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p-harmonic Flow

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Contents

Abstract

Let M and N be smooth compact Riemannian manifolds without boundary and with metrics γ and g respectively. Let m and n denote the dimensions of M and N. For a C^1-mapping $f : M \to N$ the p-energy density is defined by

$$e(f)(x) := \frac{1}{p}\, |df_x|^p \tag{1}$$

and the p-energy by

$$E(f) := \int\limits_M e(f)\, d\mu\,. \tag{2}$$

Here, p denotes a real number in the interval $[2, \infty[$, $|df_x|$ is the Hilbert-Schmidt norm with respect to γ and g of the differential

$$df_x \in T^*_x(M) \otimes T_{f(x)}(N)$$

and μ is the measure on M which is induced by the metric.

If p coincides with the dimension m of the manifold M then the energy E is conformally invariant. This fact is well-known in the case $p = 2$ where E is just the Dirichlet integral.

Variation of the energy-functional yields the Euler-Lagrange equations of the p-energy which are

$$\Delta_p f = -\left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \Gamma^l_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \tag{3}$$

in local coordinates. We make use of the usual summation convention. The operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \frac{\partial f^l}{\partial x^\alpha}\right)$$

is called p-Laplace operator (for $p = 2$ this is just the Laplace-Beltrami operator). On the right hand side of (3) the Γ^l_{ij} denote the Christoffel-symbols related to the manifold N. According to Nash's embedding theorem we can think of N as being

isometrically embedded in some Euclidean space $\mathbb{R}^k$ since N is compact. Then, if we denote by F the function f regarded as a function into $N \subset \mathbb{R}^k$, equation (3) admits a geometric interpretation, namely

$$\Delta_p F \perp T_F N$$

with Δ_p being the p-Laplace operator with respect to the manifolds M and $\mathbb{R}^k$.

For $p > 2$ the p-Laplace operator is degenerate elliptic. Another important property of the p-Laplace operator is its strong monotonicity: There holds

$$\langle -\Delta_p f + \Delta_p g, f - g \rangle \geq c\, E(f - g)\,. \tag{4}$$

for a constant $c > 0$ which only depends on the geometry of M and N.

(Weak) solutions of (3) are called (weakly) p-harmonic maps. The concept of 2-harmonic maps generalizes the notion of (closed) geodesics to higher dimensions. Moreover, if we choose $N = \mathbb{R}^k$, we see that harmonic functions simply appear as special cases of 2-harmonic maps.

The regularity theory of weakly p-harmonic maps involves an extensive part of the theory of nonlinear partial differential equations. We mention the contributions of Hardt and Lin, Giaquinta and Modica, Fusco and Hutchinson, Luckhaus, Coron and Gulliver, Fuchs, Duzaar, DiBenedetto, Friedman, Choe, and refer to their work listed in the bibliography. One of the most important recent results is due to Hélein [63], who proved regularity of weakly 2-harmonic maps.

One possibility to produce p-harmonic mappings is to investigate the heat flow related to the p-energy, i.e. to look at the flow-equation

$$\partial_t f - \Delta_p f \perp T_f N\,. \tag{5}$$

For $p = 2$ Eells and Sampson showed in their famous work [32] of 1964, that there exist global solutions of (5) provided N has nonpositive sectional curvature and that the flow tends for suitable $t_k \to \infty$ to a harmonic map. We see that under the mentioned geometric condition the heat flow also solves the homotopy problem, i.e. to find a harmonic map homotopic to a given map. Surprisingly, also a topological condition on the target may suffice to solve the homotopy problem. Lemaire [74] (and independently also Sacks-Uhlenbeck [88]) obtained this result under the assumption that $m = 2$ and $\pi_2(N) = 0$. For negative sectional curvature and arbitrary $p > 2$ the homotopy problem was solved by Duzaar and Fuchs in [29] by using different methods than the heat flow.

1984 Struwe succeeded to prove existence of weak solutions of the 2-harmonic flow in the conformal case $m = 2$ (see [107]). He also showed uniqueness and described in detail the singular set and the kind of singularities. In the higher dimensional case Y. Chen [11] (and independently Keller, Rubinstein and Sternberg [72] as well as

Shatah [96]) showed existence of global weak solutions of the 2-harmonic flow into spheres. The proof relies on the "penalizing-technique" where a penalizing term

$$F_k(f) := \frac{k}{4} \int_M |f^2 - 1|^2 d\mu$$

is added to the energy-functional. This technique together with Struwe's so called monotonicity formula (see [109]) was used in the corresponding proof for arbitrary target-manifold N, which was given by Chen and Struwe in [16]. The existence of a p-harmonic flow into spheres was shown by Chen and Hong as well as by the author (first independently, then in a joint work [14]). It turned out that the case $p > 2$ is technically much more difficult to handle than the non-degenerate case $p = 2$.

In Chapter 2 we prove existence of weak solutions of the p-harmonic flow into spheres (with no restrictions on the manifold M). The main result is Theorem 2.4.

An important ingredient in the work of Chen and Struwe on existence of the 2-harmonic flow in higher dimensions was the monotonicity formula. For $p > 2$ an analogous statement is not known. Thus, an existence proof in the case $p > 2$ with arbitrary target-manifold necessarily needs other techniques. Thereby we restrict ourselves to the conformal case $p = \dim(M)$. So, the third chapter deals with a priori estimates for that situation.

Beside a simple energy estimate we first get an a priori estimate for the L^{2p}-norm of the gradient of a solution of (5), provided the p-energy does not concentrate too much:

$$\begin{aligned}(6)\qquad & \sup\left\{E(f(t), B_R(x))\,;\, 0 \le t \le T, x \in B_{2R}(y)\right\} < \varepsilon_1 \implies \\ & \implies \int_0^T \int_{B_R(x)} |\nabla f|^{2p} d\mu\, dt < c\, E_0 \left(1 + \frac{T}{R^m}\right)\end{aligned}$$

for positive constants c and ε_1 depending only on the geometry of the problem. This kind of conditional estimate is typical for the p-harmonic flow. If we consider the equation (5) with $N = \mathbb{R}^k$ (i.e. for vanishing left side), one is tempted to apply the regularity theory for degenerate parabolic systems, which has been developed by DiBenedetto, Friedman, Choe and other authors. Unfortunately the right hand side of (5) has natural growth, since it is of the order of the p-th power of the gradient ("controllable" growth in the sense of Giaquinta would be $|\nabla f|^{p-1}$). This fact forces the mentioned conditional assertions. In order to obtain a conditional a priori bound for the gradient of the solution in L^∞_{loc} we need an intermediate step: by an iteration we first get a priori estimates for the L^q-norm of the gradient of solutions for arbitrary $q \in]2, \infty[$, again provided the above condition on non-concentration of energy is fulfilled. Once the $2p$-barrier is crossed, a Moser-iteration starts and gives the desired L^∞-bound (where again the energy concentration condition is inherited). Since it is possible to show that the energy does not concentrate too fast (i.e. if the local energy is initially small it remains small for a certain time), the crucial assumption on the local energy remains fulfilled at least for a small time-interval as long as it is fulfilled initially: in fact, for suitable constants c and $\varepsilon_0 > 0$, depending only on the geometry of the problem, there holds

$$\begin{aligned}(7)\qquad &\sup_{x\in M} E(f(0), B_{2R}(x)) < \varepsilon_0 \implies \\ &\implies E(f(t), B_R(x)) \le E(f(0), B_{2R}(x)) + cE_0^{1-\frac{1}{p}} \frac{t}{R^p}\end{aligned}$$

where E_0 denotes the initial p-energy. The a priori estimates of Chapter 3 are needed in the existence proof given in Chapter 4.

In order to show local existence of the p-harmonic flow for smooth initial data, we adapt a method of Hamilton, who proved this fact for $p = 2$ (see [59]): to get rid of the target constraint $f(M) \subset N$ we consider a totally geodesic embedding of the manifold N. Then the p-energy is approximated by the regularized energy

$$(8)\qquad E_\varepsilon(f) = \frac{1}{p} \int_M \left(\varepsilon + |df|^2\right)^{\frac{p}{2}} d\mu$$

and the corresponding heat flow is investigated. By suitable choice of the totally geodesic embedding of N the linearization of the regularized p-Laplace operator becomes uniformly elliptic in the sense of Legendre. The theory of analytic semiflows then yields local existence of the regularized flow. However, the existence interval depends on the regularization parameter ε and hence, it has to be shown that it does not shrink to zero with $\varepsilon \to 0$. This can be done with the a priori estimates of Chapter 3 which can be carried over to the regularized flow. An approximation argument then gives local existence of the p-harmonic flow for initial data in the more natural space $W^{1,p}(M)$. For small initial energy the existence is global. Finally we obtain global existence without a smallness condition and also a characterization of the singular set as well as regularity statements outside this set. Singularities may occur only at finitly many times and the number of singular times is a priori bounded in terms of the initial energy since the decrease of the p-energy at every singular time is at least $\varepsilon_1 > 0$ (this constant only depends on M and N). The main result is Theorem 4.9.

For $p = m = 2$ Struwe proved in [107] uniqueness of the harmonic flow in a class called $V(M^T, N)$ (see also Section 3.2). For $p = m > 2$ we obtain uniqueness in the space $L^\infty(0, t; W^{1,\infty}(M))$. On the other hand uniqueness of the flow in the class $L^\infty(0, T; W^{1,p}(M))$ would imply regularity of weakly p-harmonic maps: if there exists a weakly p-harmonic map $f \in W^{1,p}(M, N) \setminus W^{1,\infty}(M, N)$ with $p = \dim(M)$ then we have an example of a non-unique flow. In fact, as we show in Chapter 4 the p-harmonic flow with initial data f has a local weak solution g such that the gradient $|\nabla g|$ is finite for $t > 0$. On the other hand f (considered as constant in time) is a weak solution of the p-harmonic flow for the same initial data, but ∇f remains unbounded for any time.

In Chapter 5 we show uniqueness of the p-harmonic flow for solutions in the class $L^\infty(0, T; W^{1,\infty}(M))$.

The famous example of Chang, Ding and Ye shows the phenomenon of finite-time blow-up in the case $p = m = 2$ (see [10]). The question whether or not singularities may occur in finite time for the conformal flow with $p > 2$ remains open. However, the numerical results of Section 5.3 lend some support to the conjecture that we may have finite-time blow-up also in the case $p > 2$.

Zusammenfassung

Seien M und N glatte, kompakte Riemannsche Mannigfaltigkeiten ohne Rand mit Metriken γ respektive g. Es bezeichne m die Dimension von M und n diejenige von N. Die p-Energiedichte für eine C^1-Abbildung $f : M \to N$ ist definiert durch

$$e(f)(x) := \frac{1}{p}\,|df_x|^p \tag{1}$$

und die p-Energie durch

$$E(f) := \int\limits_M e(f)\,d\mu\,. \tag{2}$$

Dabei steht p für eine reelle Zahl aus dem Intervall $[2,\infty[$, $|df_x|$ für die Hilbert-Schmidt Norm bezüglich γ und g des Differentials

$$df_x \in T_x^*(M) \otimes T_{f(x)}(N)$$

und μ ist das durch die Metrik induzierte Maß auf M.

Falls p mit der Dimension m der Mannigfaltigkeit M übereinstimmt, ist die Energie E konform invariant, eine Tatsache, die vom Dirichlet-Integral ($p = 2$) wohlbekannt ist.

Die durch Variation des Energiefunktionals gewonnenen Euler-Lagrange Gleichungen der p-Energie lauten in lokalen Koordinaten und unter Verwendung der üblichen Summenkonvention

$$\Delta_p f = -\left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1} \gamma^{\alpha\beta}\Gamma^l_{ij}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\,, \tag{3}$$

wobei der Operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\gamma^{\alpha\beta} g_{ij}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\frac{\partial f^l}{\partial x^\alpha}\right)$$

auf der linken Seite von (3) p-Laplace Operator genannt wird und der für $p = 2$ mit dem Laplace-Beltrami Operator zusammenfällt. Auf der rechten Seite von (3) bezeichnen die Γ^l_{ij} die Christoffelsymbole der Mannigfaltigkeit N. Aufgrund der

Kompaktheit von N, ist es nach Nashs Einbettungssatz immer möglich, N in einem Euklidischen Raum $\mathbb{R}^k$ isometrisch einzubetten. Bezeichnen wir dann mit F die Funktion f aufgefaßt als Funktion nach $N \subset \mathbb{R}^k$, so erlaubt die Gleichung (3) die geometrische Deutung

$$\Delta_p F \perp T_F N,$$

wobei hier Δ_p der p-Laplace operator bezüglich den Mannifaltigkeiten M und $\mathbb{R}^k$ ist.

Der p-Laplace Operator ist für $p > 2$ degeneriert elliptisch. Eine andere wichtige Eigenschaft des p-Laplace Operators ist seine Monotonizität. Es gilt nämlich

$$\langle -\Delta_p f + \Delta_p g, f - g \rangle \geq c\, E(f - g) \tag{4}$$

für eine geometrische Konstante c.

(Schwache) Lösungen von (3) werden (schwach) p-harmonische Abbildungen genannt. Diese Abbildungen verallgemeinern im Falle $p = 2$ den Begriff der (geschlossenen) Geodäten auf höhere Dimensionen. Ferner können für $N = \mathbb{R}^k$ harmonische Funktionen als spezielle 2-harmonische Abbildungen aufgefaßt werden.

Die Regularitätstheorie von schwach p-harmonischen Abbildungen benützt weite Teile der Theorie der nichtlinearen partiellen Differentialgleichungen. Wir erwähnen hier etwa die Autoren Hardt und Lin, Giaquinta und Modica, Fusco und Hutchinson, Luckhaus, Coron und Gulliver, Fuchs, Duzaar, DiBenedetto, Friedman, Choe, und verweisen auf deren im Literaturverzeichnis aufgeführten Beiträge. Eines der wichtigsten neueren Resultate stammt von Hélein, der die Regularität von schwach 2-harmonischen Abbildungen nachgewiesen hat [63].

Eine Möglichkeit p-harmonische Abbildungen zu erzeugen besteht darin, den zur p-Energie gehörigen Wärmefluß zu untersuchen. D.h. man betrachtet Lösungen der Flußgleichung

$$\partial_t f - \Delta_p f \perp T_f N\,. \tag{5}$$

Für $p = 2$ konnten Eells und Sampson in ihrer berühmt gewordenen Arbeit [32] von 1964 zeigen, daß globale reguläre Lösungen von (5) existieren, falls N nicht-positive Schnittkrümmung aufweist, und daß sich im Limes $t_k \to \infty$ (für eine geeignete Folge t_k) eine harmonische Abbildung einstellt. Mit Hilfe des Wärmeflusses ist damit unter der genannten geometrischen Bedingung auch das Homotopie-Problem, d.h. eine zu einer gegebenen Abbildung homotope harmonische zu finden, gelöst. Erstaunlicherweise genügt für $m = 2$ bereits eine topologische Bedingung zur Lösung des Homotopie-Problems: Lemaire [74] und, von ihm unabhängig, Sacks-Uhlenbeck [88] gelangten zu diesem Resultat unter der Voraussetzung $\pi_2(N) = 0$. Für negative Schittkrümmung der Zielmannigfaltigkeit wurde das Homotopie-Problem für $p > 2$ von Duzaar und Fuchs in [29] gelöst, wobei allerdings eine ganz andere als die Wärmefluß-Methode zur Anwendung kam.

Struwe gelang 1984 der Existenzbeweis für schwache globale Lösungen des Flusses im konformen Fall $p = m = 2$ (siehe [107]). Er konnte auch Eindeutigkeit zeigen und beschrieb detailliert die singuläre Menge sowie die Art der Singularitäten. Im höherdimensionalen Fall zeigte Y. Chen [11] (und unabhängig davon auch Keller, Rubinstein und Sternberg [72], sowie von Shatah [96]) Existenz globaler schwacher Lösungen des 2-harmonischen Flusses in eine Sphäre. Die im Beweis zum Einsatz gelangte Technik, nämlich einen "penalizing"-Term

$$F_k(f) := \frac{k}{4} \int\limits_M |f^2 - 1|^2 d\mu$$

zur 2-Energie zu addieren, war dann (neben der sogenannten Monotonieformel von Struwe [109]) auch das Hilfsmittel im entsprechenden Beweis für beliebige Zielmannigfaltigkeiten N, der durch Chen und Struwe geleistet wurde [16]. Die Existenz eines p-Flusses in die Sphäre wurde von Chen und Hong sowie vom Autor (zunächst unabhängig, später in Zusammenarbeit in [14]) bewiesen. Dabei zeigte es sich bereits, daß der Fall $p > 2$ hinsichtlich technischem Aufwand erheblich schwieriger zu bewältigen ist als der nicht degenerierte Fall $p = 2$.

Im zweiten Kapitel wird die Existenz schwacher Lösungen des p-harmonischen Flusses in Sphären bewiesen (ohne weitere Bedingungen an die Mannigfaltigkeit M). Das Hauptresultat ist hier das Theorem 2.4.

Ein wichtiges Ingredienz im Beweis von Chen und Struwe ist die Monotonie-Formel für den 2-harmonischen Fluß. Im Falle $p > 2$ steht eine entsprechende Formel nicht zur Verfügung. Um einen Existenzsatz für den p-hamonischen Fluß bei allgemeiner Zielmannigfaltigkeit N zu erhalten, müssen daher andere Techniken zum Einsatz gelangen. Dabei ist es naheliegend, sich auf den konformen Fall $p = \dim(M)$ zu beschränken. Das dritte Kapitel handelt dementsprechend von a priori Abschätzungen für diesen Fall.

Neben einer einfachen Energieabschätzung bekommt man zunächst eine a priori Abschätzung für die L^{2p}-Norm des Gradienten der Lösung von (5), vorausgesetzt die Energie ist lokal nicht zu stark konzentriert: Es gilt

$$\text{(6)} \qquad \sup\{E(f(t), B_R(x))\,;\, 0 \le t \le T, x \in B_{2R}(y)\} < \varepsilon_1 \implies$$
$$\implies \int\limits_0^T \int\limits_{B_R(x)} |\nabla f|^{2p} d\mu\, dt < c\, E_0 \left(1 + \frac{T}{R^m}\right)$$

für nur von der Geometrie der Mannigfaltigkeiten M und N abhängige positive Konstanten ε_1 und c. Diese Art bedingter Abschätzungen ist typisch für den p-harmonischen Fluß. Betrachtet man die Gleichung (5) mit $N = \mathbb{R}^k$ (d.h. mit verschwindender rechter Seite), so ist man versucht, die Regularitätstheorie für degenerierte parabolische Systeme anzuwenden, wie sie etwa von DiBenedetto, Friedman, Choe und anderen entwickelt wurde. Unglücklicherweise besitzt aber die rechte Seite der Gleichung (5) im allgemeinen Fall ein natürliches Wachstum, sie ist von der Ordnung der p-ten Potenz des Gradienten ("controllable growth" im Sinne von Giaquinta wäre $|\nabla f|^{p-1}$). Dieser Umstand erzwingt die angesprochenen bedingten Aussagen. Um zu einer bedingten a priori Schranke für den Gradienten der Lösung in L^∞_{loc} zu kommen, muß ein Zwischenschritt erfolgen: durch eine Iteration erhält man zunächst eine a priori Abschätzung für die L^q-Norm des Gradienten für beliebiges $q \in]2p, \infty[$, wieder unter der Bedingung, daß die Energie lokal nicht zu stark konzentriert ist. Ist die $2p$-Schranke erst einmal übersprungen, startet eine Moser-Iteration und liefert die gewünschte L^∞-Schranke, wobei die Bedingung über die Kleinheit der lokalen Energie geerbt wird. Weil man zeigen kann, daß sich die Energie nicht zu schnell konzentrieren kann, bleibt die maßgebende Voraussetzung über die lokale Energie für eine gewisse Zeit erfüllt, falls sie es zur initialen Zeit ist: für geeignete geometrische Konstanten c und ε_0 gilt nämlich

$$(7) \qquad \sup_{x\in M} E(f(0), B_{2R}(x)) < \varepsilon_0 \implies$$
$$\implies E(f(t), B_R(x)) \le E(f(0), B_{2R}(x)) + c\, E_0^{1-\frac{1}{p}} \frac{t}{R^p}$$

wobei E_0 die Anfangsenergie bezeichnet. All diese a priori Abschätzungen werden dann im eigentlichen Existenzbeweis im Kapitel 4 benötigt.

Um lokale Existenz des p-harmonischen Flusses für glatte Anfangsdaten zu zeigen, adaptiern wir die Methode von Hamilton, von dem der entsprechende Beweis im Falle $p = 2$ stammt (siehe [59]): Um sich von der Zustandsbedingung $f(M) \subset N$ zu befreien, betrachtet man eine total geodätische Einbettung der Mannigfaltigkeit N. Die p-Energie wird dann durch die regularisierte Energie

$$(8) \qquad E_\varepsilon(f) = \frac{1}{p} \int_M \left(\varepsilon + |df|^2\right)^{\frac{p}{2}} d\mu$$

approximiert und der zu $E_\varepsilon(f)$ gehörige Wärmefluß untersucht. Durch geeignete Wahl der total geodätischen Einbettung von N erreicht man, daß die Linearisierung des regularisierten p-Laplace Operators uniform elliptisch im Legendreschen Sinne wird. Die Theorie der analytischen Halbgruppen liefert dann lokale Existenz des regularisierten Flusses. Allerdings hängt das Existenzintervall zunächst vom Regularisierungsparameter ε ab, und es muß gezeigt werden, daß es nicht mit $\varepsilon \to 0$ gegen Null schrumpft. Dies gelingt mit Hilfe der bereitgestellten a priori Abschätzungen, welche auf den regularisierten Fluß übertragen werden können. Mit Hilfe eines Approximationsargumentes gelingt es dann lokale Existenz für Anfangsdaten aus dem natürlichen Raum $W^{1,p}(M)$ zu erhalten. Für kleine Anfangsenergie ist die Existenz sogar global. Schließlich erhält man auch globale Existenz ohne die Kleinheitsbedingung, eine Charakterisierung der singulären Menge sowie Aussagen über die Regularität der Lösung außerhalb dieser Menge. Singularitäten können nur zu endlich vielen Zeiten auftreten und die Zahl singulärer Zeiten ist beschränkt durch die Anfangsenergie der Lösung, da zu jeder singulären Zeit die p-Energie mindestens um einen Betrag $\varepsilon_1 > 0$ abnimmt (diese Konstante hängt nur von den Mannigfaltigkeiten M und N ab). Das Hauptresultat ist das Theorem 4.9.

Für $p = m = 2$ hat Struwe in [107] Eindeutigkeit des 2-harmonischen Flusses in einer mit $V(M^T, N)$ bezeichneten Klasse gezeigt (siehe auch Abschnitt 3.2). Andererseits würde Eindeutigkeit des p-harmonischen Flusses in der Klasse $L^\infty(0, T; W^{1,p}(M))$ Regularität schwach p-harmonischer Abbildungen implizieren: Falls nämlich eine schwach p-harmonische Abbildung $f \in W^{1,p}(M, N) \setminus W^{1,\infty}(M, N)$ mit $p = \dim(M)$ existiert, so gibt dies Anlaß zu einem nicht eindeutigen Fluß. Wie wir in Kapitel 4 zeigen, besitzt der p-harmonische Fluß mit Anfangswerten f eine lokale schwache Lösung g derart, daß Gradient $|\nabla g|$ endlich ist für $t > 0$. Andererseits ist f (als zeitlich konstant betrachtet) eine schwache Lösung des p-harmonischen Flusses mit denselben Anfangsdaten, jedoch mit stets unbeschränktem Gradienten.

Im Kapitel 5 zeigen wir Eindeutigkeit der Lösung des p-harmonischen Flusses in der Klasse $L^\infty(0, T; W^{1,\infty}(M))$.

Die Frage, ob der konforme Fluß für $p > 2$ in endlicher Zeit Singularitäten entwickelt (so wie dies im konformen Falle für $p = 2$ das berühmte Beispiel von Chang, Ding und Ye zeigt), bleibt offen. Immerhin scheinen die numerischen Berechnungen im Abschnitt 5.3 die Vermutung zu stützen, daß auch der konforme Fluß für $p > 2$ das "finite-time blow-up" Phänomen zeigt.

Chapter 1

Introduction

1.1 The p-energy functional

Throughout this text M and N will be smooth Riemannian manifolds without boundary of dimension m and n respectively. M and N are equipped with Riemannian metrics γ and g: On each tangent space T_xM to M we are given an inner product γ_x depending differentiably on the point $x \in M$. Similarly, on the manifold N we have a family g_y of inner products on the tangent spaces T_yN. In local coordinates $x^1, \ldots, x^m$ on M the metric γ is described by a positive definite symmetric matrix function $(\gamma_{\alpha\beta})_{1\leq\alpha,\beta\leq m}$, where $\gamma_{\alpha\beta} = \langle \partial_\alpha, \partial_\beta \rangle$ denotes the inner product of the coordinate vector fields $\partial_\alpha = \frac{\partial}{\partial_\alpha}$. Thus, for general tangent vectors $\varphi = \sum\limits_{\alpha=1}^{m} \varphi^\alpha \partial_\alpha$ and $\psi = \sum\limits_{\alpha=1}^{m} \psi^\alpha \partial_\alpha$ in T_xM we get

$$\gamma_x(\varphi, \psi) = \sum_{\alpha,\beta=1}^{m} \gamma_{\alpha\beta}(x) \varphi^\alpha \psi^\beta.$$

On N the situation is similar. We usually use the summation convention that indices occurring twice in a formula are to be summed on a range that will be evident in the context in question.

The p-energy of a differentiable mapping $f : M \to N$ will be defined as the integral of the p-th power of the derivative df. To make this precise, we first recall that the differential df_x maps the tangent space T_xM linearly to $T_{f(x)}N$ for each $x \in M$. There is a generic inner product and norm on the vector space of linear mappings S, T between two inner product spaces. It is given by

$$S \cdot T := \operatorname{trace} T^* S = \operatorname{trace} S^* T, \quad |S| := \sqrt{S \cdot S}\,,$$

where the star denotes the adjoint that is built with respect to the underlying inner products. Using this norm on the space of linear mappings from T_xM with inner product γ_x to $T_{f(x)}N$ with inner product $g_{f(x)}$ we define the norm $|df_x|$ of the derivative df_x and use then the smooth measure μ on M associated with the Riemannian metric γ to integrate $|df_x|$ over M:

Definition 1.1 *The p-energy density of a C^1-mapping $f : M \to N$ is*

$$e(f)(x) := \frac{1}{p}\,|df_x|^p = \frac{1}{p}\,\left(\operatorname{trace}\left((df_x)^* df_x\right)\right)^{\frac{p}{2}}$$

and the p-energy (integral) of f is

$$E(f) := \int\limits_M e(f)\, d\mu$$

where the norm $|\cdot|$, the adjoint star and the measure μ are associated with the given Riemannian metrics on M and N.

Note that we need to know existence of df_x only μ almost everywhere on M to define $E(f)$. The p-energy integral then exists at least for compact M; otherwise it may diverge.

For concrete calculations we will need an expression of the p-energy in local coordinates: Consider linear mappings S, $T : T_xM \to T_yN$ with matrix representations (s^i_α) and (t^i_α), i.e.

$$S\partial_\alpha = s^i_\alpha \partial_i, \quad T\partial_\alpha = t^i_\alpha \partial_i\,,$$

where again ∂_α with greek index denotes the coordinate vector fields on M at x, whereas ∂_i with latin index denotes the coordinate vector fields on N in y. This gives the following expression for the inner product of S and T:

$$S \cdot T = \gamma^{\alpha\beta}(x) g_{ij}(y) s^i_\alpha t^j_\beta\,,$$

where $(\gamma^{\alpha\beta}(x))$ is the inverse of the coefficient matrix $(\gamma_{\alpha\beta}(x))$ of the Riemannian metric γ on M at x. Consequently, if $f : M \to N$ is locally given by n functions f^i of m variables, and hence

$$df_x\partial_\alpha = \partial_\alpha f^i(x)\partial_i\,,$$

we obtain the coordinate expression for the p-energy density

$$e(f)(x) = \frac{1}{p}\left(\gamma^{\alpha\beta}(x) g_{ij}(f(x))\partial_\alpha f^i(x)\partial_\beta f^j(x)\right)^{\frac{p}{2}}\,. \tag{1.1}$$

For a representation of the measure μ on M we use the local expression $d\mu = \sqrt{\gamma}\, dx$, where

$$\sqrt{\gamma} := \sqrt{|\det(\gamma_{\alpha\beta})|}$$

denotes the Jacobian of the coordinate system on M with respect to γ. So, we infer the coordinate expression for the p-energy integral of f

$$E_U(f) = \int\limits_U e(f) d\mu = \frac{1}{p}\int\limits_\Omega \left(\gamma^{\alpha\beta}(g_{ij} \circ f)\partial_\alpha f^i \partial_\beta f^j\right)^{\frac{p}{2}} \sqrt{\gamma}\, dx\,.$$

Of course, $U \subset M$ and $\Omega \subset \mathbb{R}^m$ denote the domain and the range of the coordinates on M and it is assumed that $f(U)$ is contained in the domain of the coordinates chosen on N.

Quite often it is more convenient to assume that the target manifold N is a submanifold of some Euclidean space $\mathbb{R}^k$ with induced Riemannian metric, i.e. the inner product of two tangent vectors to N at a point $y \in N$ is simply their Euclidean inner product, denoted by '$\cdot$'. Then the p-energy density of a mapping $f : M \to N$ equals the p-energy density of the same mapping but viewed as a map $F = (F^1, \ldots, F^k)$ from M into the Euclidean space $\mathbb{R}^k$. With respect to the local coordinates on M we get

$$e(f) = e(F) = \frac{1}{p} \left(\gamma^{\alpha\beta} \partial_\alpha F \cdot \partial_\beta F\right)^{\frac{p}{2}} . \tag{1.2}$$

Note, that according to Nash's embedding theorem it is always possible to find an isometrical embedding of N in some $\mathbb{R}^k$ if N is compact: see Nash [86] or Günther [58].

1.2 Conformal invariance of the p-energy

How does the p-energy behave under the action of a mapping? Let K be another smooth Riemannian manifold with metric h of the same dimension as the manifold M and consider the mappings

$$\begin{aligned} \varphi :\ & K \to M\,, \\ f :\ & M \to N\,, \\ f \circ \varphi :\ & K \to N\,. \end{aligned}$$

The chain rule implies for the derivative of the composition $f \circ \varphi$ at $y \in K$

$$d(f \circ \varphi)_y = df_{\varphi(y)} \circ d\varphi_y\,.$$

Thus, when φ is a *conformal* mapping of K to the manifold M the following is true:

$$|d(f \circ \varphi)_y| = \left(J_{\varphi_y}\right)^{\frac{1}{m}} |df_{\varphi(y)}|\,.$$

Here, $\left(J_{\varphi_y}\right)$ denotes the Jacobian of φ in the point $y \in K$. Hence, we obtain

Proposition 1.2 *The p-energy is invariant under conformal mappings $\varphi : K^m \to M^m$, i.e. for all $f \in C^1(M, N)$ there holds $E_K(f \circ \varphi) = E_{\varphi(K)}(f)$, if and only if the dimension m of the domain manifold equals p.*

This fact is of course well-known in the case of the Dirichlet integral which we get in case $p = 2$. The so called conformal capacity in higher dimensions has been studied by C. Loewner (see [79]). The conformal invariance of the p-energy is a convenient tool in considerable parts of the theory since one is allowed to switch freely between conformal charts. Especially in two dimensions we recall the fact that every Riemannian surface is locally conformal flat.

There are other reasons why the conformal case is of special interest: Another place where the dimension of the domain manifold enters the discussion are the diverse Sobolev embeddings. There it will turn out that the limiting case $\dim(M) = p$ admits a more accurate theoretical treatment.

1.3 The first variation of the p-energy

Given a class F of mappings from M to N defined by boundary conditions on ∂M (if there is a boundary) and possibly additional topological conditions, e.g. a homotopy class, we may try to minimize the p-energy within the class F. One may hope that the solution of this minimization problem is a map in F with particular analytic and geometric properties. A necessary condition that must be satisfied by a p-energy minimizing map f is, that the first variation of the p-energy must vanish at f for all variations of f with compact support in the interior of M. To compute the first variation of the p-energy we consider variations $f_t : M \to N$ of $f = f_0$ for $|t|$ small such that $f_t(x)$ is of class C^1 in the variables (x, t) and $f_t(x) = f(x)$ for all t and all x outside some compact subset $K \subset M$ which is contained in the domain U of a coordinate system $x^1, \dots, x^m$ on M with range Ω and is mapped into the domain of a fixed coordinate system $y^1, \dots, y^m$ on N by all f_t. Now, the support of the initial vector field of the variation

$$V(x) := \frac{d}{dt} f_t(x)\bigg|_{t=0} \in T_{f(x)}N$$

is contained in K. We also assume that V is of the class C^1 and $\partial_\alpha V^i = \frac{d}{dt}\partial_\alpha f_t^i\big|_{t=0}$. We remark that it is no restriction to assume $f_t^i(x) = f_i(x) + tV^i(x)$. Now, $E(f_t)$ is finite for small $|t|$ if $E(f)$ is finite and we may differentiate under the integral sign in the first variation

$$\begin{aligned}
\delta E(f, V) &:= \frac{d}{dt} E(f_t)\bigg|_{t=0} = \\
&= \frac{d}{dt} \int_\Omega \frac{1}{p} \left(\gamma^{\alpha\beta}(g_{ij} \circ f_t)\partial_\alpha f_t^i \partial_\beta f_t^j\right)^{\frac{p}{2}} \sqrt{\gamma}\, dx \Bigg|_{t=0} = \\
&= \int_\Omega \left(\gamma^{\alpha\beta}(g_{ij} \circ f_t)\partial_\alpha f_t^i \partial_\beta f_t^j\right)^{\frac{p}{2}-1} \gamma^{\alpha\beta}\Big((g_{ij} \circ f)\partial_\alpha f^i \partial_\beta V^j + \\
&\quad + \frac{1}{2}(g_{ij,k} \circ f)\partial_\alpha f^i \partial_\beta f^j V^k\Big)\sqrt{\gamma}\, dx\,.
\end{aligned}$$

Here, an index separated by a comma denotes the derivative with respect to the indexed variable, e.g. $g_{jl,j} := \frac{\partial}{\partial y^l} g_{il}$ etc. If f is of class C^2 we may use integration by parts and after a short calculation we find that vanishing first variation is in that case equivalent to

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \right) = \tag{1.3}$$
$$= -\frac{1}{2} \left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} g_{il,j} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^l}{\partial x^\beta}$$

in Ω for $j = 1, \ldots, n$. Now, we introduce the Christoffel symbols

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} (g_{ik,j} - g_{ij,k} + g_{jk,i}) \tag{1.4}$$

of the metric g with respect to the coordinates chosen on N. By means of the quantities (1.4) we can rewrite the equation (1.3) to get the equivalent form

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \frac{\partial f^l}{\partial x^\alpha} \right) = \tag{1.5}$$
$$= -\left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \Gamma^l_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}$$

in Ω for $l = 1, \ldots, n$. The systems (1.3) and (1.5) of partial differential equations are referred to as Euler-Lagrange equations of the p-energy. So, we conclude

Proposition 1.3 *For f of class C^2, coordinates on $U \subset M$ with range Ω and coordinates on N with domain containing $f(U)$ the first variation $\delta E(f, V)$ vanishes for all C^1 vector fields $V(x) \in T_{f(x)}N$ with compact support in the interior of U if and only if the Euler-Lagrange equations of the p-energy functional hold in Ω.*

Definition 1.4 *The operator on the left hand side of (1.5) is called p-Laplace operator related to the manifolds M and N and is denoted by Δ_p:*

$$\Delta_p f := \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \frac{\partial f^l}{\partial x^\alpha} \right).$$

We suppress the related manifolds M and N in the notation.

Example 1.1 Δ_2 is simply the Laplace-Beltrami operator of the manifold M and does not depend on N. ○

Δ_2 is a linear elliptic diagonal matrix operator in divergence form. For $p > 2$ ($0 < p < 2$) the operator is degenerate (singular) at $\nabla f = 0$. The right hand side of (1.5) is for $p = 2$ a quadratic form in the first derivatives of f with coefficients depending on f. These strong nonlinearities are caused by the non-Euclidean structure of the target manifold N and cannot be removed by special choices of coordinates on N unless N is locally isometric to Euclidean space $\mathbb{R}^n$. But even in this case the space of mappings from M to N does not possess a natural linear structure unless N itself is a linear space. In general, the right hand side of (1.5) is of the order of the p-th power of the gradient of f.

For $p = 2$ there is a general theory of nonlinear elliptic systems of partial differential equations with linear second order diagonal principal part of divergence form and first order nonlinearities with quadratic growth. The Euler-Lagrange equation of the 2-energy is a model for this theory. Nevertheless questions arising from the global structure of the target manifold N cannot be answered by this kind of theory.

Definition 1.5 *A mapping $f : M \to N$ is called p-harmonic if it is of class C^2 and satisfies the Euler-Lagrange equations for all choices of coordinates on M and N.*

Remark: Using variations in the *domain*, i.e. variations of the form

$$f_t(x) = f(x + t\zeta(x))$$

where $\zeta = (\zeta^1, \ldots, \zeta^m)$ with each $\zeta^j \in C_0^\infty(B_\rho(y))$, we obtain from $\frac{d}{dt} f_t(x)\big|_{t=0} = 0$ an equation whose classical solutions contain the set of the p-harmonic maps. However its weak solutions need not contain the set of weak solutions of (1.5). Weak solutions to *both* of the equations are usually referred to as "stationary p-harmonic maps" (see Evans [36]). The above discussion thus proves that p-energy minimizing implies stationary p-harmonic. Weakly p-harmonic maps admit in general far worse singularities than the energy minimizing maps. Many results are known about stationary p-harmonic maps. See e.g. Duzaar and Fuchs [28], Fuchs [42] and [45] and for some recent results on this topic Strzelecki [117] and [118].

The above intrinsic formulation of the Euler-Lagrange equations (1.3) or (1.5) is not always appropriate. If N is compact, we may think of N as being isometrically embedded in some $\mathbb{R}^k$. Let $S \subset \mathbb{R}^k$ be a tubular neighborhood of N and $\pi_N : S \to N$ the (smooth) nearest-neighbor projection. Denote $T_pN \subset T_p\mathbb{R}^k$ the tangent space to N at a point $p \in N$. Let $\varphi \in C_0^1(M, \mathbb{R}^k)$ satisfy

$$\varphi(x) \in T_{f(x)}N$$

for all $x \in M$ and φ having compact support in a single coordinate chart of M. Then φ induces a C^1-variation at $f : M \to N \subset \mathbb{R}^k$:

$$f_t = \pi_N \circ (f + t\varphi)\,.$$

Now, we can calculate the first variation and get

$$\text{(1.6)} \qquad \frac{d}{dt}E(f_t)\Big|_{t=0} = \int_M \left(\gamma^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\cdot\frac{\partial f}{\partial x^\beta}\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\cdot\frac{\partial \varphi}{\partial x^\beta}\sqrt{\gamma}\,dx = -\int_M \Delta_p f\cdot\varphi\sqrt{\gamma}\,dx$$

where μ is the measure on M and Δ_p denotes the p-Laplace operator related to M and $\mathbb{R}^k$, i.e.

$$\Delta_p f = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\gamma^{\alpha\beta}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\right).$$

Thus, the first variation vanishes at a C^2 map f if and only if

$$\text{(1.7)} \qquad \Delta_p f \perp T_f N\,.$$

We can make (1.7) more explicit if we introduce a local orthonormal frame ν_{n+1}, $\ldots,\nu_k$ for $(T_pN)^\perp$, the orthogonal complement of T_pN in $\mathbb{R}^k$. Then, by (1.7) there exist scalar functions $\lambda^{n+1},\ldots,\lambda^k$ such that

$$\text{(1.8)} \qquad -\Delta_p f = \sum_{l=n+1}^{k}\lambda^l(\nu_l\circ f)\,.$$

Multiplying (1.8) by $\nu_i\circ f$ (i fixed) and using the fact that $\dfrac{\partial f}{\partial x^\alpha}\cdot\nu_l(f)=0$ for all α, we obtain

$$\text{(1.9)} \qquad \lambda^i = \left(\gamma^{\alpha\beta}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial \nu_i^j(f)}{\partial x^\beta}\,.$$

Which one of the three given forms of the Euler-Lagrange equation describing p-harmonic mappings we use will be decided from case to case.

Example 1.2 If $M=T^m=\mathbb{R}^m/\,\mathbb{Z}^m$ and $N=S^n\subset\mathbb{R}^{n+1}$, equation (1.8) simply becomes

$$\text{(1.10)} \qquad -\nabla(|\nabla f|^{p-2}\nabla f) = |\nabla f|^p f\,.$$

If further $m=1$, i.e. $M=S^1$, it is easy to check that the constant mappings $f\equiv c\in S^n$ and the isometric embeddings $S^1\subset S^n$ are solutions of (1.10). ◯

Since we started with a coordinate free definition of the p-energy, it should be possible to derive a coordinate free formulation of the Euler-Lagrange equations. If

(f_t) is a variation of $f : M \to N$ with variational vector field $V := \left.\frac{d}{dt} f_t\right|_{t=0}$ then we find for the first variation

$$\left.\frac{d}{dt} E(f_t)\right|_{t=0} = \int_M (pe(f))^{1-\frac{2}{p}} \nabla V \cdot df \, d\mu \,. \tag{1.11}$$

Here, ∇ denotes the pull-back covariant derivative in the bundle $T^*M \otimes f^{-1}TN$. If f is of class C^2 it is not hard to obtain from (1.11) the desired coordinate free form of the Euler-Lagrange equation:

$$e(f)\tau(f) + (1 - \frac{2}{p}) df \operatorname{grad}_\gamma e(f) = 0 \,. \tag{1.12}$$

Here, $\tau(f)$ denotes the tension field of f: $\tau(f) = \operatorname{trace}_\gamma \nabla df$. For $p = 2$ equation (1.12) reduces simply to

$$\tau(f) = 0 \,,$$

a formula that is derived in Steffen [89] in a very lucid way.

Example 1.3 The identity mapping $\mathrm{id}_M : M \to M$ of a Riemannian manifold is p-harmonic. Since $d(\mathrm{id}_M)$ has constant coefficients with respect to all coordinate systems on M, $\nabla d(\mathrm{id}_M) = 0$ and the first term in (1.12) vanishes. On the other hand, the p-energy density $e(\mathrm{id}_M)$ is constant and hence the gradient in the second term disappears, too. We remark that in general id_M is not p-energy minimizing within its homotopy class (see Eells-Lemaire [31]). ○

1.4 Weakly p-harmonic maps

Using the direct method of the calculus of variations to find a function which minimizes the p-energy, the function one obtains may fail to be smooth. Thus, using this approach, one may not expect to get a p-harmonic mapping in the classical sense. However, the resulting function will be weakly p-harmonic, i.e. it will satisfy the Euler-Lagrange equations for p-harmonic mappings in the sense of distributions. We use (1.6) to formulate the definition:

Definition 1.6 *A mapping $f : M \to N \subset \mathbb{R}^k$ is called weakly p-harmonic if it satisfies the Euler-Lagrange equations for p-harmonic mappings in the sense of distributions, i.e. for each coordinate chart Ω on M there holds*

$$\int_\Omega \left(\gamma^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \cdot \frac{\partial f}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \cdot \frac{\partial \varphi}{\partial x^\beta} \sqrt{\gamma} \, dx = 0$$

for all smooth mappings φ with compact support in Ω and satisfying $\varphi(x) \in T_{f(x)}N$ for all $x \in M$.

Considering the p-energy $E(f)$ of mappings $f : M \to N$, experience from the calculus of variations shows that there is only one possible choice of topology for which the direct method to minimize energy can be expected to work successfully. Namely the weak topology with respect to a Sobolev $W^{1,p}$-norm. But here we encounter a fundamental difficulty: The target manifold N is not a linear space and hence there is no obvious definition of a Sobolev space $W^{1,p}(M, N)$ of p-integrable functions from M to N with p-integrable derivatives. For two functions $f, g : M \to N$ we could try to define the deviation in the p-integral sense by integrating up the p-th power of the Riemannian distance $d_N(f(x), g(x))^p$ of the values $f(x)$ and $g(x)$ in N with respect to the Riemannian measure on M. But there is no immediate way to define the p-integral deviation of the derivatives df and dg since $\partial_\alpha f(x) \in T_{f(x)}N$ and $\partial_\alpha g(x) \in T_{g(x)}N$ belong to two different tangent spaces of N, in general. But since we will deal only with compact manifolds, we may think of N as being embedded in some Euclidean space $\mathbb{R}^k$. Thus, the following definition becomes reasonable.

Definition 1.7 *The space of mappings of the class $W^{1,p}$ from M to N is*

$$W^{1,p}(M, N) := \{f \in W^{1,p}(M, \mathbb{R}^k);\ f(x) \in N \text{ for } \mu\text{-almost all } x \in M\}$$

equipped with the topology inherited from the topology of the linear Sobolev space $W^{1,p}(M, \mathbb{R}^k)$.

The nonlinear space $W^{1,p}(M, N)$ defined above depends on the embedding of N in the $\mathbb{R}^k$. This fact does not cause any problems if M and N are compact: In that case different embeddings give rise to homeomorphic spaces $W^{1,p}(M, N)$. Using an *isometric* embedding of N in $\mathbb{R}^k$ the p-energy density of a function $f \in W^{1,p}(M, N)$ may be defined by the expression (1.1) that we had for smooth functions. Note that the partial derivatives $\partial_\alpha f$ (f viewed as a function of M to $\mathbb{R}^k$) exist μ almost everywhere on M. Again, the scalar product of $\partial_\alpha f(x)$ and $\partial_\beta f(x)$ in the tangent space $T_{f(x)}N$ with respect to the Riemannian metric $g_{f(x)}$ of N and the Euclidean scalar product of $\partial_\alpha f(x)$ and $\partial_\beta f(x)$ viewed as vectors in $\mathbb{R}^k$ coincide. Thus, the p-energy of $W^{1,p}(M, N)$ mappings f is an intrinsic quantity, independent of the chosen isometric embedding of N. This is of course not true for the p-integral of f itself. Thus, the norm

$$\left(\int_M |f|^p d\mu + \int_M |df|^p d\mu\right)^{\frac{1}{p}}$$

in $W^{1,p}(M, N)$ remains dependent on the embedding. For an intrinsic definition of the space $W^{1,p}(M, N)$ see Federer [37]. We will content ourselves with the given definition since different embeddings lead to isomorphic spaces $W^{1,p}(M, N)$ with the same topology in the case when M and N are compact. A basic observation involving the above notions is the following proposition:

Proposition 1.8 *The p-energy functional is (sequentially) weakly lower semicontinuous on $W^{1,p}(M, N)$.*

The space $H^{1,p}(M,N)$ defined as the closure of the class of smooth functions from M to N in the $W^{1,p}$-norm is contained in $W^{1,p}(M,N)$ but does not coincide with the latter space in general (this fact gives rise to the so called "gap phenomenon" of Hardt-Lin [61]). This important observation was first made by Schoen and Uhlenbeck: see Eells and Lemaire [31] as a main reference. However, we have $H^{1,p}(M,N) = W^{1,p}(M,N)$ if $\dim(M) = p$ (see Schoen and Uhlenbeck [93], Bethuel [2] or Bethuel and Zheng [4]).

1.5 The Heat Flow Method

A basic existence problem for p-harmonic maps is the following:

Homotopy problem: *Given a map $f_0 : M \to N$ is there a p-harmonic map f homotopic to f_0?*

This question is very well understood in the case $p = 2$ of harmonic mappings. In that particular case, the answer is affirmative if the sectional curvature K^N of N is non-positive (see Eells-Sampson [32]), or—in case of a two-dimensional surface M—if the second fundamental group of N is trivial: $\pi_2(N) = 0$ (see Lemaire [74] and Sacks-Uhlenbeck [88]). Eells and Wood destroyed the hope for a more general theorem by the following counterexample in [33].

Example 1.4 If $f : T^2 \to S^2$ is 2-harmonic, then $\deg f \neq \pm 1$. ○

Another counterexample has been given by Lemaire in [74]. Thus, in general the attempt to solve the homotopy problem by minimizing E within a given homotopy class will fail: Homotopy classes are not weakly closed in $W^{1,p}(M,N)$ generally. This is made explicit by the following example, whose construction relies on the fact that the conformal group on S^p acts non-compactly on $W^{1,p}(S^p, S^p)$.

Example 1.5 Let $D_\lambda : \mathbb{R}^p \to \mathbb{R}^p, x \mapsto \lambda x$ denote the dilation by factor λ and $\pi_q : S^p \setminus \{q\} \to \mathbb{R}^p$ the stereographic projection with respect to a point $q \in S^p$. Both, D_λ and π_q are conformal mappings. Thus, by Example 1.3 and Section 1.2 the composition

$$f_\lambda = \pi_q^{-1} \circ D_\lambda \circ \pi_q \ : \ S^p \to S^p$$

is p-harmonic for any choice of λ. All the f_λ have p-energy $e_p := p^{\frac{2}{p}-1}|S^p|$. Any other homotopically nontrivial map $f : S^p \to S^p$ is necessarily surjective. Then, using the fact that the geometric mean is smaller than the arithmetic mean, and hence that $|J_f|^{\frac{2}{p}} \leq \frac{1}{p}|df|^2$, we get

$$\begin{aligned} e_p &= p^{\frac{p}{2}-1}|S^p| \leq \\ &\leq p^{\frac{p}{2}-1}\int_{S^p} |J_f|\, d\mu \leq \\ &\leq p^{\frac{p}{2}-1}\int_{S^p} \left(\frac{1}{p}\right)^{\frac{p}{2}} |df|^p d\mu = E(f)\,. \end{aligned}$$

Thus, the maps f_λ are even p-energy minimizing within the homotopy class of the identity $\mathrm{id} = f_1 : S^p \to S^p$. But on the other hand we have

$$f_\lambda \rightharpoonup f_\infty \equiv \{q\} \qquad (\lambda \to \infty)$$

weakly in $W^{1,p}(S^p, S^p)$. ○

The examples given above show that it may be difficult or impossible to solve the homotopy problem for p-harmonic maps by direct variational methods. As the key idea to get around the trouble one encounters dealing with the homotopy problem and to attack the problem from another direction, Eells and Sampson proposed in [32] to study the heat flow related to the 2-energy:

(1.13) $$f_t - \Delta_2 f = A(f)(\nabla f, \nabla f)_M \qquad \text{on } M \times [0,\infty[$$

with initial and boundary data

$$f = f_0 \text{ at } t = 0 \text{ and on } \partial M \times [0,\infty[$$

for maps $f : M \times [0,\infty[\to N \subset \mathbb{R}^k$. Here, $A(f) : T_f N \times T_f N \to (T_f N)^\perp$ is the second fundamental form of N: see Section 1.3. The idea behind this strategy is of course that a continuous deformation $f(\cdot,t)$ of f_0 will remain within the given homotopy class. Since (1.13) may be interpreted as the L^2-gradient flow for the 2-energy, one may hope that the solution $f(\cdot,t)$ for $t \to \infty$ will come to a rest at some critical point of E that is a harmonic map. For target manifold N, satisfying the geometric restrictions mentioned above, this program has been applied with success for $p = 2$.

A new approach to the homotopy problem for p-harmonic maps has been given by Duzaar and Fuchs in [29]: they extended the Eells-Sampson result to the case $p \in [2,\infty)$ by using an asymptotic analysis of the not degenerate energy $\int_M (\varepsilon + |df|^2)^{p/2} d\mu$, $\varepsilon > 0$. We will also use this regularized energy functional in Chapter 4 to study the conformal flow and we will solve the homotopy problem in the conformal case at least for small energy by constructing a regular flow.

The aim of the next chapters will be to study the heat flow related to the p-energy functional

(1.14) $$f_t - \Delta_p f = (pe(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)_M \qquad \text{on } M \times [0,\infty[$$

with initial data

$$f = f_0 \text{ at } t = 0$$

for maps $f : M \times [0,\infty[\to N \subset \mathbb{R}^k$ (compare Section 1.3).

To get familiar with the notion of p-harmonic flow we close this section with the following simple example:

Example 1.6 Consider the heat flow of S^1 into S^n with initial map f_0 being a scaled embedding of S^1 in S^n. We make the following ansatz for a solution

$$f(s,t) = (r(t)\cos(s), r(t)\sin(s), z(t), \dots, z(t))$$

with radius $r(t)$ and $r^2(t) + (n-1)z^2(t) = 1$. s is the arclength on S^1. Thus, the equation for the heat flow reduces to an ordinary differential equation for $r(t)$:

$$\frac{dr}{dt} = -r^{p-1}(1-r^2)\,. \tag{1.15}$$

For $r(0) = r_0 = 1$ we get a stationary solution of the considered heat flow, whereas an initial circle with radius $r_0 < 1$ shrinks to a point. The shrinking time is finite for $p < 2$ and infinite for $p \geq 2$ since the integral

$$\int_0^{r_0} \frac{dr}{r^{p-1}(1-r^2)}$$

diverges exactly for $p \geq 2$. For $p < 2$ and shrinking time t_0 the leading term in the expansion of $E(f(t))$ in t_0 is $c(t_0 - t)^{\frac{p}{2-p}}$ and hence the p-harmonic flow is not very smooth in this case.

The general solution of (1.15) is

$$t(r) = \begin{cases} \frac{1}{2}\log\left(\frac{r_0^2}{r^2}\,\frac{1-r^2}{1-r_0^2}\right), & \text{if } p = 2 \\[2ex] \dfrac{r_0^p r^2 F\left(1, 1-\frac{p}{2}, 2-\frac{p}{2}, r^2\right) - r^p r_0^2 F\left(1, 1-\frac{p}{2}, 2-\frac{p}{2}, r_0^2\right)}{(p-2)r^p r_0^p}, & \text{if } p \neq 2 \end{cases}$$

where F denotes the hypergeometric function. The following pictures show the behaviour of the solution for some values of p: Starting with radius $r_0 = 0.95$ at time $t = 0$ the plots show the graph of $r(t)$. ◯

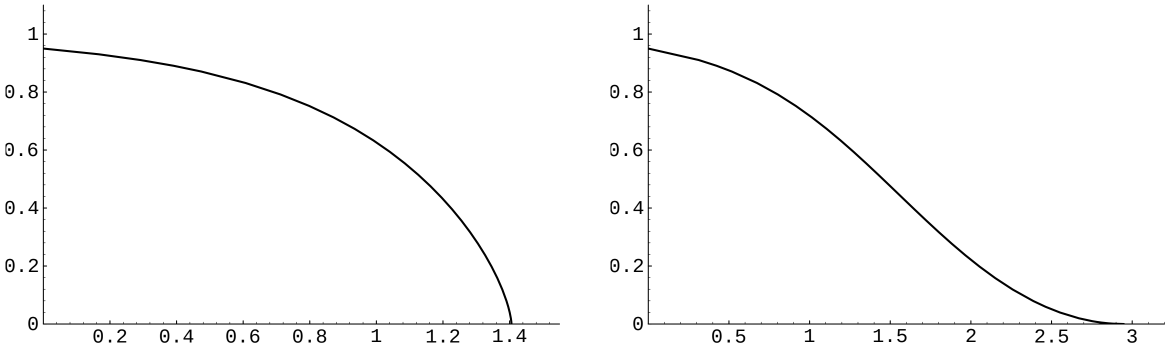

Figure 1.1: *Finite vanishing time:* $p = \frac{1}{2}$ *on the left and* $p = \frac{3}{2}$ *on the right.*

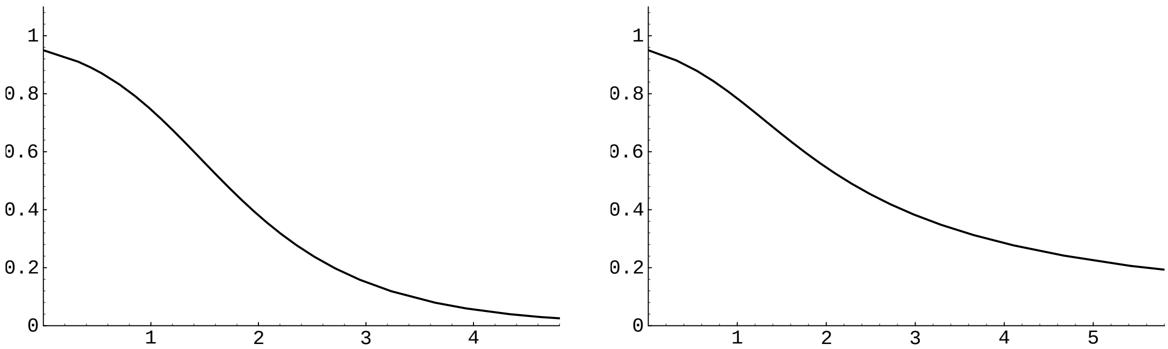

Figure 1.2: *Infinite vanishing time:* $p = 2$ *on the left and* $p = 3$ *on the right.*

Chapter 2

p-harmonic Flow into Spheres

2.1 Introduction

In this chapter M is a compact m-dimensional smooth Riemannian manifold without boundary and N is the unit sphere S^n of $\mathbb{R}^{n+1}$. In local coordinates $x^1, x^2, \ldots, x^m$ the metric γ on M is represented by the matrix $(\gamma_{\alpha\beta})_{m\times m}$. The p-harmonic mappings $f : M \to S^n$ are the C^2-solutions of the equation

$$-\Delta_p f = p\, e(f) f \tag{2.1}$$

where Δ_p denotes the p-Laplace operator related to the manifolds M and $\mathbb{R}^{n+1}$ and $e(f)$ is the p-energy density of f as defined in the previous chapter. A map $f \in W^{1,p}(M, S^n)$ satisfying (2.1) in the sense of distributions is called a weakly p-harmonic map. $W^{1,p}(M, S^n)$ is the Sobolev space defined in Section 1.4.

Example 2.1 Let $f : B_1(0) \subset \mathbb{R}^n \to S^{n-1}$ be the mapping

$$f : x \mapsto \frac{x}{|x|}\,.$$

Then, f is a weakly p-harmonic map for any $p \in [2, n[$ (for $p \geq n$ the energy gets infinite). ○

One way to produce more examples of (weakly) p-harmonic maps $f : M \to S^n$ is to study the heat flow related to the p-energy. The hope is that the flow will tend to a p-harmonic map for $t \to \infty$. If the flow is smooth this procedure also gives as a by-product the solution to the homotopy problem (see Section 1.5).

For $p = 2$ Yunmei Chen was able to show existence of global weak solutions of the heat flow of harmonic maps from compact Riemannian manifolds without boundary into spheres (see [11]). The same result was independently obtained by Keller, Rubinstein and Sternberg in [72] and also by Shatah in [96]. The main purpose of this chapter is to generalize this result to the case $2 < p < \infty$, that is to study the global existence of weak solutions of the following evolution problem for mappings $f : M \times [0, \infty[\to S^n \subset \mathbb{R}^{n+1}$

$$\partial_t f - \Delta_p f = pe(f)f \qquad \text{on } M \times [0, \infty[\tag{2.2}$$

$$f(\,\cdot\,, 0) = f_0 \qquad \text{on } M \tag{2.3}$$

$$|f| = 1 \qquad \mu\text{-a.e. on } M \times [0, \infty[, \tag{2.4}$$

where Δ_p is the p-Laplacian of M and $\mathbb{R}^{n+1}$, and $e(f)$ is the p-energy density of the mapping f. We will approximate a solution to (2.2)–(2.4) by the solutions to the penalized equation

$$\partial_t f_k - \Delta_p f_k + k \left| f_k^2 - 1 \right|^{2\alpha - 2} (f_k^2 - 1) f_k = 0 \tag{2.5}$$

on $M \times [0, \infty[$ and for $k = 1, 2, \ldots$. This equation comes from the penalized energy $E_k(f) := E(f) + F_k(f)$ for functions $f : M \to \mathbb{R}^{n+1}$ with

$$F_k(f) := \frac{k}{4\alpha} \int_M |f^2 - 1|^{2\alpha} d\mu \,.$$

The energy F_k punishes values of f which are away from S^n. Of course, the hope is that for $k \to \infty$ the stationary points of E_k will take values in the sphere and solve the problem (2.2)–(2.4). The constant α has to fulfill some technical condition, namely $\frac{1}{2} < \alpha < \frac{p}{4(m-p)} + \frac{1}{2}$ if $p \leq m$ and $\alpha = 1$ if $p > m$. For fixed k, we will use Galerkin's method to prove the existence of weak solutions of the penalized equation (2.5). The "monotonicity trick" will be used in this proof. However, due to the higher nonlinearity of the p-Laplacian for $p > 2$ we will have difficulties to prove that for $k \to \infty$, f_k converges to a map f which is a weak solution of (2.2)–(2.4). From the energy inequality we know that f_k is uniformly bounded in $L^\infty(0, T; W^{1,p}(M, \mathbb{R}^{n+1}))$ and that $\partial_t f_k$ is uniformly bounded in $L^2(0, T; L^2(M, \mathbb{R}^{n+1}))$. By a modification of Kondrachov's compactness theorem, we can prove that f_k strongly converges to f in $L^p(0, T; L^p(M, \mathbb{R}^{n+1}))$. The main difficulty is to prove that $|df_k|^{p-2} df_k$ converges to $|df|^{p-2} df$ weakly in $L^{p'}(0, T; L^{p'}(M))$. Fortunately, the term $k \left| f_k^2 - 1 \right|^{2\alpha - 2} (f_k^2 - 1) f_k$ is not "too bad" and uniformly bounded in $L^1(0, T; L^1(M, \mathbb{R}^{n+1}))$. Therefore, we can modify a compactness assertion presented by Evans in [35] to obtain the strong convergence of df_k in $L^q(0, T; L^q(M))$ for each $1 \leq q < p$ and to overcome this difficulty.

This chapter is based upon joint work with Yunmei Chen and Min-Chun Hong (see [14]).

2.2 Preliminaries and the penalized approximation equation

In this section we shall give some lemmas and prove the global existence of weak solutions to the penalized approximation equation.

First, we state a lemma so called the "decisive monotonicity trick" or "Minty's trick". A proof can be found in Zeidler [132].

Lemma 2.1 *Suppose that X is a reflexive Banach space and X^* is the dual space of X, suppose also that an operator $A : X \to X^*$ satisfies*

(i) A is monotone on X, i.e. $\langle Af - Ag, f - g\rangle \geq 0$ for all $f, g \in X$.

(ii) A is hemicontinuous, i.e. the map $t \mapsto \langle A(f + tg), h\rangle$ is continuous on $[0, 1]$ for all $f, g, h \in X$.

Then, it follows from

(iii) $f_n \rightharpoonup f$ weakly in X as $n \to \infty$, $Af_n \rightharpoonup b$ weakly in X^* as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle Af_n, f_n\rangle \leq \langle b, f\rangle$$

that

$$Af = b\,.$$

For $2 \leq p < \infty$, let $X = W^{1,p}(M, \mathbb{R}^{n+1})$ and $X^* = W^{-1,p'}(M, \mathbb{R}^{n+1})$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Then the operator $-\Delta_p$ may be considered as an operator from X to X^* by

$$-\Delta_p : \varphi \mapsto \left(\psi \mapsto \int_M |d\varphi|^{p-2}\,\mathrm{trace}(d\varphi^* d\psi)\, d\mu\right). \tag{2.6}$$

Of course, the operator $-\Delta_p$ is monotone. This is easily derived from the convexity of the p-energy functional (see Evans [35] or Brézis [5]). However, the simple monotonicity property of $-\Delta_p$ will not be enough for our purposes. We will need the so called strong monotonicity later, i.e.

$$\langle -\Delta_p f + \Delta_p g, f - g\rangle \geq c\,E(f - g) \tag{2.7}$$

holding for some constant c which only depends on the geometry of the manifold M. The strong monotonicity property of $-\Delta_p$ follows from the following lemma and the compactness of M.

Lemma 2.2 *Let $p \geq 2$. Then there holds for all $a, b \in \mathbb{R}^{nk}$*

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq 2^{p-2}|a - b|^p.$$

Proof
By a suitable rotation and dilatation, the problem reduces to two dimensions where the verification is elementary. □

In the sequel, we will often use the operator $-\Delta_p$ in its integrated form, i.e. as an operator mapping the space $L^p(0,T;W^{1,p}(M,\mathbb{R}^{n+1}))$ to its dual $L^{p'}(0,T;W^{-1,p'}(M,\mathbb{R}^{n+1}))$ by

$$-\Delta_p : \varphi \mapsto \left(\psi \mapsto \int_0^T \int_M |d\varphi|^{p-2} \operatorname{trace}(d\varphi^* d\psi)\, d\mu\, dt \right) . \tag{2.8}$$

By Lemma 2.2, it is now easy to verify that $-\Delta_p$ defined in (2.8) satisfies the assumptions (i) and (ii) on the operator A in Lemma 2.1.

Therefore we have

Corollary 2.1 *Consider $-\Delta_p$ as the operator defined in (2.8). Suppose that $\{f_n\}$ converges to f weakly in $L^p(0,T;W^{1,p}(M,\mathbb{R}^{n+1}))$ and that*

$$\limsup_{n\to\infty} \langle -\Delta_p f_n, f_n\rangle \le \int_0^T \int_M \operatorname{trace}(w^* df)\, d\mu\, dt \tag{2.9}$$

where w is the weak limit of the sequence $\{|df_n|^{p-2}df_n\}$ in $L^{p'}(0,T;L^{p'}(M,\mathbb{R}^{n+1}))$. Then it follows that the sequence $\{-\Delta_p f_n\}$ converges to $-\Delta_p f$ weakly in the space $L^{p'}(0,T;W^{-1,p'}(M,\mathbb{R}^{n+1}))$.

Proof
The weak convergence of $\{|df_n|^{p-2}df_n\}$ to w in $L^{p'}(0,T;L^{p'}(M,\mathbb{R}^{n+1}))$ implies

$$\langle -\Delta_p f_n, \varphi\rangle = \int_0^T \int_M |df_n|^{p-2} \operatorname{trace}(df_n^* d\varphi)\, d\mu\, dt \;\to$$

$$\to \int_0^T \int_M \operatorname{trace}(w^* d\varphi)\, d\mu\, dt \;=:\; \langle b, \varphi\rangle$$

for all $\varphi \in L^p(0,T;W^{1,p}(M,\mathbb{R}^{n+1}))$. In other words we have $-\Delta_p f_n \rightharpoonup b$ in $L^{p'}(0,T;W^{-1,p'}(M,\mathbb{R}^{n+1}))$. But now, (2.9) is just the additional condition we need to apply Lemma 2.1:

$$\limsup_{n\to\infty} \langle -\Delta_p f_n, f_n\rangle \le \langle b, f\rangle$$

and the assertion follows. □

Remark: The p-Laplace operator is even maximal monotone, i.e. from $\langle -\Delta_p f - g_0, f - f_0\rangle \ge 0$ for all $f \in X$ it follows $g_0 = -\Delta_p f_0$. The maximal monotonicity property follows from the hemicontinuity of the operator by the following lemma due to Browder and Minty:

Lemma 2.3 (Browder, Minty) *Let X be a real Banach space and $A : X \to X^*$ a hemicontinuous (not necessary monotone) operator which acts from X to its dual X^*. Let $f_0 \in X$ and $g_0 \in X^*$ be fixed. If $\langle Af - g_0, f - f_0\rangle \geq 0$ for all $f \in X$ then $g_0 = Af_0$.*

An easy proof can be found in Vainberg [125]. The proof we present in this section may be formulated also in terms of the concept of the maximal monotonicity.

Before we are able to prove the existence of global weak solutions of the penalized problem we mention the following useful compactness result.

Lemma 2.4 *Suppose that $\{f_l\}$ is bounded in $L^\infty(0,T;W^{1,p}(M)), 1 \leq p < \infty$ and $\{\partial_t f_l\}$ is bounded in $L^2(0,T;L^2(M))$. Then $\{f_l\}$ subconverges to f strongly in $L^r(0,T;L^r(M))$ for each r satisfying $p \leq r < \frac{mp}{m-p}$.*

Proof
By the Kondrachov compactness theorem (see e.g. Gilbarg and Trudinger [53]), we know that there exists a subsequence, still denoted by $\{f_l\}$, such that there holds

(2.10)
$$f_l \to f \text{ strongly in } L^1(0,T;L^1(M)) \text{ and } f \in L^\infty(0,T;W^{1,p}(M))$$

as $l \to \infty$. On the other hand, by using Hölder's inequality, we have the estimate

(2.11)
$$\int_0^T\int_M |f_l - f|^r \,d\mu\,dt \leq \left(\int_0^T\int_M |f_l - f| \,d\mu\,dt\right)^{\frac{p^*-r}{p^*-1}} \cdot \left(\int_0^T\int_M |f_l - f|^{p^*} \,d\mu\,dt\right)^{\frac{r-1}{p^*-1}}$$

where $p^* = \dfrac{mp}{m-p}$ denotes the Sobolev exponent. Now, (2.10) implies that the first factor in (2.11) converges to 0 for $l \to \infty$. The second factor in (2.11) is bounded because of the Sobolev inequality and the assumption that the sequence $\{f_l\}$ is bounded in $L^\infty(0,T;W^{1,p}(M))$:

$$\|f_l - f\|_{L^{p^*}(M)} \leq c\|f_l - f\|_{W^{1,p}(M)} < C \text{ for all } t \in [0,T] \text{ and all } l.$$

This proves the assertion. □

Remark: The above proof still works for the weakened assumption that the sequence $\{\partial_t f_l\}$ is bounded in $L^1(0,T;L^1(M))$.

We approximate a solution to (2.2)–(2.4) by solving the penalized equation (2.5) with the initial data

$$f_k(0,x) = f_0(x), \quad \forall x \in M, \quad k = 1,2,\dots. \tag{2.12}$$

We will use Galerkin's method and the decisive monotonicity trick respectively Minty's trick to solve the problem (2.5) and (2.12) for every fixed $k \geq 1$. We will also obtain an energy estimate.

Theorem 2.2 *For initial data $f_0 \in W^{1,p}(M,S^n)$, $2 \le p < \infty$ and every fixed $k \ge 1$, there exists a weak solution f_k for the penalized approximation equation (2.5) and (2.12). For any $t \ge 0$ this solution satisfies the energy inequality*

$$\int_0^t \|\partial_t f_k\|^2_{L^2(M)} dt + E_k(f_k(t)) \le E(f_0)\,. \tag{2.13}$$

Proof

In this proof we will suppress the index k in the notation. Assume that $\{w_j\}_{j\in\mathbb{N}}$, is a base in the space $W^{1,p}(M,\mathbb{R}^{n+1})$. For any fixed positive integer l, we try to find an approximate solution f_l in the form

$$f_l(t) = \sum_{i=1}^{l} g_{il}(t) w_i$$

such that

$$\langle f_l' - \Delta_p f_l + k\left|f_l^2 - 1\right|^{2\alpha-2}(f_l^2-1)f_l, w_j\rangle = 0 \tag{2.14}$$

or explicitly

$$\int_M \Big(f_l' w_j + |df_l|^{p-2}\,\mathrm{trace}(df_l^* dw_j) + k\left|f_l^2-1\right|^{2\alpha-2}(f_l^2-1)f_l w_j \Big)\, d\mu = 0$$

for $1 \le j \le l$, with

$$f_l(0) = f_{0l} \tag{2.15}$$

where

$$f_{0l} = \sum_{i=1}^{l} \xi_i w_i \to f_0 \text{ strongly in } W^{1,p}(M,\mathbb{R}^{n+1}) \text{ as } l \to \infty. \tag{2.16}$$

By the well-known results on the systems of ordinary differential equations, the Cauchy problem (2.14)–(2.15) has a unique short time solution f_l. Using the time derivative f_l' as a testfunction in (2.14) we find the following energy equality

$$\int_0^t \|f_l'(\tau)\|^2_{L^2(M)}\, d\tau + E_k(f_l(t)) = E_k(f_{0l})\,. \tag{2.17}$$

Then, by these uniform estimates on the local solution, there exists a global solution f_l such that $f_l \in L^\infty(0,\infty; W^{1,p}(M,\mathbb{R}^{n+1}))$, $f_l' \in L^2(0,\infty; L^2(M,\mathbb{R}^{n+1}))$ and the energy equality (2.17) holds for all $t \ge 0$. Thus, we have that

$$\{f_l'\} \text{ is a bounded set in } L^2\left(0,\infty; L^2(M,\mathbb{R}^{n+1})\right), \tag{2.18}$$

and from Poincaré's inequality and the definition of E_k we further conclude that

$$\{f_l\} \text{ is a bounded set in } L^\infty\big(0,\infty; W^{1,p}(M,\mathbb{R}^{n+1})\big) \tag{2.19}$$

and

$$\{f_l\} \text{ is a bounded set in } L^\infty\big(0,\infty; L^{4\alpha}(M,\mathbb{R}^{n+1})\big). \tag{2.20}$$

By the compactness of the spaces mentioned in (2.18)–(2.19) and Lemma 2.4, one can pass to a subsequence (without changing notation) to get

$$f_l \rightharpoonup f \text{ weakly* in } L^\infty\big(0,\infty; W^{1,p}(M,\mathbb{R}^{n+1})\big), \tag{2.21}$$

$$f_l' \rightharpoonup f' \text{ weakly in } L^2\big(0,\infty; L^2(M,\mathbb{R}^{n+1})\big) \tag{2.22}$$

and

$$f_l \to f \text{ strongly in } L^p_{loc}\big(0,\infty; L^p(M,\mathbb{R}^{n+1})\big) \text{ and a.e. on } \mathbb{R}_+ \times M \tag{2.23}$$

as $l \to \infty$. Moreover, from (2.19) we have

$$|df_l|^{p-2} df_l \rightharpoonup w \text{ weakly in } L^{p'}\big(0,T; L^{p'}(M,\mathbb{R}^{n+1})\big) \tag{2.24}$$

as $l \to \infty$. From (2.10), (2.22) and (2.24), we get that for any C^∞-function $\varphi : t \mapsto \varphi(t)$ there holds

$$k \int_0^T \int_M |f_l^2 - 1|^{2\alpha-2}(f_l^2-1) f_l w_j \varphi(t)\, d\mu\, dt \to \tag{2.25}$$

$$\to -\int_0^T \int_M f' w_j \varphi(t)\, d\mu\, dt - \int_0^T \int_M \operatorname{trace}(w^* dw_j)\varphi(t)\, d\mu\, dt$$

as $l \to \infty$. Since $\{w_j\}$ is a base in $W^{1,p}(M,\mathbb{R}^{n+1})$ and φ was arbitrary, (2.25) shows that

$$k\,|f_l^2 - 1|^{2\alpha-2}\,(f_l^2-1) f_l \to -f' - b \text{ in distributional sense}$$

with $\langle b, \psi\rangle := \int_0^T \int_M \operatorname{trace}(w^* d\psi)\, d\mu\, dt$. However, from (2.19)–(2.20) and the assumption on α, we know that

$$k\,\big||f_l|^2 - 1\big|^{2\alpha-2}\,(|f_l|^2-1) f_l \quad \text{is bounded in } L^{p'}\big(0,T; L^{p'}(M,\mathbb{R}^{n+1})\big). \tag{2.26}$$

Therefore, we conclude that

$$k\,|f_l^2 - 1|^{2\alpha-2}\,(f_l^2-1) f_l \rightharpoonup -f' - b \quad \text{weakly in } L^{p'}\big(0,T; L^{p'}(M,\mathbb{R}^{n+1})\big). \tag{2.27}$$

From (2.10) and (2.22), (2.23) and (2.27) we obtain that

$$
\begin{aligned}
(2.28) \qquad & \limsup_{l\to\infty} \langle -\Delta_p f_l, f_l\rangle = \\
&= \limsup_{l\to\infty} \Big(\langle f_l', f_l\rangle + \langle k \left| f_l^2 - 1\right|^{2\alpha-2} (f_l^2-1) f_l, f_l\rangle \Big) = \\
&= \langle f', f\rangle + (-\langle f', f\rangle - \langle b, f\rangle) = \\
&= -\int_0^T \int_M \operatorname{trace}(w^* df)\, d\mu\, dt\,.
\end{aligned}
$$

By Corollary 2.1 we get that, as $l \to \infty$,

$$(2.29) \qquad -\Delta_p f_l \rightharpoonup -\Delta_p f \text{ weakly in } L^{p'}\big(0,T; W^{-1,p'}(M, \mathbb{R}^{n+1})\big).$$

Moreover, from (2.23) and (2.26) we know that as $l \to \infty$,

$$(2.30) \qquad k \left| f_l^2 - 1\right|^{2\alpha-2} (f_l^2-1) f_l \rightharpoonup k \left| f^2 - 1\right|^{2\alpha-2} (f^2-1) f$$

weakly in $L^{p'}(0,T; L^{p'}(M, \mathbb{R}^{n+1}))$.

In the limit $l \to \infty$ the energy equality (2.17) turns into the energy inequality (2.13) and it follows from (2.22), (2.29) and (2.30) that for every $k \geq 1$, the problem (2.5) and (2.12) has a global weak solution f_k satisfying

$$(2.31) \qquad f_k \in L^\infty\big(0,\infty; W^{1,p}(M)\big)$$

and

$$(2.32) \qquad \partial_t f_k \in L^2\big(0,\infty; L^2(M)\big)\,,$$

This proves Theorem 2.2. $\square$

2.3 Global existence of weak solutions

In this section, we shall prove that the sequence $\{f_k\}$ of the solutions of the penalized equation (2.5) and (2.12), constructed in the previous section, weakly converges to a map f which is a weak solution to (2.2)–(2.4).

Definition 2.5 *A function f is said to be a global weak solution to (2.2)–(2.4), if f is defined a.e. on $M \times \mathbb{R}_+$, such that*

(D.1) $f \in L^\infty\big(0,\infty; W^{1,p}(M, \mathbb{R}^{n+1})\big)$, $\partial_t f \in L^2\big(0,\infty; L^2(M, \mathbb{R}^{n+1})\big)$,

(D.2) $|f|^2 = 1$ *a.e. on* $M \times \mathbb{R}_+$,

(D.3) f satisfies (2.2) in the sense of distributions.

First, let us state a compactness theorem which is well-known in the case $p = 2$ (see Evans in [35] or Struwe in [113]). None of the proofs for $p = 2$ allows directly a generalization to $p > 2$. Nevertheless, Evans' proof, based upon very fundamental principles, gives an idea in what direction one has to go.

Theorem 2.3 *For $k = 1, 2, \dots$, let $f_k = (f_k^1, \dots, f_k^{n+1})$ be vector functions of (x,t) on $M \times [0,T]$ satisfying the equation*

$$\partial_t f_k - \Delta_p f_k = g_k, \quad \text{on } M \times [0,T]$$

in the sense of distribution. Here, Δ_p denotes the p-Laplace operator related to the manifolds M and $\mathbb{R}^{n+1}$. Assume further that $\{f_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;W^{1,p}(M,\mathbb{R}^{n+1}))$, $\{\partial_t f_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(M,\mathbb{R}^{n+1}))$, and $\{g_k\}_{k\in\mathbb{N}}$ is bounded in $L^1(0,T;L^1(M,\mathbb{R}^{n+1}))$. Then, $\{f_k\}_{k\in\mathbb{N}}$ is precompact in $L^q(0,T;W^{1,q}(M,\mathbb{R}^{n+1}))$ for each $1 \le q < p$.

Proof
By Lemma 2.1, there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}} \subset \{f_k\}_{k\in\mathbb{N}}$ satisfying

$$f_{k_j} \rightharpoonup f \text{ weakly in } L^p\left(0,T;W^{1,p}(M,\mathbb{R}^{n+1})\right) \tag{2.33}$$

and

$$f_{k_j} \to f \text{ strongly in } L^p\left(0,T;L^p(M,\mathbb{R}^{n+1})\right). \tag{2.34}$$

Take K such that $K \ge \|df_i\|_{L^p(0,T;L^p(M))} + \|g_j\|_{L^1(0,T;L^1(M))} + \|\partial_t f_k\|_{L^1(0,T;L^1(M))}$ for all i, j, k.

For $\delta \in]0,1]$, let $E^i_\delta = \{(x,t) \in M \times [0,T];\ |f_i(x,t) - f(x,t)| \ge \delta\}$. Then

$$\int_{E^i_\delta} |df_i - df|^q \, d\mu \, dt \le (2K)^q |E^i_\delta|^{(p-q)/p}. \tag{2.35}$$

On the other hand, we define the cutoff function

$$\eta(y) = y \min\left\{\frac{\delta}{|y|}, 1\right\} \; : \; \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

which cuts every vector longer than δ at length δ. Thus, Lemma 2.2 implies that for some constant C there holds the estimate

$$\begin{aligned}
C \int\limits_{M\times[0,T]\setminus E_\delta^i} |df_i - df|^p \, d\mu\, dt \le \\
\le \int\limits_{M\times[0,T]\setminus E_\delta^i} \operatorname{trace}\big((|df_i|^{p-2}df_i - |df|^{p-2}df)^*(df_i - df)\big)\, d\mu\, dt = \\
= \int\limits_{M\times[0,T]} |df_i|^{p-2}\operatorname{trace}\left(df_i^* d\eta(f_i - f)\right)\, d\mu\, dt \\
-\delta \int\limits_{E_\delta^i} |df_i|^{p-2}\operatorname{trace}\left(df_i^* d\left(\frac{f_i - f}{|f_i - f|}\right)\right)\, d\mu\, dt \\
- \int\limits_{M\times[0,T]\setminus E_\delta^i} |df|^{p-2}\operatorname{trace}\left(df^*(df_i - df)\right)\, d\mu\, dt = \\
= I + II + III\,.
\end{aligned}$$

Now,

$$|I| \le \int_{M\times[0,T]} |(g_i - \partial_t f_i)\eta(f_i - f)|\, d\mu\, dt \le 2\delta K$$

where one uses the equation and Hölder's inequality. In local coordinates the term II may be written in the following way

$$\begin{aligned}
II &= -\int\limits_{E_\delta^i} \frac{\delta}{|f_i - f|^3} |df_i|^{p-2}\gamma^{\alpha\beta}\partial_\alpha f_i^m \big(|f_i - f|^2\partial_\beta(f_i^m - f^m) \\
&\qquad -(f_i^m - f^m)(f_i^n - f^n)\partial_\beta(f_i^n - f^n)\big)\, d\mu\, dt = \\
&= -\int\limits_{E_\delta^i} \frac{\delta}{|f_i - f|^3} |df_i|^{p-2}\gamma^{\alpha\beta}\partial_\alpha f_i^m \left(|f_i - f|^2\partial_\beta f_i^m - (f_i^m - f^m)(f_i^n - f^n)\partial_\beta f_i^n\right) + \\
&\quad + \int\limits_{E_\delta^i} \frac{\delta}{|f_i - f|^3} |df_i|^{p-2}\gamma^{\alpha\beta}\partial_\alpha f_i^m \left(|f_i - f|^2\partial_\beta f^m - (f_i^m - f^m)(f_i^n - f^n)\partial_\beta f^n\right) = \\
&= II' + II''.
\end{aligned}$$

Note that $II' \le 0$ and, by Hölder's inequality,

$$|II''| \le 2\int\limits_{E_\delta^i} |df_i|^{p-1}|df|\, d\mu\, dt \le 2K^{p-1}\left(\int\limits_{E_\delta^i} |df|^p\, d\mu\, dt\right)^{\frac{1}{p}}.$$

For III, we use the weak convergence of f_i and Hölder's inequality again to get

$$\begin{aligned} |III| &\leq \left| \int\limits_{M\times[0,T]} |df|^{p-2} \operatorname{trace}(df^* d(f_i - f))\, d\mu\, dt \right| + \\ &\quad + \left| \int\limits_{E^i_\delta} |df|^{p-2} \operatorname{trace}(df^* d(f_i - f))\, d\mu\, dt \right| \\ &\leq o(1) + 2K \left(\int\limits_{E^i_\delta} |df|^p \, d\mu\, dt \right)^{\frac{1}{p'}} . \end{aligned}$$

Thus, we obtain

(2.36)
$$C \int\limits_{M\setminus E^i_\delta} |df_i - df|^p \, d\mu\, dt \leq I + II'' + III \leq |I| + |II''| + |III| \, .$$

Choosing δ to be small, using the facts that $df \in L^p(0,T;L^p(M))$, $|E^i_\delta| \to 0$ as $i \to \infty$ and (2.35)–(2.36), the assertion follows. $\Box$

Theorem 2.4 *For initial data $f_0 \in W^{1,p}(M, S^n)$ there exists a global weak solution f to the equation (2.2)–(2.4) in the sense of (D.1)–(D.3). This solution is weakly continuous in $t > 0$ with values in $W^{1,p}(M)$, i.e. for any test function $g \in C^\infty(M)$, $h_1(t) = \int\limits_M f \cdot g \, d\mu$ and $h_2(t) = \int\limits_M df \cdot dg \, d\mu$ are in $C^{0,\frac{1}{2}}(\mathbb{R}_+)$. Furthermore for any $t \geq 0$ this solution satisfies the energy inequality*

$$\int\limits_0^t \|\partial_t f\|^2_{L^2(M)} dt + E(f(t)) \leq E(f_0) \, . \tag{2.37}$$

Proof
By Theorem 2.2 we know that there exists a global weak solution f_k to (2.2) and (2.4) such that

$$\{f_k\} \text{ is a bounded set in } L^\infty \left(0, \infty; W^{1,p}(M, \mathbb{R}^{n+1})\right) \tag{2.38}$$

and

$$\{\partial_t f_k\} \text{ is a bounded set in } L^2 \left(0, \infty; L^2(M, \mathbb{R}^{n+1})\right). \tag{2.39}$$

By the compactness of the spaces mentioned in (2.38)–(2.39), there exists a subsequence of $\{f_k\}$ (again denoted by $\{f_k\}$) such that as $k \to \infty$,

$$f_k \rightharpoonup f \text{ weakly* in } L^\infty \left(0, \infty; W^{1,p}(M, \mathbb{R}^{n+1})\right) \tag{2.40}$$

and

$$\partial_t f_k \rightharpoonup \partial_t f \text{ weakly in } L^2\left(0,\infty; L^2(M,\mathbb{R}^{n+1})\right). \tag{2.41}$$

From (2.40), (2.41) and (2.13) we conclude that (2.37) holds for f. Moreover, (2.40) and (2.41) imply by Lemma 2.4 that

$$f_k \to f \text{ strongly in } L^2_{loc}\left(0,\infty; L^2(M)\right) \text{ and a.e. on } \mathbb{R}_+ \times M. \tag{2.42}$$

On the other hand, the energy inequality (2.13) implies

$$f_k^2 \to 1 \text{ strongly in } L^1(0,T; L^1(M)). \tag{2.43}$$

The combination of (2.42) and (2.43) shows that

$$|f|^2 = 1 \text{ a.e. on } \mathbb{R}_+ \times M. \tag{2.44}$$

Now, we observe, that (2.40) and (2.42) imply for h_1 and after integration by parts for h_2

$$h_1, h_2 \in L^\infty(0,\infty;\mathbb{R});\ \frac{d}{dt}h_1^k(t), \frac{d}{dt}h_2^k(t) \in L^2(0,\infty;\mathbb{R})\,.$$

Hence, h_1, $h_2 \in W^{1,2}_{loc}(0,\infty;\mathbb{R})$, and the Sobolev embedding theorem gives the weak continuity we claimed.

To check *(D.3)*, which is the hardest part of the proof, we start with

Lemma 2.6 *Let f_k be a solution to the penalized equation (2.5) for a fixed $k \geq 1$. If the initial data satisfy $|f_0| = 1$ then*

$$|f_k| \leq 1$$

holds on $M \times [0,\infty[$.

Proof
By testing (2.5) with the function

$$f_k - \frac{f_k}{|f_k|}\min\{1, |f_k|\}$$

we find

$$\frac{1}{2}\,\partial_t \int\limits_{|f_k|\geq 1} (f_k)^2(1-\frac{1}{|f_k|})^2\,d\mu + \int\limits_{|f_k|\geq 1} |\nabla f_k|^p(1-\frac{1}{|f_k|})\,d\mu \leq 0\ .$$

Since $|f_k| = 1$ for $t = 0$ we get $|f_k| \leq 1$ for all $t \geq 0$. □

Now we finish the proof of Theorem 2.4. By Lemma 2.6 it follows from Lebesgue's theorem and (2.42) that

$$f_k \to f \qquad \text{strongly in } L^q_{loc}(\mathbb{R}_+ \times M) \tag{2.45}$$

for all $q \in [1, \infty[$.

On account of (2.38)–(2.39) and Lemma 2.6 it is now easy to check that

$$\left\{ k \left| |f_k|^2 - 1 \right|^{2\alpha - 2} (|f_k|^2 - 1) f_k \right\}_{k \in \mathbb{N}}$$

is a bounded sequence in $L^1(0, T; L^1(M, \mathbb{R}^{n+1}))$: Indeed, there holds

$$\int_0^T \int_M k \left| |f_k|^2 - 1 \right|^{2\alpha - 1} |f_k| d\mu \, dt \leq \tag{2.46}$$

$$\leq \int_0^T \int_M k \left| |f_k|^2 - 1 \right|^{2\alpha} d\mu \, dt + \int_0^T \int_M k \left| |f_k|^2 - 1 \right|^{2\alpha - 1} |f_k|^2 d\mu \, dt \, .$$

Now, the first term on the right hand side of (2.46) is bounded by means of the energy inequality (2.13). Also the second term is bounded which can be seen by testing (2.5) by f_k and using Lemma 2.6 once more.

Applying Theorem 2.3 we get

$$df_k \to df \text{ strongly in } L^q\left(0, T; L^q(M)\right) \text{ for each } q \in [1, p[. \tag{2.47}$$

Therefore, we have from (2.38) that

$$|df_k|^{p-2} df_k \rightharpoonup |df|^{p-2} df \text{ weakly in } L^{p'}_{loc}\left(0, \infty; L^{p'}(M)\right). \tag{2.48}$$

Now, by taking the wedge product of (2.5) with f_k, we get in local coordinates

$$0 = \partial_t f_k \wedge f_k - \frac{1}{\sqrt{|\gamma|}} \frac{\partial}{\partial x^\alpha} \left(|df_k|^{p-2} \gamma^{\alpha\beta} \sqrt{|\gamma|} \, \frac{\partial f_k}{\partial x^\beta} \wedge f_k \right) \tag{2.49}$$

in distributional sense. From (2.41), (2.45) and Lemma 2.6, we know that as $k \to \infty$

$$\partial_t f_k \wedge f_k \rightharpoonup \partial_t f \wedge f \quad \text{weakly in } L^2\left(0, T; L^2(M)\right). \tag{2.50}$$

The combination of (2.45), (2.48) and Lemma 2.6 leads to

$$|df_k|^{p-2} \nabla f_k \wedge f_k \rightharpoonup |df|^{p-2} \nabla f \wedge f \quad \text{weakly in } L^{p'}\left(0, T; L^{p'}(M)\right). \tag{2.51}$$

Due to (2.50) and (2.51), one can pass to the limit in (2.49) to get

$$0 = \partial_t f \wedge f - \frac{1}{\sqrt{|\gamma|}} \frac{\partial}{\partial x^\alpha} \left(|df|^{p-2} \gamma^{\alpha\beta} \sqrt{|\gamma|} \, \frac{\partial f}{\partial x^\beta} \wedge f \right). \tag{2.52}$$

in the sense of distribution. Finally, *(D.3)* will follow from (2.52) and *(D.2)* by the last lemma of this chapter:

Lemma 2.7 *A function f satisfying (D.1), (D.2) and (2.52) is a solution of the heat flow equation for a sphere (2.2) in the sense of distributions.*

Proof
In the proof of this lemma we follow Struwe, who emphasized in [115] in the case $p = 2$ that this point has to be handled with care. A short calculation shows that for every function $\tau \in C_0^\infty([0,\infty[\times M;\mathbb{R})$ the relation

$$(\partial_t f - \Delta_p f - pe(f)f) \cdot f\tau = 0 \tag{2.53}$$

automatically holds in distributional sense, provided f satisfies *(D.2)*. Note, that any function $\varphi \in C_0^\infty([0,\infty[\times M;\mathbb{R}^n)$ can be decomposed in the following way

$$\varphi = f(f\cdot\varphi) - f\wedge(f\wedge\varphi)\,. \tag{2.54}$$

Using *(D.1)* and an approximation argument we obtain that $\psi = f\wedge\varphi$ and $\tau = f\cdot\varphi$ are admissible as test functions in (2.52) and (2.53), respectively. Subtracting the resultant equations and using (2.54), we get the weak form of (2.2). □

According to Lemma 2.7 *(D.3)* is verified and hence the proof of Theorem 2.4 is finished. □

In the case $p = 2$ Chen and Struwe were able to generalize the previous result to arbitrary compact target manifold N (see [16]). The technique of their proof is based upon two ingredients. The first one is a penalizing technique similar to the one we used. Of course, the penalizing energy term has to be adapted to the geometry of N. Is it possible to use the same idea for $p > 2$? Using the penalized energy constructed by Chen and Struwe, Galerkin's method and Minty's trick, one can prove global existence of weak solutions to the penalized equation and also a maximum principle similar to our Lemma 2.6 may be obtained. But in the case $N = S^n$ an L^1-bound for the penalizing term in (2.5) was surprisingly enough to pass to the limit in the equation. There is no way to manage this miracle with a general geometry. At this point enters the second of the two mentioned ingredients for $p = 2$: It is a monotonicity formula which gives control on the local behaviour of the energy. To give an idea of this technique we consider for instance a p-harmonic map $f : B_1(0;\mathbb{R}^m) \to N$ for fixed boundary values. An interesting quantity is the normalized p-energy

$$\varphi(\rho) := \varphi(\rho, f) := \frac{1}{p}\,\rho^{p-m}\int\limits_{B_\rho(0)} |\nabla f|^p dx$$

for $0 < \rho < 1$. φ is invariant under scaling $f \to f_R(x) := f(Rx)$, i.e. there holds $\varphi(\frac{\rho}{R}, f_R) = \varphi(\rho, f)$. Moreover, f_R is p-harmonic, provided f is. If f is supposed to be smooth, we may calculate the derivative of $\varphi(\rho)$ for instance at $\rho = 1$:

$$\begin{aligned}\frac{d}{d\rho}\varphi(\rho) &= \frac{d}{d\rho}\varphi(1, f_\rho) = \int\limits_B |\nabla f|^{p-2}\nabla f\cdot\nabla\frac{d}{d\rho}f_\rho\,dx = \\ &= \int\limits_{\partial B} |\nabla f|^{p-2}(x\cdot\nabla f)\cdot\frac{d}{d\rho}f_\rho\,do - \int\limits_B \Delta_p f\cdot\frac{d}{d\rho}f_\rho\,dx\,.\end{aligned}$$

Since $\frac{d}{d\rho} f_\rho = x \cdot \nabla f \in T_f N$, the second term vanishes because f is p-harmonic and the first term gets positive. Thus, $\frac{d}{d\rho}\varphi(\rho) \geq 0$ for $\rho = 1$ and by scaling for $\rho \in]0, 1]$. Hence, what we proved is that for any $0 < \rho < r < 1$ there holds

$$\rho^{p-m} \int\limits_{B_\rho} |\nabla f|^p dx \leq r^{p-m} \int\limits_{B_r} |\nabla f|^p dx \,.$$

Schoen-Uhlenbeck [91] and Giaquinta-Giusti [50] observed that estimates of that kind hold for 2-energy minimizing harmonic mappings and can be used to obtain partial regularity. In [109] Struwe found an analogous monotonicity formula holding for the heat flow of 2-harmonic maps and used it to prove an ε-regularity theorem similar to Schoen-Uhlenbeck [91]. It turned out that Struwe's monotonicity formula was strong enough to overcome the difficulties that one encounters in the limit $k \to \infty$ of the penalized equations (see Chen-Struwe [16]). Moreover Chen and Struwe carefully discussed the regular and the singular set of the weak solutions they obtained.

Unfortunately it seems impossible to find a handy generalization of Struwe's monotonicity formula for $p > 2$. Thus, although our next chapter will deal with arbitrary target manifolds, we cannot expect results being as extensive as Chen's and Struwe's results in the case $p = 2$.

Chapter 3

A priori estimates

3.1 Introductory Remarks

The regularity of minimizing p-harmonic mappings between two compact smooth Riemannian manifolds has been widely discussed, see Hardt and Lin [61], Giaquinta and Modica [51], Fusco and Hutchinson [49], Luckhaus [80], Coron and Gulliver [22] and Fuchs in [41] who also discussed obstacle problems for minimizing p-harmonic mappings (see [40] and [43]–[46]). The results of these investigations may be summarized briefly as follows:

Suppose $1 < p < \infty$ and let M and N be two smooth compact Riemannian manifolds, M possibly having a boundary ∂M. Consider mappings $f : M \to N$ minimizing the p-energy and having fixed trace on ∂M. Such a minimizer f is locally Hölder continuous on $M \setminus Z$ for some compact subset Z of $M \setminus \partial M$ which has Hausdorff dimension at most $\dim(M) - [p] - 1$. Moreover Z is a finite set in case $\dim(M) = [p] + 1$ and empty in the case $\dim(M) < [p] + 1$. On $(M \setminus Z) \setminus \partial M$, the gradient of f is also locally Hölder continuous.

The set Z is defined as the set of points a in M for which the normalized p-energy on the ball $B_r(a)$,

$$r^{p-\dim(M)} \int\limits_{M \cap B_r(a)} |df|^p d\mu \,,$$

fails to approach zero as $r \to 0$. The technique to prove the above assertions essentially is to show that, near points $a \in (M \setminus Z) \setminus \partial M$, this normalized integral decays for $r \to 0$ like a positive power of r. Then, local Hölder continuity of f on $(M \setminus Z) \setminus \partial M$ follows by Morrey's lemma. The proof of the Hölder continuity of the gradient of f is much more difficult.

Instead of looking for minimizers of the p-energy, Uhlenbeck investigated weak solutions of systems of the form

$$\operatorname{div} \big(\rho(|\nabla f|^2)\nabla f\big) = 0$$

where ρ satisfies some ellipticity and growth condition (see Uhlenbeck [123]). Her work prompted an extensive study of quasilinear elliptic *scalar* equations having a lack of ellipticity: Evans in [34] and Lewis in [75] showed the $C^{1,\alpha}$-regularity for rather special equations. Later DiBenedetto [23] and Tolksdorf in [119], [120] proved $C^{1,\alpha}$-regularity of the solutions of rather general quasilinear equations which are allowed to have such a lack of ellipticity. It is remarkable that Ural'ceva in [124] obtained Evan's result already in 1968.

For such equations and systems, $C^{1,\alpha}$-regularity is optimal. Tolksdorf gave in [119] an example of a scalar function minimizing the p-energy and which does not belong to $C^{1,\alpha}$, if $\alpha \in]0,1[$ is chosen sufficiently close to one.

In contrast to equations, everywhere-regularity cannot be obtained for *general* elliptic quasilinear systems. The counterexample of Giusti and Miranda in [54] shows that it is generally impossible to obtain C^{α}-everywhere regularity for homogeneous quasilinear systems with analytic coefficients satisfying the usual ellipticity and growth conditions. Nevertheless almost-everywhere-regularity has been obtained for rather general classes of quasilinear elliptic systems: see Morrey [83] or Giusti-Miranda [54].

As far as the heat flow of p-harmonic maps for $p > 2$ is concerned only results in the Euclidean case are known: In [25] DiBenedetto and Friedman investigated weak solutions $f : \Omega \times (0,T] \to \mathbb{R}^m$ of the parabolic system

$$\frac{\partial f}{\partial t} - \operatorname{div}(|\nabla f|^{p-2}\nabla f) = 0\,,$$

where Ω is an open set in $\mathbb{R}^n$. The main result of DiBenedetto and Friedman is that for $\max\left\{1, \frac{2n}{n+2}\right\} < p < \infty$ weak solutions of this problem are regular in the sense that ∇f is continuous on $\Omega \times (0,T]$ with $|\nabla f(x,t) - \nabla f(\bar{x},\bar{t})| \le \omega(|x-\bar{x}| + |t-\bar{t}|^{\frac{1}{2}})$ in any compact subset $\tilde{\Omega}$ of Ω, where ω depends only on $\tilde{\Omega}$ and the norms of f in the spaces $L^\infty(0,T;L^2(\Omega))$ and $L^p(0,T;W^{1,p}(\Omega))$. In [26] the same authors obtained for $p > \max\{1, \frac{2n}{n+2}\}$ Hölder regularity of ∇f in $\Omega \times (0,T]$ using a combination of Moser iteration and De Giorgi iteration. H. Choe investigated in [18] weak solutions of the system

$$\frac{\partial f}{\partial t} - \operatorname{div}(|\nabla f|^{p-1}\nabla f) + b(x,t,f,\nabla f) = 0 \tag{3.1}$$

where b respects the growth condition

$$|b_x(x,t,f,Q)| + |b_f(x,t,f,Q)||Q| + |b_{Q^i_\alpha}(x,t,f,Q)Q^i_\alpha| \le c(1+|Q|^{p-1}) \tag{3.2}$$

and where $f \in C^0(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$. In this case, Choe proves $f \in C^{0,\alpha}_{loc}(\Omega \times (0,T])$ for some $\alpha > 0$ provided $f \in L^{r_0}_{loc}(\Omega \times (0,T])$ for some $r_0 > \frac{n(2-p)}{p}$. Notice that the p-harmonic flow does not satisfy condition (3.2): there the growth is of order $|Q|^p$. This will be one of the main difficulties in the study of the p-harmonic flow.

All other results for the heat flow concern the case $p = 2$: As already mentioned Eells and Sampson [32] derived the existence of 2-harmonic maps in the case where

$N \subset \mathbb{R}^k$ has nonpositive sectional curvature by proving existence of a global smooth solution of the related heat flow (see section 1.5). Without the assumptions on the curvature, the solution to this flow may blow up in a finite time, see Coron and Ghidaglia [21], Chen and Ding [13]. In the conformal case, i.e. $\dim(M) = 2 = p$ Struwe obtained existence and regularity of partially regular global weak solutions of the heat flow and he described in detail the behaviour of the development of singularities (see [107]). These results have been carried over to the higher dimensional case again by Struwe [109], and by Chen-Struwe in [16] (we already mentioned the methods used in this theory at the end of the previous chapter).

In the theory of 2-harmonic mappings and 2-harmonic flow important results have been obtained in the conformal case $m = 2$; for example, global regularity of harmonic maps (see Hélein [63]). It turns out that also in the theory of the p-harmonic flow the conformal situation is a considerable special case. So, in the sequel we restrict ourselves to the conformal situation $p = \dim(M) > 2$.

3.2 A Gagliardo-Nirenberg Inequality

First we generalize a lemma which plays an important role in the theory of the 2-harmonic flow where different proofs were given (see Struwe [107] or Struwe [113]).

Lemma 3.1 *Let M be a Riemannian manifold of dimension $m = p$ and ι the injectivity radius of M. Then there exist constants $c > 0$ and $R_0 \in]0, \iota]$ only depending on M, N, such that for any measurable function $f : M \times [0, T] \to N$ ($T > 0$ arbitrary), any $B_R(x) \subset M$ with $R \in]0, R_0]$ and any function $\varphi \in L^\infty(B_R(x))$ depending only on the distance from x, i.e. $\varphi(y) \equiv \varphi(|x - y|)$, and non-increasing as a function of this distance, the estimate*

$$
\begin{aligned}
(3.3)\int_0^T\int_M |f|^{2p}\varphi\, d\mu\, dt \quad &\leq \quad c \operatorname*{ess\,sup}_{0\leq t\leq T} \left(\int_{B_R(x)} |f|^p d\mu\right)^{\frac{2}{p}} \int_0^T\int_M \left|\nabla |f|^{p-1}\right|^2 \varphi\, d\mu\, dt + \\
&\quad + \frac{c}{\mu(B_R(x))} \operatorname*{ess\,sup}_{0\leq t\leq T} \int_{B_R(x)} |f|^p d\mu \cdot \int_0^T\int_M |f|^p \varphi\, d\mu\, dt
\end{aligned}
$$

holds, provided $\varphi \equiv 1$ on $B_{R/2}(x)$.

Remark: Here, $B_R(x)$ denotes the geodesic ball in M around x with radius R, i.e. $B_R(x) = \{y \in M : \operatorname{dist}_M(x, y) < R\}$, where $\operatorname{dist}_M(x, y)$ means the geodesic distance of $x, y \in M$ with respect to the given metric γ on the manifold M.

Proof
(i) Suppose first that $\varphi \equiv 1$. We may assume that the right hand side of (3.3) is finite. Let $g \in H^{1,2}(B_R(x))$ be a function with vanishing mean value $\int_{B_R(x)} g = 0$.

Then we infer from Ladyženskaya, Solonnikov, Ural'ceva [73]

$$\|g\|_{L^{\frac{2p}{p-1}}(B_R(x))} \le \beta \|\nabla g\|^{\frac{p-1}{p}}_{L^2(B_R(x))} \|g\|^{\frac{1}{p}}_{L^{\frac{p}{p-1}}(B_R(x))} . \tag{3.4}$$

We apply inequality (3.4) to the function $g = |f|^{p-1} - \lambda$ with

$$\lambda = \frac{1}{\mu(B_R(x))} \int\limits_{B_R(x)} |f|^{p-1} d\mu .$$

We get

$$\begin{aligned}
&\int\limits_0^T \int\limits_{B_R(x)} |f|^{2p} d\mu\, dt \le \\
&\le c \int\limits_0^T \int\limits_{B_R(x)} \left| |f|^{p-1} - \lambda \right|^{\frac{2p}{p-1}} d\mu\, dt + c \int\limits_0^T \int\limits_{B_R(x)} \lambda^{\frac{2p}{p-1}} d\mu\, dt \le \\
&\le c \operatorname*{ess\,sup}_{0\le t\le T} \left(\int\limits_{B_R(x)} \left| |f|^{p-1} - \lambda \right|^{\frac{p}{p-1}} d\mu \right)^{\frac{2}{p}} \int\limits_0^T \int\limits_{B_R(x)} \left| \nabla |f|^{p-1} \right|^2 d\mu\, dt + \\
&\quad + c \int\limits_0^T \frac{1}{\mu(B_R(x))^{\frac{2p}{p-1}-1}} \left(\int\limits_{B_R(x)} |f|^{p-1} d\mu \right)^{\frac{2p}{p-1}} dt .
\end{aligned} \tag{3.5}$$

The various terms on the right hand side of (3.5) are estimated in the following way:

$$\int\limits_{B_R(x)} \left| |f|^{p-1} - \lambda \right|^{\frac{p}{p-1}} d\mu \le 2^{\frac{p}{p-1}} \left(\int\limits_{B_R(x)} |f|^p d\mu + \int\limits_{B_R(x)} \lambda^{\frac{p}{p-1}} d\mu \right) \tag{3.6}$$

$$\begin{aligned}
\int\limits_{B_R(x)} \lambda^{\frac{p}{p-1}} d\mu &= \mu(B_R(x))^{\frac{-1}{p-1}} \left(\int\limits_{B_R(x)} |f|^{p-1} d\mu \right)^{\frac{p}{p-1}} \le \\
&\le \int\limits_{B_R(x)} |f|^p d\mu
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\left(\int\limits_{B_R(x)} |f|^{p-1} d\mu \right)^{\frac{2p}{p-1}} &\le \left(\int\limits_{B_R(x)} |f|^p d\mu \right)^2 \cdot \mu(B_R(x))^{\frac{2}{p-1}} \\
&\le \mu(B_R(x))^{\frac{2}{p-1}} \operatorname*{ess\,sup}_{0\le t\le T} \int\limits_{B_R(x)} |f|^p d\mu \cdot \int\limits_{B_R(x)} |f|^p d\mu
\end{aligned} \tag{3.8}$$

Plugging in the estimates (3.6)–(3.8) into (3.5) the assertion for $\varphi \equiv 1$ follows.

(ii) By linearity and (i) the assertion remains true for step functions φ which are non-increasing in radial distance and which satisfy $\varphi \equiv 1$ on $B_{R/2}(x)$. Finally, the general case follows by density of the step functions in $L^\infty(B_R(x))$ in measure.

□

Corollary 3.1 *Let M be of dimension $m = p$ and*

$$V(M^T, N) = \Big\{ f : M \times [0,T] \to N \text{ measurable:}$$

$$\operatorname*{ess\,sup}_{0 \le t \le T} \int_M |\nabla f|^p d\mu + \int_0^T \int_M \left(|\partial_t f|^2 + |\nabla^2 f|^2 |\nabla f|^{2p-4} \right) d\mu \, dt < \infty \Big\}$$

then for any $f \in V(M^T, N)$ we have

$$\nabla f \in L^{2p}(M \times [0,T], N)$$

with

$$\int_0^T \int_M |\nabla f|^{2p} d\mu \, dt \le \tag{3.9}$$

$$\le \; c \operatorname*{ess\,sup}_{(x,t) \in M \times [0,T]} \left(\int_{B_R(x)} |\nabla f|^p d\mu \right)^{\frac{2}{p}} \int_0^T \int_M |\nabla^2 f|^2 |\nabla f|^{2p-4} d\mu \, dt +$$

$$+ \frac{c}{R^p} \operatorname*{ess\,sup}_{(x,t) \in M \times [0,T]} \int_{B_R(x)} |\nabla f|^p d\mu \cdot \int_0^T \int_M |\nabla f|^p d\mu \, dt$$

for all $R \in]0, R_0]$ (R_0 as in Lemma 3.1) and a constant c only depending on M, N and p.

Proof
We choose a finite cover $\{B_R(x_i)\}$ of M with $0 < R \le R_0$ (R_0 as in Lemma 3.1) having the property that at each point $x \in M$ at most K of the balls $\{B_R(x_i)\}$ meet (K only depending on M, R_0, *not* on R) and apply Lemma 3.1 with $\varphi \equiv 1$ on each $B_R(x_i)$. Adding the resulting inequalities the desired result follows. □

3.3 Energy estimate

Section 3.2 deals with general facts about functions mapping a manifold into another. Now we need to state properties of solutions of the p-harmonic flow, i.e. mappings $f : M \times [0, \infty[\to N$ satisfying

$$f_t - \Delta_p f = (pe(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)_M \qquad \text{on } M \times [0, \infty[\tag{3.10}$$

(in distributional sense) with prescribed initial data

$$f = f_0 \text{ at } t = 0.$$

Here Δ_p is the p-Laplace operator of the smooth compact manifolds M and N and $A(f)$ is the second fundamental form of N (for the details we refer to Section 1.5).

If we think of N as being isometrically embedded in $\mathbb{R}^k$, then the equation of the p-harmonic flow takes the form

$$f_t - \Delta_p f = \sum_{i=n+1}^{k} \lambda^i (\nu_i \circ f) \tag{3.11}$$

where n is the dimension of N and ν_i is a local orthonormal frame of $(T_p)^{\perp}N$, the orthogonal complement of T_pN in $\mathbb{R}^k$. The coefficients λ^i are given by

$$\lambda^i = (pe(f))^{1-\frac{2}{p}} \gamma^{\alpha\beta} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial \nu_i^j(f)}{\partial x^\beta}$$

(see Section 1.5).

First we generalize the energy inequality we already proved in the case when N was a sphere (see Section 2.3). Notice that this lemma is true without the conformality assumption $\dim(M) = p$.

Lemma 3.2 *Let $f \in C^2(M \times [0,T], N)$ be a solution to the p-harmonic flow (3.10). Then the following energy equality holds for all t_1, t_2 with $0 \le t_1 < t_2 \le T$:*

$$\int_{t_1}^{t_2} \int_M |\partial_t f|^2 d\mu \, dt + E\left(f(t_2)\right) = E\left(f(t_1)\right) . \tag{3.12}$$

Hence, $E(f(t))$ is a non-increasing function in t.

Proof
Step 1: We multiply (3.11) by $\partial_t f$. Since $\partial_t f \in T_f N$ the right hand side vanishes.

Step 2: On the left hand side we use

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma} \left(\gamma^{\alpha\beta} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^i}{\partial x^\beta} \right)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^i}{\partial t} \right) = \Delta_p f \cdot \partial_t f + \frac{d}{dt} e(f) .$$

The divergence term vanishes if we integrate over M. Integrating over the time interval $[t_1, t_2]$ we get

$$-\int_{t_1}^{t_2} \int_M \Delta_p f f_t \, d\mu \, dt = E\left(f(t_2)\right) - E\left(f(t_1)\right) \tag{3.13}$$

and hence the desired result. Notice that (3.13) is automatic for smooth functions: we did not need that f is a solution of the p-harmonic flow. $\square$

Remark: (i) Lemma 3.2 remains true for solutions $f \in V(M^T; N)$ for almost all t_1, t_2. To be more precise: If $f \in V(M^T; N)$ is a solution of (3.10) then there exists a set $A \subset [0,T]$ of measure zero such that for $t_1, t_2 \in [0,T] \setminus A$, $t_1 < t_2$, equality (3.12) holds. To see this we first repeat the first step of the above proof and get that, after multiplying the heat flow equation by f_t (which is allowed for a.e. t), the right hand side vanishes a.e. on $M \times [0,T]$. In a second step we show that (3.13) remains true for functions in $V(M^T;N)$ for almost all t_1, t_2: Approximate $f \in V(M^T;N)$ with functions $f_k \in C^\infty(M^T;\mathbb{R}^k)$ such that

$$\partial_t f_k \rightarrow \partial f_t \quad \text{in } L^2(M^T), \tag{3.14}$$

$$\nabla f_k \rightarrow \nabla f \quad \text{in } L^{2p}(M^T) \tag{3.15}$$

and

$$\{\nabla^2 f_k |\nabla f_k|^{p-2}\} \text{ is bounded in } L^2(M^T) \tag{3.16}$$

(we can use mollification to do this; we also need corollary 3.1). Now (3.15) implies that there is a subsequence $f_{k'}$ such that $E(f_{k'}(t)) \to E(f(t))$ for almost all $t \in [0,T]$. Observing (3.14) and $\Delta_p f_{k'} \rightharpoonup \Delta_p f$ in $L^2(M^T)$ (use (3.16) and Corollary 3.1 again) we may pass to the limit on the left hand side.

(ii) From part (i) it now follows that $E(f(t))$ is continuous in t for a solution $f \in V(M^T;N)$ (after modification of f on a set of measure zero). Notice that this fact is automatic for functions $f \in V(M^T;N)$ if $p=2$ since then $V(M^T;N) \hookrightarrow C^0([0,T];W^{1,2}(M))$ (see Lions-Magenes [78] and Bergh-Löfström [1]).

3.4 L^{2p}-estimate for ∇f

As we mentioned in Section 3.1 Hi Jun Choe proved in [19] Hölder regularity for the solutions of the degenerate parabolic system

$$f_t - \nabla(|\nabla f|^{p-2}\nabla f) + b(x,f,\nabla f) = 0$$

where b satisfies a certain growth condition (see (3.2)). This growth condition is not fulfilled in the case of the p-harmonic flow and we have to proceed more carefully. First we want to prove the following a priori estimate which holds in the conformal situation. It turns out that the important quantity to control is the local energy:

Definition 3.3 *We denote by*

$$E\left(f(t), B_R(x)\right) = \int\limits_{B_R(x)} e\left(f(\,\cdot\,,t)\right)\, d\mu$$

the local energy of the function $f(\,\cdot\,,t) : M \to N$ in the geodesic ball $B_R(x) \subset M$.

Given some uniform control of the local energy, we can control the $2p$-norm of the gradient and higher derivatives.

Lemma 3.4 *Let ι denote the injectivity radius of the manifold M. If $m = \dim(M) = p$ then there exists $\varepsilon_1 > 0$ which only depends on M and N with the following property:*

If $f \in C^2(B_{3R}(y) \times [0,T[; N)$ with $E(f(t)) \leq E_0$ is a solution of the p-harmonic flow (3.10) on $B_{3R}(y) \times [0,T[$ for some $R \in]0, \frac{\iota}{3}[$ and if

$$\sup \{E(f(t), B_R(x))\,;\, 0 \leq t \leq T, x \in B_{2R}(y)\} < \varepsilon_1$$

then we have for every $x \in B_R(y)$

$$\int_0^T \int_{B_R(x)} |\nabla^2 f|^2 |\nabla f|^{2p-4} d\mu\, dt < c\, E_0 \left(1 + \frac{T}{R^m}\right) \tag{3.17}$$

and

$$\int_0^T \int_{B_R(x)} |\nabla f|^{2p} d\mu\, dt < c\, E_0 \left(1 + \frac{T}{R^m}\right) \tag{3.18}$$

for some constant c which only depends on the manifolds M and N.

Proof
For simplicity we consider the case of a flat torus $M = \mathbb{R}^m / \mathbb{Z}^m$. For a locally conformal flat manfold M we may do this anyhow since we could pass to a conformal chart. Let $\varphi \in C_0^\infty(B_{2R}(y))$ be a cutoff function satisfying $0 \leq \varphi \leq 1$, $\varphi|_{B_R(y)} \equiv 1$ and $|\nabla\varphi| < \frac{2}{R}$. The equation of the p-harmonic flow which takes the form

$$f_t - \nabla(\nabla f |\nabla f|^{p-2}) \perp T_f N \tag{3.19}$$

(see Example 1.2) is now tested by the function $\nabla(\nabla f|\nabla f|^{p-2})\varphi^p$. Using the explicit form of the right hand side in (3.11) we get

$$|f_t \Delta_p f \varphi^p - (\Delta_p f)^2 \varphi^p| \leq c |\nabla f|^p |\Delta_p f| \varphi^p \tag{3.20}$$

with a constant c only depending on N. For brevity let $Q = B_{2R}(y) \times [0,T[$. Integrating over Q we obtain

$$\begin{aligned} \int_Q \left(\frac{1}{p}\frac{d}{dt}|\nabla f|^p \varphi^p + |\Delta_p f|^2 \varphi^p\right) dx\, dt &= \\ = \int_Q \left(-\nabla(\nabla f|\nabla f|^{p-2}\varphi^p) f_t + |\Delta_p f|^2 \varphi^p\right) dx\, dt &\leq \\ = \int_Q \left(-\Delta_p f \varphi^p f_t - p\nabla f|\nabla f|^{p-2}\varphi^{p-1}\nabla\varphi f_t + |\Delta_p f|^2\varphi^p\right) dx\, dt &\leq \\ \leq \int_Q \left(p|\Delta_p f \nabla f |\nabla f|^{p-2} \varphi^{p-1}\nabla\varphi| + c|\nabla f|^p |\Delta_p f|\varphi^p\right) dx\, dt\,. & \end{aligned} \tag{3.21}$$

In the last step we used (3.20) and then equation (3.11) to substitute f_t by $\Delta_p f$. By Young's inequality we can estimate the last line in (3.21) by

$$\int_Q \left(\frac{1}{4}|\Delta_p f|^2\varphi^p + c|\nabla f|^{2p}\varphi^p + c|\nabla f|^{2p-2}|\nabla\varphi|^2\varphi^{p-2}\right)\,dx\,dt\,. \tag{3.22}$$

By integrating by parts twice, exchanging derivatives and rearranging the resulting terms we find that for arbitrary functions $f \in C^2(Q;N)$ there holds

$$\begin{aligned}\int_Q |\Delta_p f|^2\varphi^p dx\,dt &\geq \\ &\geq \frac{1}{2}\int_Q |\nabla^2 f|^2|\nabla f|^{2p-4}\varphi^p dx\,dt - c\int_Q |\nabla\varphi|^2|\nabla f|^{2p-2}\varphi^{p-2}dx\,dt\end{aligned} \tag{3.23}$$

for some constant c only depending on absolute data. Putting (3.21)–(3.23) together we obtain the estimate

$$\begin{aligned}\int_Q |\nabla^2 f|^2|\nabla f|^{2p-4}\varphi^p dx\,dt &\leq \\ &\leq c\int_{B_{2R}(y)} |\nabla f(\,\cdot\,,0)|^p dx + c\int_Q \left(|\nabla\varphi|^2|\nabla f|^{2p-2}\varphi^{p-2} + |\nabla f|^{2p}\varphi^p\right)dx\,dt\,.\end{aligned} \tag{3.24}$$

The second term on the right hand side of (3.24) may be estimated separately by Hölder's and Young's inequality:

$$\int_Q |\nabla\varphi|^2|\nabla f|^{2p-2}\varphi^{p-2}dx\,dt \leq c\int_Q \left(|\nabla\varphi|^p|\nabla f|^p + |\nabla f|^{2p}\varphi^p\right)\,dx\,dt\,. \tag{3.25}$$

Hence, from (3.24) and (3.25) it follows

$$\begin{aligned}\int_Q |\nabla^2 f|^2|\nabla f|^{2p-4}\varphi^p dx\,dt &\leq \\ &\leq c\int_{B_{2R}(y)} |\nabla f(\,\cdot\,,0)|^p dx + c\int_Q \left(|\nabla f|^{2p}\varphi^p + |\nabla\varphi|^p|\nabla f|^p\right)\,dx\,dt\,.\end{aligned} \tag{3.26}$$

From Lemma 3.1 we infer

$$\begin{aligned}\int_Q |\nabla f|^{2p}\varphi^p dx\,dt &\leq \\ \leq\ c \sup_{0\leq t\leq T, x\in B_{2R}(y)} &\left(\int_{B_R(x)} |\nabla f(\,\cdot\,,t)|^p dx\right)^{\frac{2}{p}} \int_Q |\nabla^2 f|^2|\nabla f|^{2p-4}\varphi^p dx\,dt\, + \\ &+\frac{c}{R^m}\sup_{0\leq t\leq T, x\in B_{2R}(y)} \int_{B_R(x)} |\nabla f(\,\cdot\,,t)|^p dx \int_Q |\nabla f|^p dx\,dt\,.\end{aligned} \tag{3.27}$$

Hence, for $\varepsilon_1 > 0$ small enough, the condition

$$\sup_{0\le t\le T, x\in B_{2R}(y)} \int\limits_{B_R(x)} |\nabla f(\,\cdot\,,t)|^p d\mu < \varepsilon_1$$

used in (3.27) and in (3.26) implies the estimate (3.17) and by applying Lemma 3.1 once again, we get (3.18). □

3.5 Higher regularity

Regularity and a priori estimates of solutions $f : \Omega \times (0,T] \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$, of the degenerate parabolic system

$$f_t - \nabla(\nabla f|\nabla f|^{p-2}) = 0$$

have been discussed by E. DiBenedetto and A. Friedman (see [25]). In order to obtain local L^q-regularity ($q < \infty$) they used the testfunction $\nabla(\nabla f|\nabla f|^s\zeta^2)$ (ζ is a suitable cutoff function). The same testfunction was used in the case $p = 2$ (see e.g. Struwe [113]). In order to adapt these ideas to the case of the p-harmonic flow between Riemannian manifolds we have to modify this approach somewhat.

In order to describe how much the energy is concentrated we will use the following quantity:

Definition 3.5 *For a function $f : M \times [t_1,t_2] \to N$, $f \in L^\infty(t_1,t_2;W^{1,p}(M,N))$, $\varepsilon > 0$ and $\Omega \subset M \times [t_1,t_2]$ let*

$$R^*(\varepsilon,f,\Omega) = \operatorname{ess\,sup}\left\{R \in [0,\iota] \;:\; \operatorname*{ess\,sup}_{(x,t)\in\Omega}\left(E(f(t)),B_R(x)\right) < \varepsilon\right\}$$

where ι denotes the injectivity radius of M.

Lemma 3.6 *Let $q \in]2p,\infty[$ be a given constant. If $\dim(M) = p$ then there exists a constant $\varepsilon_2 > 0$ only depending on M, N and q with the following property:*

For any solution $f \in C^2(M \times [0,T[;N)$ of the p-harmonic flow (3.10) with $R^* = R^*(\varepsilon_2,f,M\times[0,T[) > 0$ and any open set $\Omega \subset\subset M\times]0,T[$ there exists a constant C which only depends on p,q,M,N,T,E_0,R^* and $\operatorname{dist}(\Omega,M\times\{0\})$ such that

$$\int\limits_\Omega |\nabla f|^q dx\,dt < C.$$

E_0 denotes the initial energy $E_0 = E(f(\,\cdot\,,0))$.

Remark: It seems very inconvenient that the constant ε_2 may not be chosen independently of the level q of integrability we want to reach. We will obtain a better result below which uses the assertion of this lemma in a technical way.

Proof
In this proof we consider the case $M = \mathbb{R}^m / \mathbb{Z}^m$ for simplicity. We write (3.10) in the form

$$\partial_t f - \nabla(\nabla f |\nabla f|^{p-2}) = |\nabla f|^{p-2} A(f)(\nabla f, \nabla f) \tag{3.28}$$

where $A(f)(\nabla f, \nabla f)$ is the second fundamental form of N. Let us first fix a few notations:

For $(x_0, t_0) \in M \times]0, T[$ let

$$\begin{aligned} B_R &= B_R(x_0) = \{|x_0 - x| < R\} \\ Q_R &= Q_R(x_0, t_0) = B_R \times]t_0 - R^p, t_0[\\ Q_R(\sigma_1, \sigma_2) &= B_{R-\sigma_1 R} \times]t_0 - (1-\sigma_2) R^p, t_0[\quad \text{for } \sigma_i \in]0,1[. \end{aligned}$$

For R small enough such that $\overline{Q}_R \subset M \times]0, T[$ we take a cutoff function ζ with

$$\begin{aligned} &\zeta = 1 \quad \text{on } Q_R(\sigma_1, \sigma_2) \\ &\zeta = 0 \quad \text{in a neighborhood of the parabolic boundary of } Q_R \end{aligned}$$

and with

$$0 \le \zeta \le 1, \quad |\nabla \zeta| \le \frac{2}{\sigma_1 R}, \quad |\partial_t \zeta| \le \frac{2}{\sigma_2 R^p}\,. \tag{3.29}$$

We now use $-\nabla(\nabla f\, v^\alpha \zeta^2)$, with $v = |\nabla f|^2$, as a testfunction in the equation (3.28). This gives rise to the following calculations:

(i) The first term on the left gives

$$\begin{aligned} &\int_{Q_R} f_t(-\nabla(\nabla f v^\alpha \zeta^2))\, dx\, dt = \\ &= \tfrac{1}{2(\alpha+1)} \int_{Q_R} \zeta^2 \partial_t v^{\alpha+1} dx\, dt = \\ &= \tfrac{1}{2(\alpha+1)} \int_{B_R} v^{\alpha+1} \zeta^2(\,\cdot\,, t_0)\, dx - \tfrac{1}{2(\alpha+1)} \int_{Q_R} v^{\alpha+1} 2\zeta\zeta_t\, dx\, dt\,. \end{aligned}$$

(ii) For the second term on the left we observe that

$$\int_{Q_R} (f^i_k |\nabla f|^{p-2})_k (f^i_j v^\alpha \zeta^2)_j\, dx\, dt = \int_{Q_R} (f^i_k |\nabla f|^{p-2})_j (f^i_j v^\alpha \zeta^2)_k\, dx\, dt\,.$$

In order not to lose control on the various terms we proceed here in two steps.

(a) First we get

$$\int\limits_{Q_R} |\nabla f|^{p-2} f^i_{jk} \left(f^i_{jk} v^\alpha \zeta^2 + \alpha f^i_j v^{\alpha-1} v_k \zeta^2 + 2 f^i_j v^\alpha \zeta \zeta_k \right) dx\,dt =$$

$$= \int\limits_{Q_R} \left(\zeta^2 \sum_{ijk} (f^i_{jk})^2 v^{\frac{2\alpha+p-2}{2}} + \tfrac{\alpha}{2} \zeta^2 v^{\frac{2\alpha+p-4}{2}} |\nabla v|^2 + v^{\frac{2\alpha+p-2}{2}} \nabla v \zeta \nabla \zeta \right) dx\,dt.$$

(b) Second we find

$$\int\limits_{Q_R} \left(|\nabla f|^{p-2} \right)_j f^i_k \left(f^i_{jk} v^\alpha \zeta^2 + \alpha f^i_j v^{\alpha-1} v_k \zeta^2 + 2 f^i_j v^\alpha \zeta \zeta_k \right) dx\,dt =$$

$$= \int\limits_{Q_R} \Big(\tfrac{p-1}{2} |\nabla f|^{p-4} |\nabla v|^2 v^\alpha \zeta^2 + \tfrac{\alpha(p-2)}{2} |\nabla f|^{p-4} v^{\alpha-1} \zeta^2 \sum_i (\nabla f^i \nabla v)^2 +$$

$$+ (p-2) |\nabla f|^{p-4} \zeta v^\alpha \sum_i (\nabla f^i \nabla v)(\nabla f^i \nabla \zeta) \Big) dx\,dt\,.$$

(iii) On the right hand side we finally get

$$\int\limits_{Q_R} (|\nabla f|^{p-2} A(f)(\nabla f, \nabla f))_j f_j v^\alpha \zeta^2 dx\,dt = \tag{3.30}$$

$$= \int\limits_{Q_R} |\nabla f|^{p-2} (\nabla A(f))(\nabla f, \nabla f) \nabla f v^\alpha \zeta^2 dx\,dt\,.$$

In the calculation (iii) of the right hand side of (3.30) we used the fact that $f_j v^\alpha \zeta^2 \in T_f N$. Putting all the terms (i)–(iii) together and taking the supremum over the time interval $]t_0 - (1-\sigma_2)R^p, t_0[$ we get by using the monotonicity of the positive terms

$$\tfrac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int\limits_{B_R} v^{\alpha+1} \zeta^2(\,\cdot\,, t)\,dx + \tfrac{1}{2} \int\limits_{Q_R} \zeta^2 v^{\frac{p+2\alpha-2}{2}} \sum_{ijk} (f^i_{jk})^2 dx\,dt + \tag{3.31}$$

$$+ \tfrac{\alpha(p-2)}{4} \int\limits_{Q_R} \zeta^2 v^{\frac{p+2\alpha-6}{2}} \sum_i (\nabla f^i \nabla v)^2 dx\,dt + \tfrac{\alpha+p-2}{4} \int\limits_{Q_R} v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \zeta^2 dx\,dt \le$$

$$\le \tfrac{1}{\alpha+1} \int\limits_{Q_R} v^{\alpha+1} \zeta |\zeta_t| dx\,dt + \int\limits_{Q_R} v^{\frac{p+2\alpha-2}{2}} |\nabla v| \zeta |\nabla \zeta| dx\,dt +$$

$$+ (p-2) \int\limits_{Q_R} v^{\frac{p+2\alpha-6}{2}} \sum_i |\nabla f^i \nabla v| v^{\frac{3}{2}} \zeta |\nabla \zeta| dx\,dt + c \int\limits_{Q_R} \zeta^2 v^{\frac{p+2\alpha+2}{2}} dx\,dt\,.$$

Two terms on the right hand side of the inequality (3.31) need to be interpolated by the binomic inequality

$$\int\limits_{Q_R} v^{\frac{p+2\alpha-2}{2}} |\nabla v|\, \zeta\, |\nabla \zeta|\, dx\,dt \le \tag{3.32}$$

$$\le \tfrac{1}{2\varepsilon} \int\limits_{Q_R} v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 \zeta^2 dx\,dt + \tfrac{\varepsilon}{2} \int\limits_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla \zeta|^2 dx\,dt\,,$$

$$(3.33)\quad (p-2)\int\limits_{Q_R} v^{\frac{p+2\alpha-6}{2}} \sum_i |\nabla f^i \nabla v| v^{\frac{3}{2}} \zeta |\nabla\zeta|\, dx\, dt \le$$
$$\le \tfrac{p-2}{2\delta}\int\limits_{Q_R} v^{\frac{p+2\alpha-6}{2}} \zeta^2 \sum_i |\nabla f^i \nabla v|^2 dx\, dt + \tfrac{\delta(p-2)}{2}\int\limits_{Q_R} v^{\frac{p+2\alpha-6}{2}} v^3 |\nabla\zeta|^2 dx\, dt\,.$$

Thus, choosing the constants ε and δ in (3.32) and (3.33) appropriately, e.g. $\varepsilon = \frac{4}{\alpha+p-2}$ and $\delta = \frac{4}{\alpha}$, and absorbing the resulting terms we obtain from (3.31)

$$(3.34)\quad \tfrac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int\limits_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\, dx + \tfrac{\alpha+p-2}{8}\int\limits_{Q_R} \zeta^2 v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 dx\, dt \le$$
$$\le \tfrac{1}{\alpha+1}\int\limits_{Q_R} v^{\alpha+1}\zeta|\zeta_t| dx\, dt + 2\left(\tfrac{1}{\alpha+p-2} + \tfrac{p-2}{\alpha}\right)\int\limits_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla\zeta|^2 dx\, dt +$$
$$+\, c\int\limits_{Q_R} v^{\frac{p+2+2\alpha}{2}} \zeta^2 dx\, dt\,.$$

Since we trivially have

$$\int\limits_{Q_R} \zeta^2 v^{\frac{p+2\alpha-4}{2}} |\nabla v|^2 dx\, dt \ge \tfrac{8}{(p+2\alpha)^2}\int\limits_{Q_R} \left|\nabla\left(v^{\frac{p+2\alpha}{4}}\zeta\right)\right|^2 dx\, dt - \tfrac{16}{(p+2\alpha)^2}\int\limits_{Q_R} |\nabla\zeta|^2 v^{\frac{p+2\alpha}{2}} dx\, dt$$

it follows from (3.34)

$$(3.35)\quad \tfrac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int\limits_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\, dx + \tfrac{\alpha+p-2}{(p+2\alpha)^2}\int\limits_{Q_R} \left|\nabla\left(v^{\frac{p+2\alpha}{4}}\zeta\right)\right|^2 dx\, dt \le$$
$$\le \tfrac{1}{\alpha+1}\int\limits_{Q_R} v^{\alpha+1}\zeta|\zeta_t| dx\, dt + 2\left(\tfrac{1}{\alpha+p-2} + \tfrac{p-2}{\alpha} + \tfrac{\alpha+p-2}{(p+2\alpha)^2}\right)\int\limits_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla\zeta|^2 dx\, dt +$$
$$+\, c\int\limits_{Q_R} v^{\frac{p+2+2\alpha}{2}} \zeta^2 dx\, dt\,.$$

Now, for every fixed time t Hölder's inequality implies

$$\|v^{\frac{p+2\alpha+2}{4}}\zeta\|^2_{L^2(B_R)} \le \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)} \|v\|_{L^{\frac{p}{2}}(B_R)}\,.$$

So, we may split off a suitable factor in the last term on the right hand side of (3.35)

$$(3.36)\quad \tfrac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int\limits_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\, dx + \tfrac{\alpha+p-2}{(p+2\alpha)^2}\int\limits_{t_0-R^p}^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{H_0^{1,2}(B_R)} dt \le$$
$$\le \tfrac{1}{\alpha+1}\int\limits_{Q_R} v^{\alpha+1}\zeta|\zeta_t| dx\, dt + 2\left(\tfrac{1}{\alpha+p-2} + \tfrac{p-2}{\alpha} + \tfrac{\alpha+p-2}{(p+2\alpha)^2}\right)\int\limits_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla\zeta|^2 dx\, dt +$$
$$+\, c \operatorname*{ess\,sup}_{t_0-R^p<t<t_0} \left(\int\limits_{B_R} v^{\frac{p}{2}}(\,\cdot\,,t) dx\right)^{\frac{2}{p}} \int\limits_{t_0-R^p}^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)} dt\,.$$

Using the Poincaré-Sobolev inequality $\|v^{\frac{p+2\alpha}{4}}\zeta\|_{H_0^{1,2}(B_R)} \geq k\|v^{\frac{p+2\alpha}{4}}\zeta\|_{L^{2^*}(B_R)}$ to estimate the last term on the left of inequality (3.36) we see that we may absorb the last term of the right hand side provided the quantity

$$\sup\{E(f(t), B_R(x))\,;\, 0 \leq t < T, x \in M\}$$

is smaller than a suitable constant $\varepsilon_2 > 0$ which only depends on absolute data and the level $q < \infty$ of integrability that we want to reach (it is enough to choose ε_2 such that $c\varepsilon_2^{2/p} \leq \frac{k^2(p-2)}{(q+2)^4}$, and $\varepsilon_2 \leq \varepsilon_1$ which allows to use the result of Lemma 3.4 later). In this way we get from (3.36)

$$\tag{3.37} \frac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\,dx + \frac{\alpha k^2}{(p+2\alpha)^2} \int_{t_0-R^p}^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)} dt \leq$$
$$\leq \frac{1}{\alpha+1} \int_{Q_R} v^{\alpha+1}\zeta|\zeta_t| dx\,dt + 2\left(\frac{1}{\alpha+p-2} + \frac{p-2}{\alpha} + \frac{\alpha+p-2}{(p+2\alpha)^2}\right) \int_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla\zeta|^2 dx\,dt\,.$$

Now, by Hölder's inequality and a further elementary inequality we observe that for every $\lambda, \gamma > 0$ there holds

$$\tag{3.38} \lambda \int_{t_1}^{t_2} \int_{B_\rho} v^{\frac{p+2\alpha}{2}+\frac{2}{p}(\alpha+1)} dx\,dt \leq$$
$$\leq \lambda \operatorname*{ess\,sup}_{t\in]t_1,t_2[} \left(\int_{B_\rho} v^{\alpha+1} dx \right)^{\frac{2}{p}} \int_{t_1}^{t_2} \|v^{\frac{p+2\alpha}{4}}\|^2_{L^{2^*}(B_\rho)} dt \leq$$
$$\leq \lambda \left(\gamma^{\frac{p}{2}} \operatorname*{ess\,sup}_{t\in]t_1,t_2[} \int_{B_\rho} v^{\alpha+1} dx + \frac{1}{\gamma} \int_{t_1}^{t_2} \|v^{\frac{p+2\alpha}{4}}\|^2_{L^{2^*}(B_\rho)} dt \right)^{1+\frac{2}{p}}.$$

Choosing $t_1 = t_0 - (1-\sigma_1)R^p$, $t_2 = t_0$, $\rho = (1-\sigma_1)R$ and the constants λ and γ such that

$$\lambda^{\frac{p}{p+2}} \gamma^{\frac{p}{2}} = \frac{1}{4(\alpha+1)}$$
$$\frac{\lambda^{\frac{p}{p+2}}}{\gamma} = \frac{\alpha k^2}{(p+2\alpha)^2}$$

we get from (3.37) together with (3.38) and (3.29) that

$$\tag{3.39} \int_{Q_R(\sigma_1,\sigma_2)} v^{\frac{p+2\alpha}{2}+\frac{2}{p}(\alpha+1)} dx\,dt \leq C \left(\int_{Q_R} v^{\frac{p+2\alpha}{2}} dx\,dt + 1 \right)^{1+\frac{2}{p}}$$

where the constant C in (3.39) may be expressed by means of the constants α, σ_1, σ_2, R, p, k. Now, the assertion follows by iteration and a covering argument. The iteration starts e.g. with $\alpha = p/2$ and the a priori estimate of Lemma 3.4 (we should not start with $\alpha = 0$ since we used $\alpha > 0$ in the calculations of the proof). We stop the iteration as soon as $\frac{p+2\alpha}{2} + \frac{2}{p}(\alpha+1) \geq \frac{q}{2}$. It is easy to check that our choice of ε_2 remains valid during the iteration process. □

3.6 L^∞-estimate for ∇f

The next step is to find local a priori estimates for $\|\nabla f\|_\infty$ by a Moser iteration technique (see [84] and [85]). H. Choe used similar arguments in [19] to handle the case of systems of type (3.1). We start once again from the estimate (3.35) but this time the iteration is arranged in quite a different way.

Lemma 3.7 *Let $a_0 > p$ be an arbitrary constant,* $\dim(M) = p$*, $f \in C^2(M \times [0,T[; N)$ a solution of the p-harmonic flow (3.10) and $Q_R \subset M \times]0,T[$. Then there exists a constant C which only depends on p, a_0, R, M and N with the property that*

$$||\nabla f||_{L^\infty(Q_R(\frac{1}{2},\frac{1}{2}))} < C \left(1 + \int\limits_{Q_R} |\nabla f|^{2a_0} dx\, dt\right)^{\frac{1}{a_0 - p}}.$$

Proof
By the Hölder and the Sobolev inequality we have

$$(3.40) \qquad \int\limits_{Q_R(\sigma_1,\sigma_2)} v^{(\alpha+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p})} dx\, dt \leq$$

$$\leq c \operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \left(\int\limits_{B_R} v^{\alpha+1} dx\right)^{\frac{2}{p}} \int\limits_{Q_R} |\nabla(v^{\frac{2\alpha+p}{4}}\zeta)|^2 dx\, dt \leq$$

$$\leq c \left(\operatorname*{ess\,sup}_{t_0-(1-\sigma_2)R^p<t<t_0} \int\limits_{B_R} v^{\alpha+1} dx + \int\limits_{Q_R} |\nabla(v^{\frac{2\alpha+p}{4}}\zeta)|^2 dx\, dt\right)^{1+\frac{2}{p}}.$$

Using (3.35) we conclude

$$(3.41) \qquad \int\limits_{Q_R(\sigma_1,\sigma_2)} v^{(\alpha+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p})} dx\, dt \leq$$

$$\leq c \left(\int\limits_{Q_R} v^{\alpha+1}\zeta|\zeta_t| dx\, dt + \int\limits_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla\zeta|^2 dx\, dt + \alpha \int\limits_{Q_R} v^{\frac{p+2\alpha+2}{2}} \zeta^2 dx\, dt\right)^{1+\frac{2}{p}}$$

for a new constant c which does not depend on α (we assume that $\alpha \geq \frac{p-2}{2}$). Observing (3.29) estimate (3.41) simplifies to

$$\int\limits_{Q_R(\sigma_1,\sigma_2)} v^{(\alpha+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p})} dx\,dt \leq \tag{3.42}$$

$$\leq c\left(\tfrac{2}{\sigma_2 R^p}\int\limits_{Q_R} v^{\alpha+1} dx\,dt + \tfrac{4}{\sigma_1^2 R^2}\int\limits_{Q_R} v^{\frac{p+2\alpha}{2}} dx\,dt + \alpha\int\limits_{Q_R} v^{\frac{p+2\alpha+2}{2}} dx\,dt\right)^{1+\frac{2}{p}}.$$

Now, for every $\nu \in \mathbb{N}$ we put

$$\begin{aligned} R_\nu &= \frac{R_0}{2}(1+\frac{1}{2^\nu}) \\ Q_R &= Q_\nu \\ Q_R(\sigma_1,\sigma_2) &= Q_{\nu+1} \\ \sigma_1 R &= \frac{R_0}{2^{\nu+2}} \\ \sigma_2 R^p &= \frac{R_0^p}{2^{\nu+2}} \\ \alpha &= \alpha_\nu \\ \mu &= 1+\frac{2}{p} \end{aligned}$$

in (3.42) and get after some elementary manipulations

$$\int\limits_{Q_{\nu+1}} v^{(\alpha_\nu+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha_\nu+2}{2\alpha_\nu+p})} dx\,dt \leq \tag{3.43}$$

$$\leq \widetilde{c}\left(\frac{4^\nu}{R_0^p}|Q_\nu| + \left(\alpha_\nu + \frac{4^\nu}{R_0^p}\right)\int\limits_{Q_\nu} v^{\frac{p+2\alpha_\nu+2}{2}} dx\,dt\right)^{1+\frac{2}{p}}.$$

In order to iterate (3.43) we define

$$\begin{aligned} a_{\nu+1} &:= (\alpha_\nu+\frac{p}{2})(1+\frac{2}{p}\cdot\frac{2\alpha_\nu+2}{2\alpha_\nu+p}) \\ a_\nu &:= \frac{p+2\alpha_\nu+2}{2}. \end{aligned}$$

We see that

$$a_{\nu+1} = \mu a_\nu - 2$$

and hence that

$$a_\nu = \mu^\nu(a_0-p)+p\,.$$

Thus, $a_\nu \to \infty$ as $\nu \to \infty$—provided $a_0 > p$. Moreover we have $\alpha_\nu = \mu^\nu(a_0-p)+\frac{p}{2}-1$ such that we obtain from (3.43) by simple estimates

$$\int\limits_{Q_{\nu+1}} v^{a_{\nu+1}} dx\,dt \leq K 4^{\nu\mu}\left(1+\int\limits_{Q_\nu} v^{a_\nu} dx\,dt\right)^{\mu} \tag{3.44}$$

with $K = \widetilde{c}\,(|M| + a_0 + \frac{1}{R_0^p})^\mu$. Now we use once more the fact that $\mu \leq 2$ and hence that $(1+x)^\mu \leq 4(1+x^\mu)$ for all $x \geq 0$. Defining

$$I_\nu = \int\limits_{Q_\nu} v^{a_\nu}\, dx\, dt$$

(3.44) thus implies for $\nu \in \mathbb{N}_0$

$$I_{\nu+1} \leq 4 \cdot 16^\nu K (1 + I_\nu^\mu)\,. \tag{3.45}$$

Using (3.45) we can easily prove by induction that

$$I_\nu \leq L^{b_\nu}(1 + I_0^{\mu^\nu})$$

with $L = 4K + 16$ and a sequence b_ν satisfying

$$b_{\nu+1} = \mu b_\nu + 2\mu + \nu\,, \quad b_0 = 0\,.$$

It is an interesting exercise to find an explicit formula for b_ν:

$$b_\nu = \mu^\nu(2 + b_0 + p + \frac{p^2}{4}) - (2 + p + \frac{p^2}{4} + \frac{\nu p}{2})\,.$$

An inductive proof of this formula is not hard to do. Now we immediately obtain the limits

$$\begin{aligned} \frac{\mu^\nu}{a_\nu} &\to A = \frac{1}{a_0 - p} \\ \frac{b_\nu}{a_\nu} &\to B = \frac{2 + p + \frac{p^2}{4}}{a_0 - p}\,. \end{aligned}$$

This implies

$$||\nabla f||_{L^\infty(Q_{R_0}(\frac{1}{2},\frac{1}{2}))} = \sup_{\nu \in \mathbb{N}} I_\nu^{1/a_\nu} \leq L^B(1 + I_0^A)\,.$$

□

As a corollary of the Lemmas 3.6 and 3.7 we immediately obtain

Corollary 3.2 *If* $\dim(M) = p$ *then there exists a constant* $\varepsilon_3 > 0$ *only depending on* M *and* N *with the following property:*

For any solution $f \in C^2(M \times [0,T[; N)$ *of the p-harmonic flow (3.10) with* $R^* = R^*(\varepsilon_3, f, M \times [0,T[) > 0$ *and any open set* $\Omega \subset\subset M \times]0,T[$ *there exists a constant* C *which only depends on* p, E_0, M, N, R^* *and* $\operatorname{dist}(\Omega, M \times \{0\})$ *such that*

$$\|\nabla f\|_{L^\infty(\Omega)} < C.$$

$E_0 = E(f(\,\cdot\,, 0)$ *again denotes the initial energy.*

3.7 Energy Concentration

In the theory of the harmonic flow an energy concentration theorem plays a fundamental role (see Struwe [107]). A modified form of this theorem also holds in the case $p > 2$. It tells us that the local energy cannot concentrate to fast.

Theorem 3.3 *If* $\dim(M) = p$ *and* $f \in C^2(M \times [0,\infty[; N)$ *is a solution of the p-harmonic flow (3.10) then there exist constants* c, $\varepsilon_0 > 0$ *which only depend on the geometry of the manifolds* M *and* N*, and there exists a time* $T_0 > 0$ *which depends in addition on* E_0 *and* $R^*(\varepsilon_0, f, M \times \{0\})$*, with the following properties: If the initial local energy satisfies*

$$\sup_{x\in M} E(f(0), B_{2R}(x)) < \varepsilon_0$$

then it follows

$$E(f(t), B_R(x)) \le E(f(0), B_{2R}(x)) + c\, E_0^{1-\frac{1}{p}} \frac{t}{R^p} \tag{3.46}$$

for all $(x,t) \in M \times [0, T_0]$*. Here* E_0 *denotes the initial energy* $E_0 = E(f(\,\cdot\,, 0))$.

Proof
We make the same assumptions on M as in the proof of Lemma 3.6. We choose a testfunction $\varphi \in C_0^\infty(B_{2R}(x))$ which satisfies

$$\begin{aligned} 0 \le \;\; \varphi \;\; &\le 1 \\ \varphi \;\; &\equiv \;\; 1 \text{ on } B_R(x) \\ |\nabla\varphi| \;\; &\le \;\; \frac{2}{R} \end{aligned}$$

where x is an arbitrary point in M and $4R \le \iota$ (ι the injectivity radius of M). Then we test

$$f_t - \nabla(\nabla f |\nabla f|^{p-2}) \perp T_f N$$

by the testfunction $f_t\varphi^2$ and obtain

$$\begin{aligned} 0 &= \int_0^T \int_{B_{2R}(x)} \left(f_t^2\varphi^2 - \nabla(\nabla f|\nabla f|^{p-2}) f_t \varphi^2\right) dx\, dt = \\ &= \int_0^T \int_{B_{2R}(x)} \left(f_t^2\varphi^2 + \nabla f|\nabla f|^{p-2}(\nabla f_t\varphi^2 + 2\varphi\nabla\varphi f_t)\right) dx\, dt\,. \end{aligned} \tag{3.47}$$

Thus,

$$\int_0^T \int_{B_{2R}(x)} \left(f_t^2 + \frac{1}{p}\frac{d}{dt}|\nabla f|^p\right)\varphi^2 dx\, dt = -2\int_0^T \int_{B_{2R}(x)} \nabla f|\nabla f|^{p-2}\varphi\nabla\varphi f_t\, dx\, dt\,.$$

This implies

$$E(f(T),B_R(x)) - E(f(0),B_{2R}(x)) \le \tag{3.48}$$
$$\le \frac{1}{p}\int\limits_{B_{2R}(x)} |\nabla f|^p\varphi^2 dx\Big|_0^T =$$
$$= -\int\limits_0^T\int\limits_{B_{2R}(x)} f_t^2\varphi^2 dx\,dt - 2\int\limits_0^T\int\limits_{B_{2R}(x)} \nabla f|\nabla f|^{p-2}\varphi\nabla\varphi f_t\,dx\,dt\,.$$

The second term on the right hand side of (3.48) may be estimated by Young's inequality by the first term and

$$\int\limits_0^T\int\limits_{B_{2R}(x)} |\nabla f|^{2p-2}|\nabla\varphi|^2 dx\,dt\,.$$

In this way we get from (3.48)

$$E(f(T),B_R(x)) - E(f(0),B_{2R}(x)) \le \tag{3.49}$$
$$\le \frac{c}{R^2}\int\limits_0^T\int\limits_{B_{2R}(x)} |\nabla f|^{2p-2}dx\,dt \le$$
$$\le c\frac{T^{\frac{1}{p}}}{R}\left(\int\limits_0^T\int\limits_{B_{2R}(x)} |\nabla f|^{2p}dx\,dt\right)^{1-\frac{1}{p}}$$

using Hölder's inequality in the last step. Now, there exists a number L only depending on the geometry of M but not on R such that every ball $B_{2R}(x)$ may be covered by at most L balls $B_R(x_i)$. Now remember that in Lemma 3.4 we found a constant $\varepsilon_1 > 0$ with the property that

$$\sup_{t\in[0,T],y\in B_{4R}(x)}\int\limits_{B_{2R}(y)} |\nabla f|^p dx < \varepsilon_1 \Longrightarrow \tag{3.50}$$
$$\Longrightarrow \int\limits_0^T\int\limits_{B_{2R}(x)} |\nabla f|^{2p}dx\,dt < cE_0\left(1+\frac{T}{(2R)^p}\right)\,.$$

Now we choose

$$\varepsilon_0 = \frac{\varepsilon_1}{4L}$$

and suppose that $R_0 < \frac{\iota}{4}$ is such that there holds

$$\sup_{x\in M} E(f(0),B_{2R_0}(x)) < \varepsilon_0\,.$$

Then we choose T_0 as

$$T_0 = \min\left\{\left(\frac{\varepsilon_1 R_0}{4Lc(2cE_0)^{1-\frac{1}{p}}}\right)^p\,,\ 1\right\}\,.$$

Now we claim: For all $(x,t) \in M \times [0,T_0]$ and all $R \le R_0$ there hold

(i)
$$\int_0^t \int_{B_{2R}(x)} |\nabla f|^{2p} dx\, dt \le cE_0 \left(1 + \frac{t}{(2R)^p}\right)$$

and

(ii)
$$\sup_{0\le\tau\le t} \int_{B_{2R}(x)} |\nabla f(\,\cdot\,,\tau)|^p dx \le \varepsilon_1\,.$$

To see this let $T < T_0$ such that for $t \in [0,T]$ (i) and (ii) hold. Then it follows from (3.49) and (3.50) together with (ii) that

$$\begin{aligned}
\sup_{(x,t)\in M\times[0,T]} & E(f(t), B_{2R_0}(x)) \le \\
&\le L \sup_{(x,t)\in M\times[0,T]} E(f(t), B_{R_0}(x)) \le \\
&\le L\left(\sup_{x\in M} E(f(0), B_{2R_0}(x)) + c\,\frac{T^{\frac{1}{p}}}{R_0}\left(cE_0(1+\frac{T}{(2R_0)^p})\right)^{1-\frac{1}{p}}\right) \le \\
&\le \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4} = \frac{\varepsilon_1}{2}
\end{aligned}$$

due to the special choice of ε_0, R_0 and T_0. Thus, (ii) and consequently (i) hold on some larger interval $[0, T+\delta]$. On the other hand the interval where (i) and (ii) hold is closed and nonempty. Hence, (i) and (ii) hold on $[0,T_0]$. Using (i) in the formula (3.49) the assertion follows after a short calculation with a new constant c. □

Chapter 4

Existence Results

The aim of this chapter will be to prove existence results for the heat flow of the p-energy for functions $f : M \to N$ where M and N are compact smooth Riemannian manifolds of dimension $m = p$ and n, respectively, as before. Recall that if we think of N as being isometrically embedded in some $\mathbb{R}^k$, the equation of the heat flow is

$$f_t - \Delta_p f = (p\, e(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)_M \tag{4.1}$$

with initial data $f_0 : M \to N$

$$f = f_0 \quad \text{at time } t = 0 \tag{4.2}$$

where we think of f as a map from M to $\mathbb{R}^k$ with constraint $f(M) \subset N$ for all time. Here Δ_p will continue to denote the p-Laplace operator related to M and $\mathbb{R}^k$, $e(f)$ is the p-energy density and $A(f) : T_f N \times T_f N \to (T_f N)^\perp$ is the second fundamental form of N.

In order to prove existence results for problem (4.1) we encounter two main difficulties:

(1) We have to assure that the image of f remains contained in N for all time: $f(M) \subset N \subset \mathbb{R}^k$ $\forall t \geq 0$.

(2) The p-Laplace operator is degenerate for $p > 2$.

In Chapter 2 we considered the case $N = S^n$. Due to the special geometry of the sphere we overcame the first difficulty by the "penalty trick". In case $p = 2$ the same technique has been applied successfully for general N (see Chen-Struwe [16]). For $p > 2$ this does not seem to work any more. Thus, we will try to solve the problem with Hamilton's technique of a totally geodesic embedding of N in $\mathbb{R}^k$ (see [59]). Using this special kind of embedding, Hamilton was able to show local existence of the 2-harmonic flow.

The second difficulty will be attacked by regularizing the p-energy: We will consider

$$E_\varepsilon(f) = \frac{1}{p}\int_M (\varepsilon + |df|^2)^{\frac{p}{2}}\,d\mu.$$

We will then apply the theory of analytic semigroups to the corresponding regularized operator. Due to the a priori estimates of Chapter 3 it will be possible to pass to the limit $\varepsilon \to 0$.

4.1 Totally geodesic embedding of N in $\mathbb{R}^k$

In a first step we will work with a special embedding of N in $(\mathbb{R}^k, h)$: We equip $\mathbb{R}^k$ with a metric h such that

1. N is embedded isometrically, i.e. the metric g on N equals the metric induced by h.
2. The metric h equals the Euclidean metric outside a large ball B.
3. There exists an involutive isometry $\iota : T \to T$ on a tubular neighborhood T of N corresponding to multiplication by -1 in the orthonormal fibers of N and having precisely N for its fixed point set.

Such an embedding is called totally geodesic: The h-geodesic curve γ connecting x, $y \in N$ (x, y close enough) will always be contained in N. This follows from the (local) uniqueness of geodesics and the fact that with γ the curve $\iota \circ \gamma$ is another geodesic joining x and y.

A totally geodesic embedding can be accomplished as follows: We start with the standard Nash-embedding of $N \subset \mathbb{R}^k$ and choose a tubular neighborhood T of N: $T = T_{2\delta} = \{x \in \mathbb{R}^k \;:\; \mathrm{dist}(x, N) < 2\delta\}$ (δ small enough and dist the Euclidean distance). Then we choose locally in $N \times]-2\delta, 2\delta[^{k-n}$ the metric $\tilde{h}_{ij} = g_{ij} \otimes \delta_{ij}$ (or like Hamilton in [59] we just take the average of any extension of g under the action of ι). Then we smooth out $\tilde{h}$ by taking a positive C^∞ function ψ with support in $T_{2\delta}$ and $\psi \equiv 1$ on T_δ and by defining $h_{ij} = \psi\,\tilde{h}_{ij} + (1-\psi)\delta_{ij}$.

4.2 The regularized p-energy

The p-energy as defined in Section 1.1 gives rise to the p-Laplace operator which is degenerate for $p > 2$ and singular in case $p < 2$. For that reason it makes sense to consider a regular approximation of the p-energy:

Definition 4.1 *For $\varepsilon > 0$ the regularized p-energy density of a C^1-mapping $f : M \to N$ is*

$$e_\varepsilon(f)(x) := \frac{1}{p}\left(\varepsilon + |df_x|^2\right)^{\frac{p}{2}} = \frac{1}{p}\left(\varepsilon + \mathrm{trace}\left((df_x)^* df_x\right)\right)^{\frac{p}{2}}$$

and the regularized p-energy of f is

$$E_\varepsilon(f) := \int_M e_\varepsilon(f)\, d\mu$$

where the norm $|\cdot|$, the adjoint star and the measure μ are associated with the given Riemannian metrics on M and N.

As in Chapter 1 we derive the equation for the heat flow of the regularized p-energy. We find as the intrinsic form

$$f_t - \Delta_p^\varepsilon f = 0 \tag{4.3}$$

where in local coordinates

$$\Delta_p^\varepsilon f = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\varepsilon + \gamma^{\alpha\beta} g_{ij} \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1} \gamma^{\alpha\beta}\frac{\partial f^l}{\partial x^\alpha}\right) + \\ + \left(\varepsilon + \gamma^{\alpha\beta} g_{ij}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1} \gamma^{\alpha\beta}\Gamma^l_{ij}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta} \tag{4.4}$$

(g and γ are the metrics of N and M respectively, Γ^l_{ij} denote the Christoffel symbols related to g). It will be necessary below to attach the related target-manifold in the notation; we will write ${}^g\Delta_p^\varepsilon$ or ${}^N\Delta_p^\varepsilon$ for that.

The extrinsic form of this equation which we use if N is isometrically embedded in the Euclidean space $\mathbb{R}^k$ is

$$f_t - \Delta_p^\varepsilon f = (p\, e_\varepsilon(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)_M \tag{4.5}$$

where in local coordinates

$$\Delta_p^\varepsilon f = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\left(\varepsilon + \gamma^{\alpha\beta}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\right).$$

In the next section we will look for a solution of the flow of the regularized p-energy.

4.3 The flow of the regularized p-energy

Let us first state a theorem of A. Lunardi (see [81]). We will use a version which was formulated by V. Vespri in [126].

Theorem 4.1 *Consider the following Cauchy problem in a Banach space X*

$$\begin{aligned} f_t &= \psi(f(t)) \qquad \text{for } t \geq 0 \\ f(0) &= f_0 \end{aligned} \tag{4.6}$$

where ψ is a C^2 function from Y to X, Y is a continuously embedded subspace of X and $f_0 \in Y$. Assume that

(i) the linear operator $D\psi(f_0) : Y \to X$ generates an analytic semigroup in X and

(ii) $\psi(f_0) \in \overline{Y}$

then there exists a strict solution f (in the sense of Lunardi [81]) of (4.6) on a time interval $[0,\tau]$, $\tau > 0$, and $f \in C^1([0,\tau], X) \cap C^0([0,\tau], Y)$. Moreover f is unique.

Now we will apply Lunardi's Theorem to the regularized flow. To do this, consider a totally geodesic embedding of N in $(\mathbb{R}^k, h)$ and the regularized p-Laplace operator ${}^h\Delta_p^\varepsilon : Y \to X$ with $X = L^2(M, \mathbb{R}^k)$ and $Y = W^{2,2}(M, \mathbb{R}^k) \cap C^{1,\alpha}(M, \mathbb{R}^k)$ for some $\alpha > 0$ (the fact that ${}^h\Delta_p^\varepsilon$ maps Y in X is checked directly in the definition (4.4)). Y is equipped with the norm $|||\cdot||| = \|\cdot\|_{W^{2,2}} + \|\cdot\|_{C^{1,\alpha}}$ and is hence continuously embedded in X. On the other hand $\overline{Y} = X$ (the closure is taken with respect to the L^2-norm in X) such that for any initial data $f_0 \in Y$ (ii) is satisfied. By expanding ${}^h\Delta_p^\varepsilon(f_0 + k)$ we find the first derivative of the regularized p-Laplace operator:

(4.7)

$$\begin{aligned} D({}^h\Delta_p^\varepsilon)(f_0) : Y &\to X \\ k &\mapsto \tfrac{1}{\sqrt{\gamma}}\big(\sqrt{\gamma}(\tfrac{p}{2}-1)(p\, e_\varepsilon(f_0))^{1-\frac{4}{p}}\gamma^{\sigma\rho}(h_{ij,s}k^s f^i_{0,\sigma} f^j_{0,\rho} + \\ &\quad +2h_{ij} f^i_{0,\sigma} k^j_{,\rho})\gamma^{\alpha\beta} f^l_{0,\alpha} + \sqrt{\gamma}(p\, e_\varepsilon(f_0))^{1-\frac{2}{p}}\gamma^{\alpha\beta}k^l_{,\alpha}\big)_{,\beta} + \\ &\quad +(\tfrac{p}{2}-1)(p\, e_\varepsilon(f_0))^{1-\frac{4}{p}}\gamma^{\alpha\beta}\Gamma^l_{ij} f^i_{0,\alpha} f^j_{0,\beta}\gamma^{\sigma\rho}(h_{ij,s}k^s f^i_{0,\sigma} f^j_{0,\rho} + \\ &\quad +2h_{ij} f^i_{0,\sigma} k^j_{,\rho}) + (p\, e_\varepsilon(f_0))^{1-\frac{2}{p}}\gamma^{\sigma\rho}(\Gamma^l_{ij,s}k^s f^i_{0,\sigma} f^j_{0,\rho} + 2\Gamma^l_{ij} f^i_{0,\sigma} k^j_{,\rho}) \end{aligned}$$

Here upper indices denote components whereas ${}_{,\alpha}$ means $\frac{\partial}{\partial x_\alpha}$. It is not difficult to check that for $f_n \to f$ in $(Y, |||\cdot|||)$ we have $D({}^h\Delta_p^\varepsilon)(f_n) \to D({}^h\Delta_p^\varepsilon)(f)$ in $L(Y, X)$ and hence that the mapping $f \mapsto D({}^h\Delta_p^\varepsilon)(f)$ is continuous. Analogously we find that ${}^h\Delta_p^\varepsilon$ has a continuous second derivative. The important facts about the operator $D({}^h\Delta_p^\varepsilon)(f_0)$ are

(1) The coefficients of $D({}^h\Delta_p^\varepsilon)(f_0)$ are of class $C^{0,\alpha}(M)$.

(2) For $\varepsilon > 0$ the operator $D({}^h\Delta_p^\varepsilon)(f_0)$ is elliptic in the sense that the main part satisfies a uniform strong Legendre-condition.

(1) is obvious. To see (2) we introduce normal coordinates in a neighborhood of $z_0 := f_0(x_0)$, i.e. $h_{ij}(z_0) = \delta_{ij}$. Then the main part of $D({}^h\Delta_p^\varepsilon)(f_0)$ in x_0 is

$$A_{il}^{\rho\beta} k^i_{,\rho\beta} = (p\, e_\varepsilon(f_0))^{1-\frac{4}{p}} \left((p-2)\gamma^{\sigma\rho} f^i_{0,\sigma} \gamma^{\alpha\beta} f^l_{0,\alpha} + (p\, e_\varepsilon(f_0))^{\frac{2}{p}} \gamma^{\alpha\beta} \delta^\rho_\alpha \delta^l_i \right) k^i_{,\rho\beta}$$

such that we get for an arbitrary vector ξ^i_α and a uniform constant $\nu > 0$ depending on the geometry of M

$$A_{il}^{\rho\beta} \xi^i_\rho \xi^l_\beta = (p\, e_\varepsilon(f_0))^{1-\frac{4}{p}} \left((p-2)(\mathrm{trace}(\gamma \xi f_0))^2 + (p\, e_\varepsilon(f_0))^{\frac{2}{p}} \langle \xi, \xi \rangle \right) \geq \nu |\xi|^2$$

where $\langle \cdot\,, \cdot \rangle$ denotes the inner product induced by γ.

Another way to force ellipticity consists in using the observation that the main part of $D({}^h\Delta_p^\varepsilon)(f_0)$ depends only on the metric h but not on its derivatives. For the Euclidean metric $h_{ij} = \delta_{ij}$ the required ellipticity is obvious (see above). Then ellipticity follows at least for metrics h which are close enough to the Euclidean metric. But we *can* choose h as close as we want to the Euclidean metric by choosing the δ-neighborhood of N in the construction of the totally geodesic embedding small (compare Section 4.1).

As Vespri has shown in [126] the conditions (1) and (2) guarantee that $D({}^h\Delta_p^\varepsilon)(f_0) : Y \to X$ generates an analytic semigroup in X. In fact, Vespri showed that under these conditions there exist constants C, $\omega > 0$ such that for all $\varphi \in X$ and every complex number λ satisfying $\mathrm{Re}(\lambda) > \omega$ there exists a solution $u \in Y$ of

$$\left(\lambda - D({}^h\Delta_p^\varepsilon)(f_0)\right) u = \varphi$$

with

$$\|u\|_X \leq \frac{C}{|\lambda|} \|\varphi\|_X \,.$$

(As a general reference for the theory of semigroups see e.g. Goldstein [55].)

Thus, we may apply Lunardi's Theorem and get that there exists a unique local solution $f \in C^0([0,\tau], W^{2,2}(M) \cap C^{1,\alpha}(M)) \cap C^1([0,\tau], L^2(M))$ of (4.3) with initial data $f_0 \in Y$. Furthermore we have

Theorem 4.2 *If* $\mathrm{Im}(f_0) \subset N$, *then* $\mathrm{Im}(f(\,\cdot\,, t)) \subset N$ *for all* $t \in [0,\tau]$.

Proof
Let ι still denote the involutive isometry on the tubular neighborhood T of $N \subset \mathbb{R}^k$ defined in Section 4.1. We proceed by contradiction. If the image of f does not always remain in N, we can restrict ourselves to a smaller interval $M \times [0, \tau']$, $\tau' \leq \tau$, such that the image of f does not always remain in N but in the tubular neighborhood T of N. Since $\iota : T \to T$ is an isometry, the composition $\iota \circ f$ is another solution of (4.3). Since ι is the identity on N this solution has the same initial value as f, so by the uniqueness of the solution we have $\iota \circ f = f$. This shows that the image of f must remain in the fixed point set N of ι. $\square$

Theorem 4.3 *If* (N, g) *is a totally geodesic embedded submanifold of* $(\mathbb{R}^k, h)$ *and* $f : M \to N \subset \mathbb{R}^k$ *then* ${}^g\Delta_p^\varepsilon f = {}^h\Delta_p^\varepsilon f$.

Proof
We refer to Hamilton [59], Section IV.5, page 108. The proof there is given for $p = 2$ and $\varepsilon = 0$, but it carries over to our situation. □

Thus, according to Theorem 4.2 we find that for initial data $f_0 : M \to N$, $f_0 \in Y$, the solution f of (4.3) we found above satisfies $f(M) \subset N$ for all $t \in [0, \tau]$ and is, by applying Theorem 4.3 twice, a solution of (4.1).

Notice that since we have constructed the local solution for *fixed* ε the existence interval $[0, \tau(\varepsilon)]$ might depend on ε. So, we will have to show that $\tau(\varepsilon) \not\to 0$ as $\varepsilon \to 0$.

4.4 An ε-independent existence interval

First we formulate the a priori estimates of Chapter 3 for solutions of the heat flow of the energy E_ε.

Theorem 4.4 *Let* $f : M \times [0, \tau] \to N \subset \mathbb{R}^k$, $f \in C^0([0, \tau], W^{2,2}(M) \cap C^{1,\alpha}(M)) \cap C^1([0, \tau], L^2(M))$, *be a solution of (4.1) with initial value* f_0. *We assume that* $\varepsilon \le 1$. *Then the following is true*

(i) $$\int_0^t \int_M |\partial_t f|^2 d\mu\, dt + E_\varepsilon(f(t)) = E_\varepsilon(f_0) \le E_1(f_0) \qquad \textit{(energy inequality)}$$

(ii) There exist constants C, $\varepsilon_0 > 0$, *only depending on* M *and* N *(but not on* ε *and* f_0*), and* $T_0 > 0$ *depending in addition on* $E_1(f_0)$ *and* $R^*(\varepsilon_0, f, M \times \{0\})$, *such that the condition*

$$\sup_{x \in M} E_1(f_0, B_{2R}(x)) < \varepsilon_0$$

implies

$$E_\varepsilon(f(t), B_R(x)) \le E_1(f_0, B_{2R}(x)) + C E_1(f_0)^{1-\frac{1}{p}} \frac{t}{R^p}$$

for all $(x, t) \in M \times [0, \min\{\tau, T_0\}]$.

(iii) There exists a constant $\varepsilon_1 > 0$ *only depending on* M *and* N *(but not on* ε *and* f_0*) such that*

$$R^* = R^*(\varepsilon_1, f, M \times [0, \tau]) > 0$$

implies for every $\Omega \subset M \times [0, \tau]$ *with* $\operatorname{dist}(\Omega, M \times \{0\}) = \mu > 0$

$$\|\nabla f\|_{L^\infty(\Omega)} \le C$$

where C *is a constant that depends on* p, $E_1(f_0)$, M, N, R^* *and* μ.

(iv) For the same constant ε_1 as in (iii) we have that

$$R^* = R^*(\varepsilon_1, f, M \times [0,\tau]) > 0$$

implies

$$\|\nabla f\|_{L^\infty(M\times[0,\tau])} \le C$$

where C is a constant that depends on p, M, N, R^* and the L^∞-norm of the initial value ∇f_0.

Proof

(i)–(iii): We obtain these assertions by repeating the corresponding proofs of Chapter 3. Notice however that the regularized energy E_ε is not conformally invariant, so every argument based on this fact would break down.

(iv) Here we repeat the three steps of Section 3.4–3.6: We obtain the L^{2p}-estimate for ∇f as in Section 3.4 by testing the equation by $-\varphi^p \Delta_p^\varepsilon f$.

Using the L^{2p}-estimate for ∇f we get the L^q-estimates for $q < \infty$ as in Section 3.5 by an iteration argument: In order to obtain the estimate up to $t = 0$ we have to choose the cutoff function ζ independent of t, i.e. $\sigma_2 = 0$ (in the notation of Section 3.5). Using the testfunction

$$-\nabla(\nabla f(\varepsilon + v)^\alpha \zeta^2)$$

(with $v = |\nabla f|^2$) in (4.1) we find after some calculation the analogue of (3.37)

$$\begin{aligned}\tfrac{1}{4(\alpha+1)} \operatorname*{ess\,sup}_{0<t<t_0} \int_{B_R} v^{\alpha+1}\zeta^2(\,\cdot\,,t)\,dx + \tfrac{\alpha k^2}{(p+2\alpha)^2} \int_0^{t_0} \|v^{\frac{p+2\alpha}{4}}\zeta\|^2_{L^{2^*}(B_R)} dt \le \\ \le \tfrac{1}{\alpha+1} \int_{B_R} v^{\alpha+1}(\,\cdot\,,0)\zeta^2 dx + 2\left(\tfrac{1}{\alpha+p-2} + \tfrac{p-2}{\alpha} + \tfrac{\alpha+p-2}{(p+2\alpha)^2}\right) \int_{Q_R} v^{\frac{p+2\alpha}{2}} |\nabla\zeta|^2 dx\,dt\,.\end{aligned}$$

The iteration yields

$$\int_{Q_R} v^q dx\,dt \le C$$

where C now depends on the initial datum $\int_{B_R} v^{\alpha_0+1}(\,\cdot\,,0)dx$ (α_0 denotes the initial value of the iteration).

Finally as in Section 3.6 we use a Moser-iteration to get the L^∞-bound for ∇f. Using the same testfunction as in the last step, we obtain the analogue of (3.41)

$$\begin{aligned}\int_{Q_R(\sigma_1)} v^{(\alpha+\frac{p}{2})(1+\frac{2}{p}\frac{2\alpha+2}{2\alpha+p})} dx\,dt \le \\ \le c\left(\int_{B_R} v^{\alpha+1}(\,\cdot\,,0)\zeta^2 dx\,dt + \int_{Q_R} v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2 dx\,dt + \alpha\int_{Q_R} v^{\frac{p+2\alpha+2}{2}}\zeta^2 dx\,dt\right)^{1+\frac{2}{p}}.\end{aligned}$$

Iteration yields

$$\|\nabla f\|_{L^\infty(Q_R(\frac{1}{2}))} \leq C$$

where C now depends on the initial datum $\|\nabla f_0\|_{L^\infty(M)}$. $\square$

Now we combine (ii) and (iv): Let $R = R^*(\frac{\min\{\varepsilon_0,\varepsilon_1\}}{2}, f_0, M \times \{0\})/3$. Hence, R is a radius with the property

$$\sup_{x\in M} E_1(f_0, B_{2R}(x)) < \frac{\min\{\varepsilon_0, \varepsilon_1\}}{2}.$$

Then (ii) implies that for all $t \leq T^*$,

$$T^* = T^*(f_0) = \frac{\varepsilon_1 R^p}{2CE_1(f_0)^{1-\frac{1}{p}}},$$

the hypothesis of (iv) and hence the conclusion holds:

$$\|\nabla f\|_{L^\infty(M\times[0,t])} \leq C \tag{4.8}$$

for $t \leq \min\{\tau, T^*\}$ with a constant depending on M, N, p, and R.

Now, for the solution $f : M \times [0,\tau] \to N$ of (4.5) constructed in Section 4.3 we define $B : M \times [0,\tau] \to \mathbb{R}^k$, $B = (p\, e_\varepsilon(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)_M$. Then f is a solution of

$$f_t - \Delta_p^\varepsilon f = B$$

and the results of DiBenedetto [24] apply: We have that $\|B\|_{L^\infty(M\times[0,\tau])} \leq C(f_0)$ and the operators $\{\Delta_p^\varepsilon\}_{\varepsilon\in[0,1]}$ satisfy the structural conditions of [24], Section VIII.1(ii) (notice that $B(x,t)$ does not depend on ∇f). Then we infer from [24], Section IX.1, that ∇f is Hölder continuous: there is a Hölder exponent $\tilde{\alpha} > 0$, independent of ε and τ, with

$$\|\nabla f\|_{C^{0,\tilde{\alpha}}(M\times[0,\tau])} \leq C(f_0)$$

(see also [24], Section VIII.1, p.216).

In order to find an ε-independent existence interval for the solution we will also need control of the second spatial derivatives in the L^2 norm:

By squaring the equation (4.5) and using an analogous estimate for the L^2-norm of $\Delta_p^\varepsilon f$ as in 3.23 of Section 3.4 (Chapter 3) we get

$$\begin{aligned} \varepsilon^{p-2}\int_0^\tau\int_M |\nabla^2 f|^2 dx\,dt &\leq \int_0^\tau\int_M |\nabla^2 f|^2(\varepsilon + |\nabla f|^2)^{p-2} dx\,dt \leq \\ &\leq \int_0^\tau\int_M |\Delta_p^\varepsilon f|^2 dx\,dt \leq \\ &\leq c\int_0^\tau\int_M |f_t|^2 dx\,dt + c\int_0^\tau\int_M |\nabla f|^{2p} dx\,dt \end{aligned}$$

with a constant c dependent on the geometry of the problem. Thus, combining (4.8) and Theorem 4.4(i) we get

$$\int_0^t \int_M |\nabla^2 f|^2 dx\, dt \le C(\varepsilon, f_0) \tag{4.9}$$

for $t \le \min\{\tau, T^*\}$.

But we need more, namely an $L^2(M)$-bound for $\nabla^2 f$ which is uniform in t. For simplicity we carry out the calculation in the case of the flat torus $M = \mathbb{R}^p / \mathbb{Z}^p$. Then the regularized p-Laplace operator is

$$\Delta_p^\varepsilon f = \nabla \left(\nabla f (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1} \right) .$$

We have (formally)

$$\begin{aligned} \int_M |\Delta_p^\varepsilon f(\,\cdot\,, t)|^2 dx = & \\ = \int_M |\Delta_p^\varepsilon f(\,\cdot\,, 0)|^2 dx + \frac{d}{dt} \int_0^t \int_M |\Delta_p^\varepsilon f|^2 dx\, dt = & \\ = \int_M |\Delta_p^\varepsilon f(\,\cdot\,, 0)|^2 dx + 2 \int_0^t \int_M \Delta_p^\varepsilon f \cdot \nabla (\nabla f_t (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1} + & \\ + (\tfrac{p}{2} - 1) \nabla f (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-2} \nabla f \cdot \nabla f_t) dx\, dt. & \end{aligned} \tag{4.10}$$

Now we multiply the gradient of (4.1) by $(\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1}$ and test the resulting equation with $-\nabla \Delta_p^\varepsilon f$ and obtain

$$\begin{aligned} & \int_0^t \int_M \Delta_p^\varepsilon f \cdot \nabla (\nabla f_t (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1}) dx\, dt + \\ & \quad + \int_0^t \int_M |\nabla \Delta_p^\varepsilon f|^2 (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1} dx\, dt = \\ = & \quad - \int_0^t \int_M (\nabla \Delta_p^\varepsilon f)(\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1} \nabla ((p\, e_\varepsilon(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f)) \le \\ \le & \quad \tfrac{1}{2} \int_0^t \int_M |\nabla \Delta_p^\varepsilon f|^2 (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1} dx\, dt + \\ & \quad + \tfrac{1}{2} \int_0^t \int_M (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-1} |\nabla ((p\, e_\varepsilon(f))^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f))|^2 dx\, dt\, . \end{aligned} \tag{4.11}$$

In the last step we used Young's inequality. On the other hand we build the inner product of the gradient of (4.1) and ∇f, multiply with $(\frac{p}{2} - 1) \nabla f (\varepsilon + |\nabla f|^2)^{\frac{p}{2}-2}$ and finally test the resulting equation again with $-\nabla\Delta_p^\varepsilon f$ to obtain

$$
\begin{aligned}
(4.12)\quad & \int_0^t \int_M (\Delta_p^\varepsilon f)\cdot\nabla(\nabla f(\tfrac{p}{2}-1)\nabla f\cdot\nabla f_t(\varepsilon+|\nabla f|^2)^{\frac{p}{2}-2})dx\,dt+\\
&+\int_0^t \int_M (\nabla\Delta_p^\varepsilon f)((\nabla\Delta_p^\varepsilon f)\cdot\nabla f(\tfrac{p}{2}-1)(\varepsilon+|\nabla f|^2)^{\frac{p}{2}-2}\nabla f)dx\,dt=\\
&=-\int_0^t \int_M (\nabla\Delta_p^\varepsilon f)\cdot(\nabla((p\,e_\varepsilon(f))^{1-\frac{2}{p}}A(f)(\nabla f,\nabla f))\cdot\nabla f(\tfrac{p}{2}-1)(\varepsilon+|\nabla f|^2)^{\frac{p}{2}-2}\nabla f)dx\,dt\le\\
&\le\tfrac{p-2}{4}\int_0^t \int_M (\nabla\Delta_p^\varepsilon f\cdot\nabla f)^2(\varepsilon+|\nabla f|^2)^{\frac{p}{2}-2}dx\,dt+\\
&+\tfrac{p-2}{4}\int_0^t \int_M |\nabla((p\,e_\varepsilon(f))^{1-\frac{2}{p}}A(f)(\nabla f,\nabla f))|^2|\nabla f|^2(\varepsilon+|\nabla f|^2)^{\frac{p}{2}-2}dx\,dt\,.
\end{aligned}
$$

In the last step we again used Young's inequality. Combining (4.10) with (4.11) and (4.12) and using the L^∞-estimates for ∇f together with (4.9) we obtain for $t\le\min\{\tau,T^*\}$

$$\|\nabla^2 f(\,\cdot\,,t)\|_{L^2(M)}\le C$$

where C depends on ε, $\|\nabla^2 f_0\|_{L^2(M)}$ and $\|\nabla f_0\|_{L^\infty(M)}$.

Remark: The calculations above only make sense, if $\nabla^3 f$ and ∇f_t are in suitable L^q-spaces. We justify these calculations for a solution f with the regularity we have, by working with the function

$$f_h(x,t)=\frac{1}{h}\int_t^{t+h} f(x,s)ds$$

and by replacing spatial differentiation by building finite-difference quotients wherever this is necessary (see e.g. [25]).

Now we can prove that $[0,T^*(f_0)]$ is the existence interval for the solution of the heat flow of the energy E_ε independent of ε by "continuous induction":

Let

$$
\begin{aligned}
I=\{t\in[0,T^*]\,:\,&\exists\tilde\alpha>0,\exists f\text{ a solution of (4.1) on }[0,t]\text{ with}\\
&f\in C^0([0,t],W^{2,2}(M)\cap C^{1,\tilde\alpha}(M))\cap C^1([0,t],L^2(M))\text{ and initial value }f_0\}\,.
\end{aligned}
$$

Then the interval I is not empty (according to Section 4.3). But I is also open: If $t\in I$, then we can extend the solution beyond t by solving the flow with initial value $f(t)$ which is possible as we have seen in Section 4.3. On the other hand by our time independent bounds for the quantities $\|\nabla^2 f\|_{L^2(M)}$, $\|\nabla f\|_{C^{0,\tilde\alpha}(M)}$ and $\|f_t\|_{L^2(M\times[0,t])}$ on $[0,T^*]$ the interval I is also closed and hence $I=[0,T^*]$.

Thus, we have

Theorem 4.5 *There exists a constant $\varepsilon_2 > 0$ depending on M and N with the following property:*

For arbitrary $f_0 : M \to N \subset \mathbb{R}^k$, $f_0 \in C^{1,\alpha}(M) \cap W^{2,2}(M)$ there exists a time $T^* > 0$ only depending on $E(f_0)$, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and the geometry of the problem such that for every $\varepsilon \in]0,1]$ there exists a solution $f \in C^0([0,T^*], W^{2,2}(M) \cap C^{1,\tilde\alpha}(M)) \cap C^1([0,T^*], L^2(M))$ of (4.1) with initial value f_0. Moreover there exist ε-independent bounds for the following quantities:

$$\begin{aligned} \|f_t\|_{L^2(M\times[0,T^*])} &\le C(E_1(f_0)) \\ \|\nabla f\|_{L^\infty(M\times[0,T^*])} &\le C(\|\nabla f_0\|_{L^\infty(M)}) \\ \|\nabla f\|_{C^{0,\tilde\alpha}(M\times[0,T^*])} &\le C(\|\nabla f_0\|_{C^{0,\alpha}(M)}) \end{aligned}$$

Of course the constants C also depend on p, M and N. The constant $\tilde\alpha$ depends on α and $\|\nabla f_0\|_{L^\infty(M)}$.

Combining Theorem 4.4 (iii) with DiBenedetto's result in [24], Theorem 1.1', Chapter IX, we obtain also

Theorem 4.6 *For the solution f from Theorem 4.5 we have for every open $\Omega \subset M \times [0,T^*]$ with* $\operatorname{dist}(\Omega, M\times\{0\}) = \mu > 0$

$$\|\nabla f\|_{C^{0,\beta}(\Omega)} \le C$$

for some constants C (depending on p, $E_1(f_0)$, M, N, $R^*(\varepsilon_2, f_0, M\times\{0\})$ and μ) and $\beta \in]0,1[$ (depending on p, M and N).

4.5 The limit $\varepsilon \to 0$

Let f_ε denote the solution of the heat flow of the energy E_ε (with initial value f_0) on the time interval $[0,T^*]$, that we have constructed in the previous section. The aim is to pass in the distributional form of (4.1) on $[0,T^*]$ to the limit. Due to the ε-independent bounds given in Theorem 4.5 we know at least that $\{f_\varepsilon\}_{\varepsilon\in]0,1]}$ is bounded in $W^{1,2}(M\times[0,T^*],N)$. Thus, we can choose a sequence $\varepsilon_k \to 0$ such that

$$f_{\varepsilon_k} \rightharpoonup f \qquad \text{weakly in } W^{1,2}(M\times[0,T^*],N).$$

But by the bound for the $C^{0,\tilde\alpha}$-norm of ∇f_ε we can pass to a subsequence if necessary, and obtain (observing that $C^{0,\tilde\alpha}(M\times[0,T^*]) \subset C^{0,\tilde\alpha/2}(M\times[0,T^*])$ compactly) that

$$\nabla f_\varepsilon \to \nabla f \qquad \text{strongly in } C^{0,\tilde\alpha/2}(M\times[0,T^*]).$$

Now, for a C_0^∞ testfunction φ we can pass to the limit $\varepsilon_k \to 0$ in

$$\int_0^{T^*}\!\!\int_M \partial_t f_\varepsilon \varphi \, d\mu\, dt + \int_0^{T^*}\!\!\int_M \frac{1}{\sqrt\gamma}\left(\sqrt\gamma\left(\varepsilon + \gamma^{\alpha\beta}\frac{\partial f_\varepsilon^j}{\partial x^\alpha}\frac{\partial f_\varepsilon^j}{\partial x^\beta}\right)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\frac{\partial f_\varepsilon}{\partial x^\alpha}\right)\frac{\partial\varphi}{\partial x^\beta} d\mu\, dt =$$

$$= \int_0^{T^*}\!\!\int_M \varphi\,(p\, e_\varepsilon(f_\varepsilon))^{1-\frac{2}{p}} A(f_\varepsilon)(\nabla f_\varepsilon, \nabla f_\varepsilon)\,.$$

Thus, we have proved the following theorem

Theorem 4.7 *There exits a constant $\varepsilon_2 > 0$ depending on M and N with the following property:*

For arbitrary $f_0 : M \to N \subset \mathbb{R}^k$, $f_0 \in C^{1,\alpha}(M) \cap W^{2,2}(M)$ there exists a time $T^* > 0$ only depending on $E(f_0)$, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and the geometry of the problem, and a local weak solution $f : M \times [0, T^*] \to N$ of

$$\begin{aligned} f_t - \Delta_p f &\perp T_f N \\ f(\,\cdot\,, 0) &= f_0 \,. \end{aligned}$$

f satisfies the energy inequality. Furthermore $\|\nabla f\|_{C^{0,\tilde{\alpha}}(M\times[0,T^*])} \le C(\|\nabla f_0\|_{C^{0,\alpha}(M)})$ and $\|\nabla f\|_{L^\infty(M\times[0,T^*])} \le C(\|\nabla f_0\|_{L^\infty(M)})$. The constants C also depend on p, M and N. Locally, for every open $\Omega \subset M \times [0, T^*]$ with $\operatorname{dist}(\Omega, M \times \{0\}) = \mu > 0$, there holds $\|\nabla f\|_{C^{0,\beta}(\Omega)} \le C$ for some constants C (depending on p, $E(f_0)$, M, N, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and μ) and $\beta \in]0, 1[$ (depending on p, M and N).

For small initial data, i.e. if $\|\nabla f_0\|_{L^p(M)} \le \varepsilon_1$, the existence is global (ε_1 is the constant from Theorem 4.4).

Proof
We have already seen that f is a weak solution of the flow on $[0, T^*]$. The energy inequality follows in the limit $\varepsilon \to 0$ from Theorem 4.4 (i) and the bounds for $\|\nabla f\|_{C^{0,\tilde{\alpha}}(M\times[0,T^*])}$ and $\|\nabla f\|_{L^\infty(M\times[0,T^*])}$ from Theorem 4.5. The local bound for $\|\nabla f\|_{C^{0,\beta}(\Omega)}$ follows from Theorem 4.6. □

4.6 Short Time existence for non-smooth initial data

We can now prove short time existence for a wider class of initial values:

Theorem 4.8 *There exits a constant $\varepsilon_2 > 0$ depending on M and N with the following property:*

For given initial value $f_0 \in W^{1,p}(M, N)$ there exists a time $T^* > 0$ only depending on $E(f_0)$, $R^*(\varepsilon_2, f_0, M \times \{0\})$ and the geometry of the problem and a weak solution $f : M \times [0, T^*] \to N$ of

$$\begin{aligned} f_t - \Delta_p f &\perp T_f N \\ f(\,\cdot\,, 0) &= f_0 \,. \end{aligned}$$

f satisfies the energy inequality. Locally, for every open $\Omega \subset M \times [0, T^*]$ with $\operatorname{dist}(\Omega, M \times \{0\}) = \mu > 0$, there holds $\|\nabla f\|_{C^{0,\beta}(\Omega)} \le C$ for some constants C

(depending on p, $E(f_0)$, M, N, $R^*(\varepsilon_2, f_0, M\times\{0\})$ and μ) and $\beta\in]0,1[$ (depending on p, M and N).

For small initial data, i.e. if $\|\nabla f_0\|_{L^p(M)}\le\varepsilon_1$, the existence is global.

Proof

From Bethuel-Zheng [4] we infer that $C^\infty(M,N)$ is dense in $W^{1,p}(M,N)$. Hence, we can approximate the given f_0 by smooth functions: there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C^\infty(M,N)$ such that $\varphi_n\to f_0$ in $W^{1,p}(M,N)$. Let f_n denote the solution of

$$\begin{aligned}\partial_t f-\Delta_p f &\perp T_fN\\ f(\,\cdot\,,0) &= \varphi_n\,.\end{aligned}$$

Notice that for every $\varepsilon>0$ there exists a radius $R>0$ such that $\sup_{x\in M}E(\varphi_n, B_R(x))\le\varepsilon$ for all $n\in\mathbb{N}$. According to the construction of the time T^* in Section 4.4 there exists an existence interval $[0,T^*]$, $T^*>0$, valid for every solution f_n. Using the energy inequality of Theorem 4.7 for the solutions f_n we find a subsequence (still denoted by f_n) such that

$$f_n\rightharpoonup f \text{ weakly* in } L^\infty\left(0,T^*;W^{1,p}(M,N)\right)$$

and

$$\partial_t f_n\rightharpoonup\partial_t f \text{ weakly in } L^2\left(0,T^*;L^2(M)\right).$$

The local estimate on $\|\nabla f_n\|_{C^{0,\beta}(M\times[t,T^*])}$ for $t>0$ also from Theorem 4.7 implies (after passing to another subsequence) that

$$\nabla f_n\to\nabla f \text{ strongly in } C^{0,\beta/2}(M\times[t,T^*]).$$

Choosing $t=\frac{T^*}{n}$ we obtain, by iterated extraction of subsequences and passing to the diagonal sequence, that

$$\nabla f_n\to\nabla f \text{ strongly in } C^{0,\beta/2}(M\times[t,T^*])$$

for all $t>0$. This allows to go to the limit $n\to\infty$ in the weak form of the equation for f_n.

The energy inequality and the local estimate for $\|\nabla f\|_{C^{0,\beta}(\Omega)}\le C$ also follow in the limit. □

4.7 Global existence and partial regularity

Once we have established local existence for initial data in $W^{1,p}(M,N)$, we can try to extend the local solution beyond an occurring singularity. It will be possible to find an a priory bound for the number of singular times which enables us to obtain global existence by repeating the extension finitely many times.

Theorem 4.9 *For given initial value $f_0 \in W^{1,p}(M,N)$ there exists a weak solution $f : M \times [0,\infty[\to N$ of*

$$\begin{aligned} f_t - \Delta_p f &\perp T_f N \\ f(\,\cdot\,,0) &= f_0\,. \end{aligned}$$

f satisfies the energy inequality and is in $W^{1,p}(M)$ weakly continuous in time. There exists a set $\Sigma = \cup_{k=1}^{K} \Sigma_k \times \{T_k\}$, $\Sigma_k \subset M$, $0 < T_k \le \infty$, such that on every open set $\Omega \subset M \times [0,\infty[$ with $\operatorname{dist}(\Omega, (M \times \{0\}) \cup \Sigma) = \mu > 0$ there holds $\|\nabla f\|_{C^{0,\beta}(\Omega)} \le C$ for some constants C (depending on p, $E(f_0)$, M, N and μ) and $\beta \in]0,1[$ (depending on p, M and N). The number K of singular times is a priori bounded by $K \le \varepsilon_1^{-1} E(f_0)$ and the singular points (x, T_k) are characterized by the condition $\limsup_{t \nearrow T_k} E(f(t), B_R(x)) \ge \varepsilon_1$ for any $R > 0$. At every singular time T_k the decrease of the p-energy is at least ε_1:

$$E(f(T_k)) \le \liminf_{t \nearrow T_k} E(f(t)) - \varepsilon_1.$$

Proof

We can extend the local solution of the last section, which is defined on $[0, T^*]$, to a maximal interval $[0, T_1[$ where T_1 is characterized by

(i) $\nabla f \in C^{0,\beta}_{\mathrm{loc}}(M \times]0, T_1[)$,

(ii) there exists $x \in M$ such that $\limsup_{t \nearrow T_1} E(f(t), B_R(x)) \ge \varepsilon_1$ for any $R > 0$.

In fact, if f is a solution on $I = [0,t[$ or $I = [0,t]$ such that $\nabla f \in C^{0,\beta}_{\mathrm{loc}}(M \times I)$ and for all $x \in M$ there exists $R > 0$ with $\limsup_{\tau \nearrow t} E(f(\tau), B_R(x)) \le \varepsilon_1$, then there holds $R^* = R^*(\varepsilon_1, f, M \times I) > 0$. According to the construction in Section 4.4 we set

$$T^* = \frac{\varepsilon_1 (R^*)^p}{4CE(f_0)^{1-\frac{1}{p}}}$$

and can then find a solution of the p-harmonic flow with initial value $f(\,\cdot\,, t - \frac{T^*}{2})$ on the time interval $[t - \frac{T^*}{2}, t + \frac{T^*}{2}]$. (This extension of the solution is unique as we will see in the next Chapter.)

Now, the energy inequality implies that

$$f(\,\cdot\,,t) \rightharpoonup f(\,\cdot\,,T_1) \qquad \text{weakly in } W^{1,p}(M). \tag{4.13}$$

In fact, since $f_t \in L^2(M \times [0,T_1[)$ we have that $f_t(x,\,\cdot\,) \in L^2(0,T_1[)$ for a.e. $x \in M$, and hence we can write $f(x,T_1) = f(x,0) + \int_0^{T_1} f_t(x,s)\,ds$ for almost all $x \in M$ and $f(x,t) \to f(x,T_1)$ a.e. $x \in M$ as $t \nearrow T_1$. On the other hand, since $\|f(\,\cdot\,,t)\|_{W^{1,p}(M)}$ is bounded on $[0,T_1[$ we have (4.13) at least for a sequence $t_k \nearrow T_1$. To prove that we have convergence for an arbitrary sequence $t_k \nearrow T_1$ we argue as follows: Let $\phi, \psi \in L^{p'}(M)$ be fixed. Then we have for $\widetilde{\phi}, \widetilde{\psi} \in C^\infty(M)$

$$\left|\int_M \left((f(x,t)-f(x,T_1))\phi(x)+\nabla(f(x,t)-f(x,T_1))\psi(x)\right)d\mu\right| \le$$

$$\le \int_M |f(x,t)-f(x,T_1)|\,|\widetilde{\phi}(x)|d\mu + \int_M |f(x,t)-f(x,T_1)|\,|\phi(x)-\widetilde{\phi}(x)|d\mu +$$

$$+\left|\int_M \nabla(f(x,t)-f(x,T_1))\widetilde{\psi}(x)d\mu\right| + \int_M |\nabla(f(x,t)-f(x,T_1))(\psi(x)-\widetilde{\psi}(x)|d\mu \le$$

$$\le \max_{x\in M}|\widetilde{\phi}(x)| \int_M |f(x,t)-f(x,T_1)|d\mu +$$

$$+\left(\int_M |f(x,t)-f(x,T_1)|^p d\mu\right)^{\frac{1}{p}} \left(\int_M |\phi(x)-\widetilde{\phi}(x)|^{p'} d\mu\right)^{\frac{1}{p'}} +$$

$$+\max_{x\in M}|\nabla\widetilde{\phi}(x)| \int_M |f(x,t)-f(x,T_1)|d\mu +$$

$$+\left(\int_M |\nabla(f(x,t)-f(x,T_1))|^p d\mu\right)^{\frac{1}{p}} \left(\int_M |\phi(x)-\widetilde{\phi}(x)|^{p'} d\mu\right)^{\frac{1}{p'}} .$$

Let some $\varepsilon > 0$ be given. Since $f(\,\cdot\,,t)$ is bounded in $W^{1,p}(M)$ we can make the second and the fourth term each smaller than $\frac{\varepsilon}{4}$ by choosing $\widetilde{\phi}$ close to ϕ and $\widetilde{\psi}$ close to ψ in $L^{p'}(M)$. Then, by choosing t close to T_1 the first and the third term become smaller than $\frac{\varepsilon}{4}$ (this follows by Lebesgue's theorem).

On the other hand we have seen that at time T_1 there exist points $x \in M$ such that

$$\limsup_{t\nearrow T_1} E(f(\,\cdot\,,t),B_R(x)) \ge \varepsilon_1 \tag{4.14}$$

for any $R > 0$. Now for such a point x satisfying (4.14) and for $R > 0$ let $M_R = M \setminus B_R(x)$ and $E_0 = E(f(0))$. Then we have

$$\begin{aligned} E_0-\varepsilon_1 &\ge E_0 - \limsup_{t\nearrow T_1} E(f(t),B_R(x)) \ge \\ &\ge \liminf_{t\nearrow T_1} \left(E(f(t)) - E(f(t),B_R(x))\right) = \\ &= \liminf_{t\nearrow T_1} E(f(t),M_R) \ge \\ &\ge E(f(T_1),M_R) \xrightarrow{R\searrow 0} E(f(T_1),M) \ge 0\,. \end{aligned} \tag{4.15}$$

From (4.15) we conclude that we have

$$E(f(T_1)) \le E_0 - \varepsilon_1$$

for the energy at time T_1.

Now, for any $\Omega \subset\subset M\times]0,T_1]\setminus\Sigma_1\times\{T_1\}$ (Σ_1 the set of singular points, i.e. the set of points satisfying 4.14)), there exists a radius $R > 0$ depending on Ω such that

$$\sup_{(x,t)\in\Omega} E(f(t),B_R(x)) < \varepsilon_1\,.$$

Thus, our solution f on $[0,T_1[$ extends to a solution on $M\times[0,T_1]\setminus\Sigma_1\times\{T_1)\}$ with $\nabla f\in C^{0,\beta}(\Omega)$.

In view of (4.13) we can use $f(\,\cdot\,,T_1)$ as new initial value and iterate this process. Piecing all the resulting solutions together, we obtain a global solution as asserted. Applying (4.15) at every occurring singular time T_k we conclude that we are a priory given an upper bound for the number K of singular times $T_1,\dots,T_K$, namely

$$K\le\frac{E(f_0)}{\varepsilon_1}\,.$$

□

Chapter 5

Uniqueness Results

5.1 Preliminaries

In the case $p = 2$ Struwe proved the uniqueness of the harmonic flow in the conformal case within the class $V(M^T, N)$ (see Section 3.2 and Struwe [107]). In the nonconformal case uniqueness fails to be true even in the case $p = 2$ as counterexamples of Coron show (see Coron [20]). So, we may expect uniqueness results for $p > 2$ only in the conformal situation. In order to start we state the following technical lemma which is quite similar to Lemma 2.2.

Lemma 5.1 *Let $p \geq 2$. Then there holds for all $a, b \in \mathbb{R}^k$*

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq c|a - b|^2 \big||a| + |b|\big|^{p-2}.$$

with a constant $c > 0$ which only depends on the inner product.

Proof
By a suitable rotation and dilatation, the problem reduces to two dimensions where the verification is elementary. In the case of the standard inner product the best possible constant is $c = \frac{1}{2^{p-2}}$. □

5.2 Uniqueness in the class $L^\infty(0, T; W^{1,\infty}(M))$

In this section we show that if two solutions of the p-harmonic flow coincide at time $t = 0$ they coincide on the time interval $[0, T[$ provided the L^∞-norm of the gradients remains bounded during that time.

Theorem 5.1 *Let f_1, f_2 be weak solutions of*

$$\begin{cases} \frac{\partial f}{\partial t} - \Delta_p f & \perp & T_f N \\ f|_{t=0} & = & f_0 \end{cases}$$

and suppose that for $t \in [0,T[$ there holds $|\nabla f_1| + |\nabla f_2| \le C < \infty$. Then $f_1 \equiv f_2$ on $M \times [0,T]$.

Proof
We test the difference of the equations for f_1 and for f_2 with the testfunction $v = f_1 - f_2$ and get

(5.1) $\frac{1}{2} \int\limits_M |v(\,\cdot\,,t)|^2 d\mu +$

$$+ \int\limits_0^t \int\limits_M \Big((\gamma^{\alpha\beta} \tfrac{\partial f_1}{\partial x^\alpha} \tfrac{\partial f_1}{\partial x^\alpha})^{\frac{p}{2}} - 1 \tfrac{\partial f_1}{\partial x^\alpha} - (\gamma^{\alpha\beta} \tfrac{\partial f_2}{\partial x^\alpha} \tfrac{\partial f_2}{\partial x^\alpha})^{\frac{p}{2}} - 1 \tfrac{\partial f_2}{\partial x^\alpha} \Big) \gamma^{\alpha\beta} \big(\tfrac{\partial f_1}{\partial x^\beta} - \tfrac{\partial f_2}{\partial x^\beta} \big) \, d\mu \, dt \le$$

$$\le c \int\limits_0^t \int\limits_M \big(|v| \, |\nabla v| |\nabla F|^{p-1} + v^2 |\nabla F|^p \big) \, d\mu \, dt$$

with the shorthand notation $|\nabla F| := |\nabla f_1| + |\nabla f_2|$.
The second term II on the left hand side of (5.1) is estimated by

$$II \ge c \int\limits_0^t \int\limits_M |\nabla v|^2 |\nabla F|^{p-2} d\mu \, dt \,. \tag{5.2}$$

This inequality follows from Lemma 5.1.

Now we interpolate the term on the right hand side of (5.2) by using Young's inequality for the first term on the right of (5.1):

$$\begin{aligned} &\int\limits_0^t \int\limits_M |v| |\nabla v| |\nabla F|^{p-1} \, d\mu \, dt \le \\ &\qquad \le \frac{\varepsilon}{2} \int\limits_0^t \int\limits_M |\nabla v|^2 |\nabla F|^{p-2} \, d\mu \, dt + \frac{1}{2\varepsilon} \int\limits_0^t \int\limits_M v^2 |\nabla F|^p \, d\mu \, dt \,. \end{aligned} \tag{5.3}$$

Putting all the above inequalities (5.1)–(5.3) together, we get

(5.4)

$$\int\limits_M |v(\,\cdot\,,t)|^2 d\mu + \int\limits_0^t \int\limits_M |\nabla v|^2 |\nabla F|^{p-2} \, d\mu \, dt \le c \int\limits_0^t \int\limits_M |v|^2 |\nabla F|^p \, d\mu \, dt \,.$$

Using the assumption $|\nabla F| < C$ on $M \times [0,T[$ we get from (5.4)

$$\int\limits_M |v(\,\cdot\,,t)|^2 d\mu \le c \int\limits_0^t \int\limits_M v^2 d\mu \, dt \tag{5.5}$$

with a new constant c. The right hand side of (5.5) is increasing in t and hence we get

$$\sup_{t'\in[0,t]} \int_M |v(\,\cdot\,,t')|^2 d\mu \le c\,t \sup_{t'\in[0,t]} \int_M |v(\,\cdot\,,t')|^2 d\mu\,.$$

Thus, for $t < \frac{1}{c}$ we get $v \equiv 0$ for $t' \in [0,t]$. Iteration of the argument proves the assertion (notice that the constant c remains the same during the iteration process). □

By a slight modification of the above proof, we get the following theorem

Theorem 5.2 *Let f_1 and f_2 be weak solutions of*

$$\begin{cases} \frac{\partial f}{\partial t} - \Delta_\varepsilon f & \perp & T_f N \\ f|_{t=0} & = & f_0 \end{cases}$$

for initial data $f_0 \in W^{1,\infty}(M)$. Then there exists $\varepsilon_1 > 0$ such that

$$\int_M |\nabla f_0|^p d\mu < \varepsilon_1$$

implies $f_1 = f_2$ for all $t \ge 0$.

Proof
We repeat the proof of the above local uniqueness theorem and point out that

$$\sup_{t\in[0,T]} \int_{B_R} |\nabla f_i|^p \, d\mu < \varepsilon_1$$

and hence by Theorem 4.4 $\nabla f_i \in L^\infty(M \times [0,T])$ holds for all $T > 0$. Thus, the assertion follows by iteration. □

Since we did not make use of the term $\int_0^t \int_M |\nabla v|^2 |\nabla F|^{p-2}\, d\mu\, dt$ in (5.4) the question arises whether we could prove uniqueness in the class $L^\infty(0,T;W^{1,p}(M))$ in the conformal case without the quite strong assumption that ∇f is bounded. This question is connected to the following problem which is answered only in case $p=2$ by Hélein:

A regularity problem: *Suppose M is of dimension $m = p$ and $f \in W^{1,p}(M,N)$ is weakly p-harmonic. Is it then true or not that $f \in W^{1,\infty}(M,N)$ (and consequently $f \in C^{1,\alpha}(M,N)$)?*

If this conjecture is false, i.e. if there is a weakly p-harmonic map $f \in W^{1,p}(M,N) \setminus W^{1,\infty}(M,N)$ with $p = \dim(M)$ then we have an example of a non-unique flow in the class $L^\infty(0,T;W^{1,p}(M))$: In fact, as we showed in the previous chapter the p-harmonic flow with initial data f has a local weak solution g such that the gradient

$|\nabla g|$ is finite for $t > 0$. On the other hand f (considered as constant in time) is a weak solution of the p-harmonic flow for the same initial data, but ∇f remains unbounded for any time. In other words: Uniqueness of the flow in the class $L^\infty(0,T;W^{1,p}(M))$ would imply regularity of weakly p-harmonic maps.

5.3 Finite-time blow-up

A question arising in connection with the previous section is whether or not in the conformal case singularities develop in finite time. In the case $m = p = 2$ Hélein's regularity result for harmonic maps suggests that this might not be the case. On the other hand, results of Lemaire [74] and Wente [129] are pointing in the opposite direction. Finally, the answer was given by Chang, Ding and Ye in [10], where an example of finite time blow-up is presented. For $p > 2$ there are recent regularity results for weakly p-harmonic maps from the p-dimensional ball into a sphere, which are for this rather special geometry the analogue of Hélein's result (see Strzelecki [117] and [118]). Unfortunately it seems quite difficult to modify the example of Chang, Ding and Ye to the case $p > 2$, since the p-Laplace operator in the chosen symmetry looks a lot more difficult than in the regular case: In fact for $p = 4$, the ansatz

$$f : B_1^4(0) \to S^4,\ \ (x,t) \mapsto \left(\frac{x}{|x|}\sin(h(|x|,t), \cos(h(|x|,t)\right)$$

with a new unknown function h satisfying $h(0,t) = 0$, leads to

(5.6)
$$h_t = 3\left(h_{rr} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2}\right)\left(h_r^2 + \frac{\sin^2(h)}{r^2}\right) + \frac{3\sin(2h)}{r^2}\left(h_r^2 - \frac{\sin^2(h)}{r^2}\right)$$

for the function $h(r,t)$. As for $p = 2$ the functions

$$\psi_\lambda(r) = \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right) = 2\arctan\left(\frac{r}{\lambda}\right)$$

build a family of solutions of the stationary equation (5.6): using this function in the ansatz we obtain the composition of a dilatation in $\mathbb{R}^4$ and the inverse stereographic projection (and hence a conformal mapping) which is a weakly 4-harmonic map. But it seems difficult to modify this solution to obtain an explicit subsolution which blows up in finite time. Especially the function

$$g(r,t) = \psi_\mu(r^a) + \psi_{l(t)}(r)$$

with $l(t) \to 0$ for $t \to T$ (i.e. $g_r(0,t) \to \infty$ for $t \to T$) which was used by Chang, Ding and Ye does not seem to work any more. To obtain at least an idea of the behaviour of the solution of (5.6) we make some numerical calculations.

We apply a Crank-Nicolson type method: Let $r_i = i\Delta r$ be the discretization in the radial direction and suppose that in these points the numerical approximation h_i

at some time $t \geq 0$ is known. Let h'_i denote the unknown values at the next time step t'. Then, we approximate the derivatives in (5.6) by the following difference quotients:

$$\begin{aligned} h_t &\approx \frac{1}{t'-t}(h'_i - h_i) \\ h_r &\approx \frac{1}{4\Delta r}(h_{i+1} - h_{i-1} + h'_{i+1} - h'_{i-1}) \\ h_{rr} &\approx \frac{1}{2(\Delta r)^2}(h_{i+1} - 2h_i + h_{i-1} + h'_{i+1} - 2h'_i + h'_{i-1})\,. \end{aligned}$$

Hence, the collocation points of best common approximation are $(r_i, (t'+t)/2)$. The resulting system of nonlinear equations for the unknown values h'_i may be solved by Newton's method. The length of the time step is dynamically adapted dependent on the convergence of Newton's method.

The following pictures have been produced on a Sun SPARCstation 10 in several hours running time. The first graphic shows the solution in r (front axis) and t (backward axis). As initial value we have chosen the function of Chang, Ding and Ye.

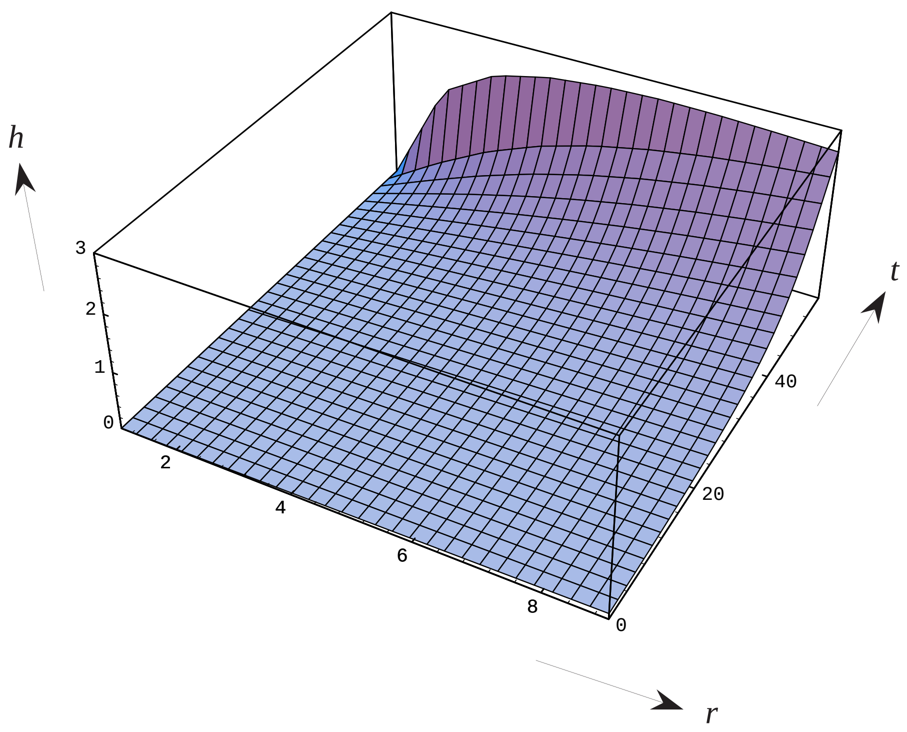

Figure 5.1: *Numerical solution of (5.6)*

The next picture is probably more instructive: it shows the numerical approximation of $h_r(0,t)$ as a function of time.

So, at least the numerical solution shows the phenomenon of finite-time blow-up.

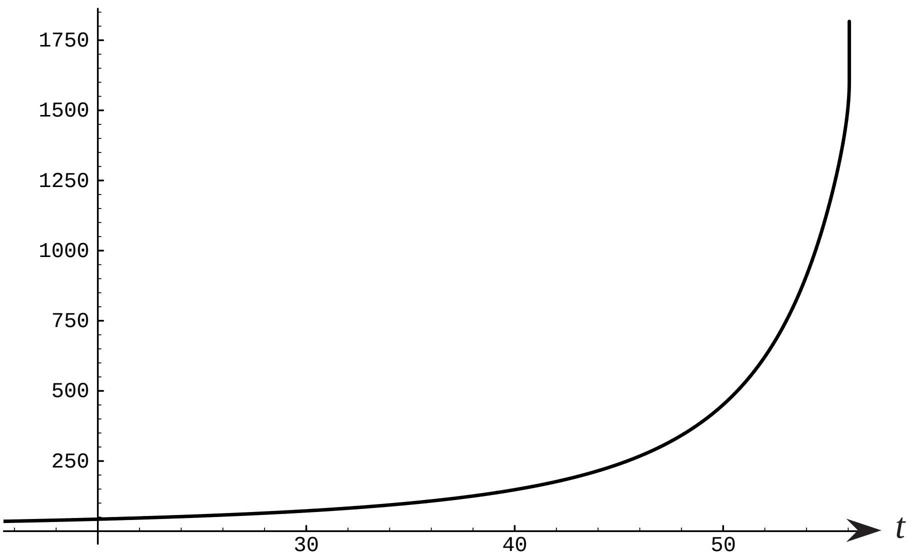

Figure 5.2: *Finite-time blow-up of h_r at $r = 0$*

Curriculum vitae

Am 25. April 1964 wurde ich als Sohn von Gebhard und Hermine Hungerbühler-Hässig in Flawil im Kanton St. Gallen (Schweiz) geboren. In eben diesem kleinen Dorf besuchte ich ab 1971 die Primar- und später die Sekundarschule. Anschließend besuchte ich ab 1980 das Mathematisch-Naturwissenschaftliche Gymnasium der Kantonsschule St. Gallen, wo ich 1984 die Maturität (Typus C) erwarb.

Im Oktober 1984 begann ich das Mathematikstudium an der Abteilung IX der ETH Zürich. Unterbrochen durch mehrere Militärdienste erlangte ich schließlich im Herbst 1989 das Diplom in Mathematik mit einer Diplomarbeit bei Prof. Christian Blatter über ein Thema der Integralgeometrie. Wenig später erwarb ich auch den Fähigkeitsausweis zum höheren Lehramt. Seit Herbst 1989 bin ich als Assistent am Mathematikdepartement der ETH Zürich tätig. Als Organisator der Assistenz II konnte ich neben den üblichen Assistentenpflichten auch manchen Blick hinter die Kulissen der Hochschule werfen. Als Assistentenvertreter nahm ich lange Zeit an der Abteilungskonferenz der Abteilung IX, später an der Departementskonferenz des Mathematikdepartementes teil. Im Wintersemester 1992/93 hatte ich die Gelegenheit, die Vorlesung Analysis III über partielle Differentialgleichungen an der Abteilung IIIB für Elektrotechnik zu halten. Im Sommersemester 1993 nahm ich gerne den Vorschlag von Prof. Michael Struwe auf, eine Nachdiplomvorlesung von Prof. Leon Simon über harmonische Abbildungen für eine Veröffentlichung als Buch auszuarbeiten (erscheint im Birkhäuser Verlag in der ETH-Lectures-Reihe). Daneben entstand meine Dissertation im Laufe von vier Jahren unter Leitung von Prof. Struwe. In diese Zeit fallen auch einige Veröffentlichungen auf verschiedenen anderen Gebieten der Mathematik. Besonders anregend empfand ich zwei Besuche bei Tagungen am Forschungsinstitut in Oberwolfach, die mir ebenfalls durch Prof. Struwe ermöglicht wurden. Er gab mir dort auch die Gelegenheit, über meine Arbeit einen Vortrag zu halten.

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Index

Symbols

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Citations

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Keywords