilde{v} \in L^{p_0}(0,T;\tilde{B}_0),\ \frac{d\tilde{v}}{dt} \in L^{p_1}(0,T;\tilde{B}_1)\}$$

and by the usual Aubin Lemma (see [Li-69, Chap. 1, Sect. 5.2]) it follows that there exists a subsequence $\tilde{v}_\nu$ which converges strongly in $L^{p_0}(0,T;\tilde{B})$. By isometry of B and $\tilde{B}$, the claim follows. □

Appendix II

Let u be an arbitrary function in $L^p(0,T;W_0^{1,p}(\Omega))$. We want to construct a sequence $v_k \in L^p(0,T;W_0^{1,p}(\Omega))$ which has the following properties:

(i) $v_k \to u$ in $L^p(0,T;W_0^{1,p}(\Omega))$.

(ii) $v_k(t) \in \operatorname{span}(w_1, w_2, \ldots, w_k)$ for $0 \leqslant t \leqslant T$.

To construct the sequence $\{v_k\}_k$, we take $\varepsilon > 0$ (with the intention to let $\varepsilon \to 0$) and a standard mollifier δ_η in space-time. The function u is extend by zero outside $\Omega \times [0,T] \subset \mathbb{R}^{n+1}$. Choosing $\eta > 0$ small enough, we may achieve that

$$\|u * \delta_\eta - u\|_{L^p(0,T;W^{1,p}(\Omega))} < \varepsilon.$$

Now, for a smooth function $\varphi \in C^\infty(\bar{\Omega} \times [0,T])$ and $j \in \mathbb{N}$ let

$$Q_j(\varphi)(x,t) := \varphi(x, i\tfrac{T}{j}) \qquad \text{if } t \in [i\frac{T}{j}, (i+1)\frac{T}{j})$$

denote the step function approximation of φ in time. We fix $j \in \mathbb{N}$ large enough such that we have

$$\|u * \delta_\eta - Q_j(u * \delta_\eta)\|_{L^p(0,T;W^{1,p}(\Omega))} < \varepsilon,$$

Finally, we choose k large enough, such that

$$\|Q_j(u * \delta_\eta) - P_k \circ Q_j(u * \delta_\eta)\|_{L^p(0,T;W^{1,p}(\Omega))} < \varepsilon,$$

where (as before) P_k denotes the $W^{s,2}(\Omega)$-projection onto $\operatorname{span}(w_1, w_2, \ldots, w_k)$ (notice that this is possible, since $t \mapsto Q_j(u * \delta_\eta)$ takes only finitely many values on $[0,T]$).

Combination yields

$$\|u - P_k \circ Q_j(u * \delta_\eta)\|_{L^p(0,T;W^{1,p}(\Omega))} < 3\varepsilon$$

and hence the sequence $v_k = P_{k(\varepsilon)} \circ Q_{j(\varepsilon)}(u * \delta_{\eta(\varepsilon)})$ for $\varepsilon \to 0$ is a sequence with the properties (i)–(ii).

Appendix III

Here, we want to prove, that

$$u_k(\cdot,T) \rightharpoonup u(\cdot,T) \qquad \text{weakly in } L^2(\Omega).$$

and that

$$u(\cdot,0) = u_0.$$

Since $\{u_k\}_k$ is bounded in $L^\infty(0,T;L^2(\Omega))$, it is clear, that for a (not relabeled) subsequence

$$u_k(\cdot,T) \rightharpoonup z \qquad \text{weakly in } L^2(\Omega),$$

and we have to show $z = u(\cdot,T)$. To shorten the notation, we write from now on $u(T)$ instead of $u(\cdot,T)$ et cetera.

In order to prove the claim, note that (again, after possible choice of a further subsequence)

$$-\operatorname{div}\sigma(x,t,u_k,Du_k) \rightharpoonup \chi \qquad \text{weakly in } L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Now, we claim that for arbitrary $\psi \in C^\infty([0,T])$ and $v \in W_0^{1,p}(\Omega)$

$$\int_\Omega z\psi(T)v dx - \int_\Omega u_0\psi(0)v dx = \langle f-\chi, \psi v\rangle + \int_0^T\int_\Omega \psi' v u dx. \tag{4.31}$$

Since $\cup_{n\in\mathbb{N}}\operatorname{span}(w_1,\dots,w_n)$ is dense in $W_0^{1,p}(\Omega)$, it suffices to verify (4.31) for $v \in \operatorname{span}(w_1,\dots,w_n)$. Then, by testing (4.6) by $v\psi$, we have for $m \geqslant n$

$$\underbrace{\int_0^T\int_\Omega \partial_t u_m v\psi dx dt}_{= \int_\Omega u_m(T)\psi(T)v dx - \int_\Omega u_m(0)\psi(0)v dx - \int_0^T\int_\Omega u_m v\psi' dx dt} + \int_0^T\int_\Omega \sigma(x,t,u_m,Du_m) : Dv\psi dx dt = \langle f, v\psi\rangle.$$

Then, (4.31) follows by letting m tend to infinity. By choosing $\psi(0) = \psi(T) = 0$ in (4.31), we have in particular

$$\langle f-\chi, \psi v\rangle = -\int_0^T\int_\Omega \psi' v u dx = \int_0^T\int_\Omega \psi v u' dx,$$

and hence

$$u' + \chi = f.$$

Using this and (4.31) we have on the other hand

$$\begin{aligned}
\int_\Omega z\psi(T)v dx - \int_\Omega u_0\psi(0)v dx &= \langle u', \psi v\rangle + \int_0^T\int_\Omega \psi' v u dx \\
&= \int_\Omega u\psi v dx\Big|_0^T \\
&= \int_\Omega u(T)\psi(T)v dx - \int_\Omega u(0)\psi(0)v dx. \qquad (4.32)
\end{aligned}$$

Choosing $\psi(T) = 1$, $\psi(0) = 0$ in (4.32), we obtain that $u(0) = u_0$ and for $\psi(T) = 0$, $\psi(0) = 1$, we get $u(T) = z$, as claimed.

Appendix IV

In this section, we want to assume that σ does not depend on x and u, and to replace condition (P2) by the following more classical quasimonotonicity condition:

(P2') For all fixed $t \in [0, T)$ the map $\sigma(t, F)$ is strictly quasimonotone in the variable F.

Here, by strictly quasimonotone we mean the following:

Definition 4.5 *A function $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is said to be strictly quasimonotone, if there exist constants $c > 0$ and $r > 0$ such that*

$$\int_\Omega (\eta(Du) - \eta(Dv)) : (Du - Dv)dx \geqslant c \int_\Omega |Du - Dv|^r dx$$

for all $u, v \in W_0^{1,p}(\Omega)$.

We want to prove:

Theorem 4.6 *If $\sigma(t, Du)$ satisfies the conditions (P0), (P1) and (P2') for some $p \in (\frac{2n}{n+2}, \infty)$, then the parabolic system (4.1)–(4.3) has a weak solution $u \in L^p(0, T; W_0^{1,p}(\Omega))$ for every $f \in L^{p'}(0, T; W^{-1,p}(\Omega))$ and every $u_0 \in L^2(\Omega)$.*

Since in this case, we do not have to deal with x and u dependence of σ, the following simple proof is possible.

Proof
Let u_k and v_k be constructed as in the proof of Theorem 4.1. Then, by using $u_k - v_k$ as a test function in (4.6), we obtain

$$\begin{aligned} \langle f, u_k - v_k\rangle - \int_0^T \int_\Omega (u_k - v_k)\partial_t u_k dxdt = \\ = \int_0^T \int_\Omega \sigma(t, Du_k) : (Du_k - Dv_k)dxdt = \\ = \int_0^T \int_\Omega (\sigma(t, Du_k) - \sigma(t, Dv_k)) : (Du_k - Dv_k)dxdt + \\ + \int_0^T \int_\Omega \sigma(t, Dv_k) : (Du_k - Dv_k)dxdt. \end{aligned} \tag{4.33}$$

The first term on the left of (4.33) $\langle f, u_k - v_k\rangle$ converges to zero as $k \to \infty$, since $u_k - v_k \rightharpoonup 0$ in $L^p(0,T;W_0^{1,p}(\Omega))$. For the second term on the left of (4.33) we have seen in Section 4.6 that

$$\liminf_{k\to\infty} -\int_0^T \int_\Omega (u_k - v_k)\partial_t u_k dxdt \leqslant 0$$

for $k \to \infty$. The last term on the right of (4.33) converges to zero for $k \to \infty$ since $\sigma(t, Dv_k) \to \sigma(t, Du)$ in $L^{p'}(0,T;L^{p'}(\Omega))$ (at least for a subsequence) and $Du_k - Dv_k \rightharpoonup 0$ in $L^p(0,T;L^p(\Omega))$. We conclude:

$$o(1) = \int_0^T \int_\Omega (\sigma(t, Du_k) - \sigma(t, Dv_k)) : (Du_k - Dv_k)dxdt \geqslant$$
$$\geqslant c\int_0^T \int_\Omega |Du_k - Dv_k|^r dxdt.$$

This implies $Du_k \to Du$ in measure for a suitable subsequence. The rest of the proof is as in case (d) in Section 4.7. □

Acknowledgment

This work has been carried out while I was at the Max-Planck Institute for Mathematics in the Sciences in Leipzig. I am most grateful to Professor Stefan Müller and Professor Eberhard Zeidler for their support and for providing and maintaining such a stimulating place.

I would like to thank Professor Stefan Müller and Professor Michael Struwe for their continuing interest in this work and their helpful remarks. I am especially grateful to Professor Jens Frehse for initiating my interest in parabolic systems with weak monotonicity assumptions. Finally I would like to thank Jan Kristensen and Mikhail Sytchev for inspiring discussions and many useful hints.

Bibliography

[Bald-84] E. J. Balder: A general approach to lower semicontinuity and lower closure in optimal control theory. SIAM J. Control Optim. **22** (1984), 570–598.

[Bald-91] E. J. Balder: On equivalence of strong and weak convergence in L_1-spaces under extreme point conditions. Israel J. Math. **75** (1991), 21–47.

[Ball-76/77] J. M. Ball: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. **63** (1976/77), no. 4, 337–403.

[Ball-77] J. M. Ball: Constitutive inequalities and existence theorems in nonlinear elastostatics. Nonlinear analysis and mechanics: Heriot-Watt Symposium (Edinburgh, 1976), Vol. I, 187–241. Res. Notes in Math., No. 17, Pitman, London, 1977.

[Ball-89] J. M. Ball: A version of the fundamental theorem for Young measures. In: Partial differential equations and continuum models of phase transitions: Proceedings of an NSF-CNRS joint seminar. Springer, 1989.

[BaMu-89] J. M. Ball, F. Murat: Remarks on Chacon's biting lemma. Proc. Amer. Math. Soc. **107** (1989), no. 3, 655–663.

[Bar-76] V. Barbu: Nonlinear semigroups and differential equations in Banach spaces. Noordhoff International Publishing, Leiden, 1976.

[BoMu-92] L. Boccardo, F. Murat: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal. **19** (1992), no. 6, 581–597.

[Bré-68] H. Brézis: Équations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier **18** (1968), 115–175.

[Bré-73] H. Brézis: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[BréBro-79] H. Brézis, F. E. Browder: Strongly nonlinear parabolic initial-boundary value problems. Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 1, 38–40.

[BrCh-80] J. K. Brooks, R. V. Chacon: Continuity and compactness of measures. Adv. in Math. **37** (1980), no. 1, 16–26.

[Bro-63] F. E. Browder: Nonlinear elliptic boundary value problems. Bull. Amer. Math. Soc. **69** (1963), 862–874.

[Bro-63a] F. E. Browder: Variational boundary value problems for quasi-linear elliptic equations. I–III. Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 31–37, 592–598, 794–798.

[Bro-68] F. E. Browder: Existence theorems for nonlinear partial differential equations. 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968) 1–60 AMS, Providence, R.I.

[Bro-68/76] F. E. Browder: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968), pp. 1–308. Amer. Math. Soc., Providence, RI, 1976 (preprint version 1968).

[Bro-97] F. E. Browder: On a sharpened form of the Leray-Lions' ellipticity criterion. C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 9, 999–1004.

[ChZh-92] J. Chabrowski, K.-W. Zhang: Quasi-monotonicity and perturbated systems with critical growth. Indiana Univ. Math. J. **41** (1992), no. 2, 483–504.

[De-85] K. Deimling: Nonlinear functional analysis. Springer-Verlag, New York, 1985.

[DoHuMü-97] G. Dolzmann, N. Hungerbühler, S. Müller: Non-linear elliptic systems with measure-valued right hand side. Math. Z. **226** (1997), 545–574.

[Du-76] J. A. Dubinskiĭ: Nonlinear elliptic and parabolic equations (Russian). Itogi Nauki i Tekhniki, Sovremennye problemy matematikii **9** (1976), 1–130.

[DuSc-88] N. Dunford, J. T. Schwartz: Linear operators. Wiley-Interscience Publication, New York, 1988.

[Ev-90] L. C. Evans: Weak convergence methods for nonlinear partial differential equations. CBMS Regional Conf. Ser. in Math., no. 74, Am. Math. Soc., Providence, RI, 1990.

[EvGa-92] L. C. Evans, R. F. Gariepy: Measure theory and fine properties of functions. Boca Raton, CRC Press, 1992.

[FoMüPe-98] I. Fonseca, S. Müller, P. Pedregal: Analysis of concentration and oscillation effects generated by gradients. SIAM J. Math. Anal. **29** (1998), no. 3, 736–756.

[Fr-99] J. Frehse: private communication.

[Fu-87] M. Fuchs: Regularity theorems for nonlinear systems of partial differential equations under natural ellipticity conditions. Analysis **7** (1987), no. 1, 83–93.

[FuFu-80] S. Fučik, A. Kufner: Nonlinear differential equations. Elsevier, New York, 1980.

[GGZ-74] H. Gajewski, K. Gröger, K. Zacharias: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, Band 38. Akademie-Verlag, Berlin, 1974.

[Hu-97] N. Hungerbühler: A refinement of Ball's Theorem on Young measures. New York J. Math. **3** (1997), 48–53.

[Hu-99] N. Hungerbühler: A remark on quasilinear elliptic systems in divergence form. New York J. Math. **5** (1999), 83–90.

[Ka-60] E. Kamke: Das Lebesgue-Stieltjes-Integral. Teubner, Leipzig, 1960.

[KiPe-91a] D. Kinderlehrer, P. Pedregal: Characterizations of Young measures generated by gradients. Arch. Rational Mech. Anal. **115** (1991), no. 4, 329–365.

[KiPe-91b] D. Kinderlehrer, P. Pedregal: Caractérisation des mesures de Young associées à un gradient. C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), no. 11, 765–770.

[KiPe-94] D. Kinderlehrer, P. Pedregal: Remarks about gradient Young measures generated by sequences in Sobolev spaces. Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), 131–150, Pitman Res. Notes Math. Ser., 299, Longman Sci. Tech., Harlow, 1994.

[KiPe-94] D. Kinderlehrer, P. Pedregal: Young measures generated by sequences in Sobolev spaces. J. Geom. Anal. **4**, No. 1 (1994), 59–90.

[Kl-79] R. Kluge: Nichtlineare Variationsungleichungen und Extremalaufgaben. Verl. der Wiss., Berlin, 1979.

[Kr-94] J. Kristensen: Lower semicontinuity of variational integrals. PhD Thesis. Technical University of Denmark, 1994.

[Kr-99] J. Kristensen: Lower semicontinuity in spaces of weakly differentiable functions. Math. Ann. **313** (1999), 653–710.

[La-80] R. Landes: On Galerkin's method in the existence theory of quasilinear elliptic equations. J. Funct. Anal. **39** (1980), no. 2, 123–148.

[La-85] R. Landes: On weak solutions of quasilinear parabolic equations. Nonlinear Anal. **9** (1985), no. 9, 887–904.

[La-90] R. Landes: A note on strongly nonlinear parabolic equations of higher order. Differential Integral Equations **3** (1990), no. 5, 851–862.

[LaMu-80] R. Landes, V. Mustonen: On pseudomonotone operators and nonlinear noncoercive variational problems on unbounded domains. Math. Ann. **248** (1980), no. 3, 241–246.

[LaMu-87] R. Landes, V. Mustonen: A strongly nonlinear parabolic initial-boundary value problem. Ark. Mat. **25** (1987), no. 1, 29–40.

[LaMu-94] R. Landes, V. Mustonen: On parabolic initial-boundary value problems with critical growth for the gradient. Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), no. 2, 135–158.

[Lang-76] A. Langenbach: Monotone Potentialoperatoren in Theorie und Anwendung. Hochschultext. Springer-Verlag, Berlin-New York, 1977.

[LeLi-65] J. Leray, J.-L. Lions: Quelques résulatats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder. Bull. Soc. Math. France **93** (1965), 97–107.

[Li-69] J.-L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris 1969.

[Mi-62] G. J. Minty: Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29** (1962), 341–346.

[Mi-63] G. J. Minty: On a "monotonicity" method for the solution of non-linear equations in Banach spaces. Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1038–1041.

[Mu-78] F. Murat: Compacité par compensation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 489–507.

[Ne-83] J. Neăs: Introduction to the theory of nonlinear elliptic equations. Teubner, Leipzig, 1983.

[PS-78] D. Pascali, S. Sburlan: Nonlinear mappings of monotone type. Academiei, Bucuresti, 1978.

[Pe-70] W. V. Petryshyn: Nonlinear equations involving noncompact operators. 1970 Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968) 206–233 Amer. Math. Soc., Providence, R.I.

[Pe-80] W. V. Petryshyn: Solvability of linear and quasilinear elliptic boundary value problems via the A-proper mapping theory. Numer. Funct. Anal. Optim. **2** (1980), 591–635.

[Pe-81] W. V. Petryshyn: Variational solvability of quasilinear elliptic boundary value problems at resonance. Nonlinear Anal. **5** (1981), 1095–1108.

[Sh-97] R. E. Showalter: Monotone operators in Banach space and nonlinear partial differential equations. Providence, Rhode Island: AMS 1997. (Mathematical surveys and monographs; vol. 49)

[Si-87] J. Simon: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.

[Sk-73] I. Skrypnik: Nonlinear elliptic equations of higher order (Russian). Naukova Dumka, Kiev, 1973.

[Sk-86] I. Skrypnik: Nonlinear elliptic boundary value problems. Teubner, Leipzig 1986.

[Ta-79] L. Tartar: Compensated compactness and applications to partial differential equations. In: Nonlinear analysis and mechanics: Heriot-Watt Symp., Vol. 4, Edinburgh 1979, Res. Notes Math. **39** (1979), 136–212.

[Ta-82] L. Tartar: The compensated compactness method applied to systems of conservation laws. In: Systems of nonlinear partial differential equations. Ed. J. M. Ball. NATO ASI Series Vol. C111, Reidel 1982.

[Vai-72] M. M. Vainberg: Variational method and method of monotone operators in the theory of nonlinear equations. Nauka, Moscow (Russian). English edition: Wiley, New York, 1973.

[Val-90] M. Valadier: Young measures. Methods of nonconvex analysis (A. Cellina, ed.), Lecture Notes in Math., vol. 1446, Springer Verlag, Berlin, 1990, 152–188.

[Val-94] M. Valadier: A course on Young measures. Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1993). Rend. Istit. Mat. Univ. Trieste **26** (1994), suppl., 349–394 (1995).

[Vi-61] I. M. Višik: Boundary value problems for quasilinear strongly elliptic equatuions (Russian). Dokl. Akad. Nauk SSSR **138** (1961), 518–521.

[Vi-62] I. M. Višik: Solubility of boundary-value problems for quasi-linear parabolic equations of higher orders. Mat. Sb. (N.S.) **59** (101) (1962) suppl. 289–325.

[Vi-63] I. M. Višik: Quasi-linear strongly elliptic systems of differential equations of divergence form. Trudy Moskov. Mat. Obšč. **12** (1963), 125–184.

[Yo] K. Yosida: Functional analysis. Grundlehren der mathematischen Wissenschaften; 123. 6th ed. Berlin: Springer, 1980.

[Zh-88] K.-W. Zhang: On the Dirichlet problem for a class of quasilinear elliptic systems of partial differential equations in divergence form. Partial differential equations (Tianjin, 1986), 262–277, Lecture Notes in Math., 1306, Springer, Berlin-New York, 1988.

[Zh-92] K.-W. Zhang: Remarks on perturbated systems with critical growth. Nonlinear Anal. **18** (1992), no. 12, 1167–1179.

[Ze-90] E. Zeidler: Nonlinear monotone operators (Nonlinear functional analysis and its applications; II/B). Springer, New York, 1990.

[Zy-77] A. Zygmund: Trigonometric series. Cambridge University Press, 1977.