





# Three Conics Determine a Cubic

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## Abstract

Given a cubic  $K$  in the real projective plane. Then for each point  $P$  there is a conic  $C_P$  associated to  $P$ . The conic  $C_P$  is called the *polar conic* of  $K$  with respect to the *pole*  $P$ . We investigate the situation when three conics  $C_1$ ,  $C_2$ , and  $C_3$  are polar conics of  $K$  with respect to the poles  $P_1$ ,  $P_2$ , and  $P_3$ , respectively. In particular, we give an elementary proof—without using any results from algebraic geometry—that any three conics  $C_1$ ,  $C_2$ ,  $C_3$  in general position, satisfying only a non-degeneracy condition, determine a unique cubic  $K$  and three points  $P_1$ ,  $P_2$ ,  $P_3$ , such that  $C_1$ ,  $C_2$ ,  $C_3$  are polar conics of  $K$  with respect to the three poles  $P_1$ ,  $P_2$ ,  $P_3$ . This can be seen as a higher degree variant of von Staudt’s Theorem.

**Keywords** Pencils · Conics · Polars · Polar conics of cubics

**Mathematics Subject Classification** 51A05 · 51A20

## 1 Introduction

This work proceeds the article [3], in which it is shown that two given conics  $C_0$  and  $C_1$  can always be considered as polar conics of a cubic  $K$  curve with respect to corresponding poles  $P_0$  and  $P_1$ . However, even though  $P_1$  is determined by  $P_0$ , neither the cubic nor the point  $P_0$  is determined by the two conics  $C_0$  and  $C_1$ . This changes if we start with three conics  $C_1$ ,  $C_2$ ,  $C_3$  in general position. In this situation, there is a

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unique cubic  $K$  and uniquely determined points  $P_1, P_2, P_3$  such that  $C_1, C_2, C_3$  are the polar conics of  $K$  with respect to the three poles  $P_1, P_2, P_3$ . Instead of formulating the result in the abstract language of algebraic geometry, we propose an elementary and explicit approach that shows a concrete method to calculate the resulting cubic curve  $K$  and the poles  $P_1, P_2, P_3$ , starting from the three given conic sections  $C_1, C_2, C_3$ . In particular, the condition for uniqueness and existence becomes visible in this way.

Our result can be seen as a higher degree variant of von Staudt's Theorem which says that given three lines  $\ell_1, \ell_2, \ell_3$  and three points  $P_1, P_2, P_3$  in perspective position determine a unique conic  $C$  such that the points  $P_i$  are the poles of the lines  $\ell_i$  with respect to  $C$  (see [7, p. 135, Sect. 241]).

The setting in which we work is the same as in [3], but for the sake of completeness we recall the notation and terminology. We will work in the real projective plane  $\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\} / \sim$ , where  $X \sim Y \in \mathbb{R}^3 \setminus \{0\}$  are equivalent, if  $X = \lambda Y$  for some  $\lambda \in \mathbb{R}$ . Points  $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{0\}$  will be denoted by capital letters, the components with the corresponding small letter, and the equivalence class by  $[X]$ . However, since we mostly work with representatives, we often omit the square brackets in the notation. A non-degenerate conic in this setting is then given by an equation of the form  $\langle X, AX \rangle = 0$ , where  $A$  is a regular, real, symmetric  $3 \times 3$ -matrix with mixed signature, i.e.,  $A$  has eigenvalues of both signs, and  $\langle \cdot, \cdot \rangle$  denotes the standard inner product of  $\mathbb{R}^3$ .

Let  $f$  be a non-constant homogeneous polynomial in the variables  $x_1, x_2, x_3$  of degree  $n$ . Then  $f$  defines a projective algebraic curve

$$C_f := \{[X] \in \mathbb{RP}^2 : f(X) = 0\},$$

of degree  $n$ . For a point  $P \in \mathbb{RP}^2$ ,

$$Pf(X) := \langle P, \nabla f(X) \rangle,$$

is also a homogeneous polynomial in the variables  $x_1, x_2, x_3$ . If the homogeneous polynomial  $f$  is of degree  $n$ , then  $C_{Pf}$  is an algebraic curve of degree  $n - 1$ . The curve  $C_{Pf}$  is called the *polar curve* of  $C_f$  with respect to the *pole*  $P$ ; sometimes we call it the *polar curve* of  $P$  with respect to  $C_f$ . In particular, when  $C_f$  is a cubic curve (i.e.,  $f$  is a homogeneous polynomial of degree 3), then  $C_{Pf}$  is a conic, which we call the *polar conic* of  $C_f$  with respect to the *pole*  $P$ , and when  $C_f$  is a conic, then  $C_{Pf}$  is a line, which we call the *polar line* of  $C_f$  with respect to the *pole*  $P$  (see, for example, the classical book of Wieleitner [8] or Dolgachev [2, Chap. 3] for a modern view). Note that  $C_{Pf}$  is defined and can be a regular curve even if  $C_f$  is singular or reducible. For some historical background, for the geometric interpretation of poles and polar lines, for the iterated construction of polar curves, as well as for the analytical method used today, see Monge [5, Sect. 3], Bobillier [1], and Joachimsthal [4, p. 373], or [3].

## 2 Algebraic Curves and Multilinear Forms

Let  $C_f$  be a conic given by the non-constant homogeneous polynomial

$$f(x_1, x_2, x_3) := \sum_{1 \leq i < j \leq 3} c_{ij} x_i x_j.$$

Then, the symmetric matrix

$$T := \begin{pmatrix} c_{11} & c_{12}/2 & c_{13}/2 \\ c_{12}/2 & c_{22} & c_{23}/2 \\ c_{13}/2 & c_{23}/2 & c_{33} \end{pmatrix},$$

has the property that a point  $X$  belongs to  $C_f$  (i.e.,  $f(X) = 0$ ), if and only if  $\langle X, T(X) \rangle = 0$ . Thus, the conic  $C_f$  is represented by the matrix  $T$ . Since the expression  $\langle X, T(Y) \rangle$  defines a bilinear form  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $(X, Y) \mapsto \langle X, T(Y) \rangle$ , we can consider the matrix  $T$  also as a purely covariant tensor of rank 2 (i.e., a tensor whose rank of covariance is 2 and whose rank of contravariance is 0). More precisely, if we consider the matrix  $T$  as a  $(0, 2)$ -tensor, where for  $X = (x_1, x_2, x_3)$  and  $Y = (y_1, y_2, y_3)$  we define

$$T(X, Y) := \sum_{1 \leq i, j \leq 3} a_{ij} x_i y_j,$$

then the expression  $\langle X, T(X) \rangle = 0$  is equivalent to  $T(X, X) = 0$ . In order to obtain the coefficients of the  $(0, 2)$ -tensor  $T = (a_{ij})_{1 \leq i, j \leq 3}$  from a conic  $C_f$  defined by a non-constant homogeneous polynomial  $f$ , we just set

$$a_{ij} := \frac{1}{2!} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{for all } 1 \leq i, j \leq 3.$$

The next result shows that this relation between a conic  $C_f$  and the corresponding  $(0, 2)$ -tensor  $T_f = (a_{ij})_{1 \leq i, j \leq 3}$  can be generalised to algebraic curves of arbitrary degree.

**Lemma 2.1** *Let  $\Gamma_f$  be an algebraic curve of degree  $d$  given by the non-constant homogeneous polynomial*

$$f(x_1, x_2, x_3) := \sum_{1 \leq i_1 \leq \dots \leq i_d \leq 3} c_{i_1 \dots i_d} \cdot x_{i_1} \cdot \dots \cdot x_{i_d},$$

and let  $T_f = (a_{i_1 \dots i_d})_{1 \leq i_1, \dots, i_d \leq 3}$ , where

$$a_{i_1 \dots i_d} := \frac{1}{d!} \cdot \frac{\partial^d f}{\partial x_{i_1} \dots \partial x_{i_d}} \quad \text{for all } 1 \leq i_1, \dots, i_d \leq 3.$$

Then  $T_f$  is a symmetric  $(0, d)$ -tensor and a point  $X$  is on the curve  $\Gamma_f$  if and only if

$$T_f(\underbrace{X, \dots, X}_{d\text{-times}}) = 0.$$

**Proof** Since for every rearrangement  $\pi$  of the sequence  $\langle i_1, \dots, i_d \rangle$  we have

$$\frac{\partial^d f}{\partial x_{i_1} \dots \partial x_{i_d}} = \frac{\partial^d f}{\partial x_{\pi(i_1)} \dots \partial x_{\pi(i_d)}} \text{ and therefore } a_{i_1 \dots i_d} = a_{\pi(i_1) \dots \pi(i_d)},$$

we get that the tensor  $T_f$  is symmetric. Furthermore, assume that the monomial  $c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$  appears in  $f$ . Then  $n_1 + n_2 + n_3 = d$  and

$$\frac{1}{d!} \cdot \frac{\partial^d (c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3})}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} = \frac{n_1! \cdot n_2! \cdot n_3!}{d!} \cdot c_{n_1 n_2 n_3}.$$

Now, it is easy to see that the number of coefficients  $a_{i_1 \dots i_d}$  such that for  $1 \leq i \leq 3$  the number  $i$  appears  $n_i$ -times in the sequence  $\langle i_1, \dots, i_d \rangle$  is given by the trinomial coefficient

$$\binom{d}{n_1, n_2, n_3} = \frac{d!}{n_1! \cdot n_2! \cdot n_3!}.$$

This shows that for any point  $X$  we have  $T_f(X, \dots, X) = 0$  if and only if  $f(X) = 0$ , or in other words,  $X$  is on the curve  $\Gamma$ . □

Let us turn our attention now to polar curves. For this, we consider first polar curves of conics  $C_f$  with corresponding  $(0, 2)$ -tensor  $T_f = (a_{ij})_{1 \leq i, j \leq 3}$ . Above we have seen that for a given point  $P \in \mathbb{R}P^2$ , a point  $X$  is on the polar curve  $C_{Pf(X)}$  of  $C_f$  with respect to the pole  $P$  if and only if

$$Pf(X) := \langle P, \nabla f(X) \rangle = 0.$$

Now, for  $P, X \in \mathbb{R}P^2$ , a short calculation shows that  $Pf(X) = 2 \cdot T_f(P, X)$ , and hence, we get

$$Pf(X) = 0 \iff T_f(P, X) = 0.$$

Since  $T_f$  is symmetric, we have  $T_f(P, X) = T_f(X, P)$ , which shows that if  $X$  is a point on the polar curve of  $C_f$  with respect to the pole  $P$ , then  $P$  is a point on the polar curve of  $C_f$  with respect to the pole  $X$ . The next result shows that also this result can be generalised to algebraic curves of arbitrary degree.

**Lemma 2.2** *Let  $\Gamma_f$  be an algebraic curve of degree  $d$  given by the non-constant homogeneous polynomial  $f$ , let  $T_f$  be the corresponding symmetric  $(0, d)$ -tensor, and let  $P \in \mathbb{R}P^2$  be a point. Then*

$$Pf(X) = 0 \iff T_f(P, \underbrace{X, \dots, X}_{(d-1)\text{-times}}) = 0.$$

In particular, a point  $X \in \mathbb{RP}^2$  is on the polar curve of  $\Gamma_f$  with respect to the pole  $P$  if and only if  $T_f(P, X, \dots, X) = 0$ .

**Proof** Notice first that for  $P = (p_1, p_2, p_3)$  and  $X = (x_1, x_2, x_3)$  we have:

$$\begin{aligned} T_f(P, X, \dots, X) &= \sum_{j=1}^3 p_j \cdot \left( \sum_{1 \leq i_2, \dots, i_d \leq 3} a_{j i_2 \dots i_d} \cdot x_{i_2} \cdot \dots \cdot x_{i_d} \right) \\ &= \sum_{j=1}^3 \sum_{1 \leq i_2, \dots, i_d \leq 3} a_{j i_2 \dots i_d} \cdot p_j \cdot x_{i_2} \cdot \dots \cdot x_{i_d}. \end{aligned}$$

Now, assume again that the monomial  $c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$  appears in  $f$ . Then, for each  $1 \leq j \leq 3$  we have

$$\frac{\partial (c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3})}{\partial x_j} = n_j \cdot c_{n_1 n_2 n_3} \cdot x_1^{n'_1} \cdot x_2^{n'_2} \cdot x_3^{n'_3},$$

where  $n'_j = n_j - 1$  and  $n'_i = n_i$  for  $i \neq j$ . Without loss of generality we assume that  $j = 1$  and  $n_1 \geq 1$ . Now, it is easy to see that the number of coefficients  $a_{1 i_2 \dots i_d}$  such that for  $1 \leq i \leq 3$ , the number  $i$  appears  $n_i$ -times in the sequence  $\langle 1, \dots, i_d \rangle$  is given by the trinomial coefficient

$$\binom{d-1}{n_1-1, n_2, n_3} = \frac{(d-1)!}{(n_1-1)! \cdot n_2! \cdot n_3!} = \frac{n_1}{d} \cdot \frac{d!}{n_1! \cdot n_2! \cdot n_3!}.$$

This shows that for any points  $P, X \in \mathbb{RP}^2$  we have

$$d \cdot T_f(P, X, \dots, X) = \langle P, \nabla f(X) \rangle,$$

in particular, we get

$$Pf(X) = 0 \iff T_f(P, X, \dots, X) = 0.$$

□

It is obvious how the iterated construction of polar curves is carried out: If, for example,  $P, Q, R \in \mathbb{RP}^2$  are given and  $\Gamma_f$  is an algebraic curve of degree  $d \geq 3$ , then the polar curve of the polar curve of the polar curve of  $\Gamma_f$  with respect to the points  $P, Q, R$ , respectively, is given by the zeros of the  $(0, d - 3)$ -tensor  $T_f(P, Q, R, X, \dots, X)$ . Notice that since  $T_f$  is symmetric, the order of  $P, Q, R$  is irrelevant. As a consequence, we obtain the following

**Fact 2.3** Let  $K$  be a cubic curve, let  $P_1, P_2, P_3 \in \mathbb{RP}^2$ , and for  $1 \leq j \leq 3$  let  $T_j$  be the  $(0, 2)$ -tensor of the polar conic of  $K$  with respect to the point  $P_j$ . Then for  $1 \leq j_1, j_2 \leq 3$  we have

$$T_{j_1}(P_{j_2}, X) = 0 \iff T_{j_2}(P_{j_1}, X) = 0,$$

in particular, if we consider the tensors  $T_j$  as  $3 \times 3$ -matrices, we obtain that

$$[P_{j_1}] = [(T_{j_2}^{-1} \cdot T_{j_1}) P_{j_2}].$$

The question that we want to treat below, is embedded in a more general problem, namely the study of the relation of a hypersurface and its Hessian variety. In a recent work Sendra-Arranz [6] investigated the Hessian correspondence for the cases of hypersurfaces of degree 3 and 4 in an  $n$ -dimensional projective space. In particular, he showed that for degree 3 and dimension  $n = 1$ , the Hessian correspondence is two to one, and that for degree 3 and  $n \geq 2$ , and for degree 4, it is birational (see [6, Sects. 2.3, 2.4]). In particular, by introducing the variety of  $k$ -gradients as the variety of  $k$ -planes containing all the first order derivatives of a polynomial, he obtains algorithms which allow to reconstruct a hypersurface of degree 3 from its Hessian variety in the cases  $n \geq 1$ , and for degree 4 if  $n$  is even. More specifically, Sendra-Arranz proves in his Proposition 2.18 that for  $n \geq 2$  a cubic can be recovered by the pencil spanned by its polars. Our main result in Theorem 2.4 is less general, but provides more specific information about the special case of degree 3 in 2 dimensions. Namely, what we show is that three conics in general position (i.e., three points of the Hessian variety) determine a unique cubic. More precisely, given three different conics  $C_1, C_2, C_3$  which satisfy a non-degeneracy condition, we show how to construct the unique cubic  $K$  such that for three points  $P_1, P_2, P_3 \in \mathbb{RP}^2$  determined by the three conics, the conic  $C_j$  (for  $1 \leq j \leq 3$ ) is the polar conic of  $K$  with respect to the pole  $P_j$ . The construction we provide in the next section proves our main result, Theorem 2.4.

**Theorem 2.4** Let  $C_1, C_2, C_3$  be three non-degenerate conics and let  $T_1, T_2, T_3$  be the corresponding  $(0, 2)$ -tensors given by  $3 \times 3$ -matrices. Assume that the matrices  $T_1, T_2, T_3$  satisfy the following condition:

(C) For all  $P \in \ker(T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3)$ , we have  $\det(T_1 P, T_2 P, T_3 P) \neq 0$ . Then there are exactly three points  $P_1, P_2, P_3$ , determined by the conics  $C_1, C_2, C_3$ , and a unique cubic curve  $K$ , such that for  $1 \leq j \leq 3$ ,  $C_j$  is the polar conic of  $K$  with respect to the pole  $P_j$ . The cubic  $K$  only depends on the two-dimensional pencil

$$\mathcal{P} = \{\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 : (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \setminus (0, 0, 0)\},$$

generated by  $C_1, C_2, C_3$ : If  $C_1, C_2, C_3$  are replaced by any other conics  $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$  in  $\mathcal{P}$  satisfying condition (C), then the same cubic  $K$  results.

**Remark 1** With respect to condition (C), we would like to mention a few facts.

- (a) First, condition (C) is symmetric in the three indices: To see this, notice that  $P \in \ker(T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3)$  is equivalent to

$$Q = T_1^{-1} T_2 P \in \ker(T_1 T_2^{-1} T_3 - T_3 T_2^{-1} T_1).$$

Replacing  $P$  in the determinant by the expression  $T_2^{-1} T_1 Q$  yields

$$\begin{aligned} 0 \neq \det(T_1 P, T_2 P, T_3 P) &= \det(T_1 T_2^{-1} T_1 Q, T_1 Q, T_3 T_2^{-1} T_1 Q) \\ &= \det(T_1 T_2^{-1} T_1 Q, T_1 Q, T_1 T_2^{-1} T_3 Q) \\ &= \det(T_1 T_2^{-1}) \det(T_1 Q, T_2 Q, T_3 Q). \end{aligned}$$

- (b) Observe also that (C) implies that  $T_3 T_1^{-1} T_2 \neq T_2 T_1^{-1} T_3$ : Indeed, assume that  $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3 = 0$ . Then the kernel of  $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3$  is  $\mathbb{R}^3$ . However, for  $P = (x_1, x_2, x_3)$ ,  $\det(T_1 P, T_2 P, T_3 P) = 0$  is a homogeneous cubic polynomial in the three variables  $x_1, x_2, x_3$ , which always has non-trivial solutions.
- (c) Consider the following example:

$$T_1 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Notice that  $T_1$  does not belong to the pencil spanned by  $T_2$  and  $T_3$ . Here, we have that  $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3 = 0$  and hence the kernel of  $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3$  is  $\mathbb{R}^3$ . But  $\det(T_1 P, T_2 P, T_3 P) = 0$  whenever the second coordinate of  $P$  is 0. So, the example shows that condition (C) can be violated even in the case when the pencil of  $T_1, T_2, T_3$  is two-dimensional. On the other hand, it is easy to see that condition (C) implies that the pencil of  $T_1, T_2, T_3$  is two-dimensional.

### 3 Constructing a Cubic from Three Conics

Let  $C_1, C_2, C_3$  be three non-degenerate conics and let  $T_1, T_2, T_3$  be the corresponding  $(0, 2)$ -tensors given by  $3 \times 3$ -matrices  $T_1, T_2, T_3$  which satisfy condition (C) of Theorem 2.4.

**Example** Let  $C_1, C_2, C_3$  be given by the following three non-constant homogeneous polynomials  $f_1, f_2, f_3$ , respectively:

$$\begin{aligned} f_1(X) &= x_1^2 + x_2^2 + 4x_1x_3, \\ f_2(X) &= 2x_1^2 + 2x_1x_2 + 2x_2^2 + 6x_1x_3 + 6x_2x_3, \\ f_3(X) &= x_1^2 + 6x_1x_2 + x_2^2 + 2x_1x_3 - 6x_2x_3. \end{aligned}$$

Figure 1 shows these three conics. Notice that all three conics meet in the origin, which is not excluded by the condition (C), as we will see below. Notice also that one

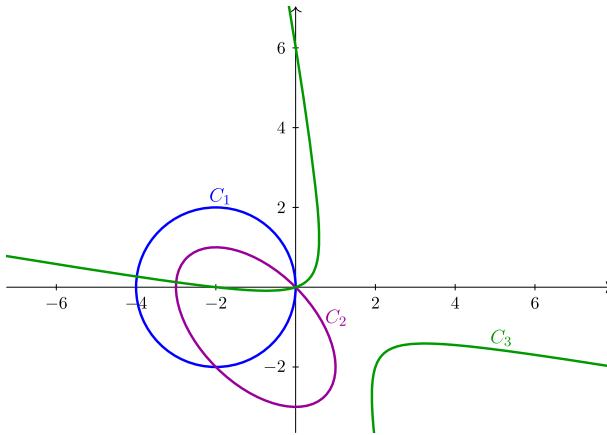


Fig. 1 The three conics  $C_1, C_2,$  and  $C_3$  of the example

of the conics is a circle, which is not a restriction since we can transform any conic by a projective transformation into a circle.

Then the corresponding matrices are:

$$T_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & -3 \\ 1 & -3 & 0 \end{pmatrix}.$$

It is easy to verify that the matrices  $T_1, T_2, T_3$  satisfy condition (C): Observe that  $\ker(T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3) = [P]$  for  $P = (\frac{6}{5}, -\frac{24}{5}, 1)$ .

Let us turn back to our general construction and construct the three points  $P_1, P_2, P_3$ : By Fact 2.3, the points  $P_1, P_2, P_3$  satisfy the following three necessary conditions

$$T_2 P_1 = T_1 P_2, \quad T_3 P_2 = T_2 P_3, \quad T_1 P_3 = T_3 P_1,$$

which is equivalent to

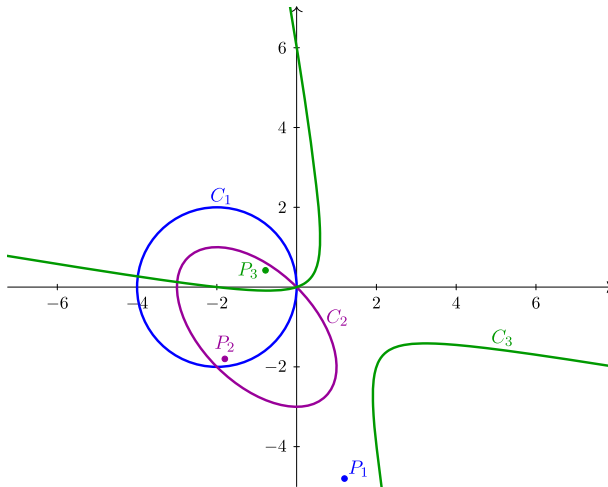
$$(T_1^{-1} T_2) P_1 = P_2, \quad (T_2^{-1} T_3) P_2 = P_3, \quad (T_3^{-1} T_1) P_3 = P_1,$$

and implies that  $P_1$  satisfies

$$(T_3^{-1} T_1)(T_2^{-1} T_3)(T_1^{-1} T_2) P_1 = P_1. \tag{1}$$

Since the matrices  $T_j$  are symmetric, for  $M := T_3 T_1^{-1} T_2$  we have  $M^T = T_2 T_1^{-1} T_3$ . Therefore, Eq. (1) is equivalent to  $M P_1 = M^T P_1$ , which is equivalent to  $(M - M^T) P_1 = 0$ . Now, condition (C) ensures that  $M \neq M^T$  (see Remark 1(b)).





**Fig. 2** The three conics  $C_1, C_2, C_3$  of the example with the three poles  $P_1, P_2, P_3$

Since  $(M - M^T)$  is a non-zero, real, anti-symmetric  $3 \times 3$ -matrix, it has exactly one eigenvalue equal to zero. In fact, if

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

is an anti-symmetric matrix, then the eigenvalues of  $A$  are  $0$  and  $\pm i\sqrt{a^2 + b^2 + c^2}$  and an eigenvector to the eigenvalue  $0$  is  $(c, -b, a)^T$ .

Hence, the pole  $P_1$  is uniquely determined by Eq. (1), and we obtain  $P_2 = (T_1^{-1} T_2)P_1$  and  $P_3 = (T_1^{-1} T_3)P_1$ . Before we proceed, let us compute the points  $P_1, P_2, P_3$  in our example.

**Example** With respect to  $T_1, T_2, T_3$  we get  $P_1 = \left(\frac{6}{5}, -\frac{24}{5}, 1\right)$ ,  $P_2 = \left(-\frac{27}{5}, -\frac{27}{5}, 3\right)$ , and  $P_3 = \left(\frac{39}{5}, -\frac{21}{5}, -10\right)$ , which correspond to the affine points  $\bar{P}_1 = \left(\frac{6}{5}, -\frac{24}{5}\right)$ ,  $\bar{P}_2 = \left(-\frac{27}{15}, -\frac{27}{15}\right)$ , and  $\bar{P}_3 = \left(-\frac{39}{50}, \frac{21}{50}\right)$ , respectively. Figure 2 shows the conics with their poles.

The goal of our construction is to find a  $(0, 3)$ -tensor  $T_K$  of a cubic  $K$ , such that we have

$$T_K(P_j, X, X) = T_j(X, X) \quad \text{for } 1 \leq j \leq 3.$$

Since by condition (C), the points  $P_1, P_2, P_3$  are not incident with a projective line, we may choose  $\{P_1, P_2, P_3\}$  as a new basis. In other words, for  $\tilde{P}_1 = (1, 0, 0)$ ,

$\tilde{P}_2 = (0, 1, 0)$ , and  $\tilde{P}_3 = (0, 0, 1)$ , we map  $P_i \mapsto \tilde{P}_i$  (for  $1 \leq i \leq 3$ ). For  $1 \leq i \leq 3$ , let  $T_i = (a^i_{jk})_{1 \leq j, k \leq 3}$  and let  $\tilde{T}_i$  be the  $(0, 2)$ -tensors (i.e., the conics  $\tilde{C}_i$ ) in this new basis. Since for any  $1 \leq i, j, k \leq 3$  we have  $T_i(P_j, P_k) = T_i(P_k, P_j) = T_j(P_k, P_i)$ , we also have

$$\tilde{T}_i(\tilde{P}_j, \tilde{P}_k) = \tilde{T}_i(\tilde{P}_k, \tilde{P}_j) = \tilde{T}_j(\tilde{P}_k, \tilde{P}_i). \tag{2}$$

Now, let  $T_{\tilde{K}} = (\tilde{a}_{ijk})_{1 \leq i, j, k \leq 3}$  be a  $(0, 3)$ -tensor defined by stipulating

$$\tilde{a}_{ijk} := \tilde{T}_i(\tilde{P}_j, \tilde{P}_k) \quad \text{for } 1 \leq i, j, k \leq 3.$$

Then, by Eq. (2), the tensor  $T_{\tilde{K}}$  is symmetric and has the property that for  $1 \leq i \leq 3$ ,

$$T_{\tilde{K}}(\tilde{P}_i, X, X) = \tilde{T}_i(X, X).$$

For the corresponding cubic  $\tilde{K}$  we therefore have that  $\tilde{C}_i$  is the polar conic of  $\tilde{K}$  with respect to the pole  $\tilde{P}_i$ .

Since every point  $\tilde{Q} = (q_1, q_2, q_3) \in \mathbb{RP}^2$  can be written as  $\tilde{Q} = q_1 P_1 + q_2 P_2 + q_3 P_3$ , we have

$$\begin{aligned} T_{\tilde{K}}(\tilde{Q}, X, X) &= q_1 T_{\tilde{K}}(\tilde{P}_1, X, X) + q_2 T_{\tilde{K}}(\tilde{P}_2, X, X) + q_3 T_{\tilde{K}}(\tilde{P}_3, X, X) \\ &= q_1 \tilde{T}_1(X, X) + q_2 \tilde{T}_2(X, X) + q_3 \tilde{T}_3(X, X), \end{aligned}$$

which shows that the polar conic of  $\tilde{K}$  with respect to the point  $\tilde{Q}$  belongs to the pencil spanned by the conics  $\tilde{T}_1, \tilde{T}_2$  and  $\tilde{T}_3$ .

Now, the re-transformed cubic  $K$  has the property that the conics  $C_1, C_2, C_3$  are the polar conics of  $K$  with respect to the poles  $P_1, P_2, P_3$ , respectively. Furthermore, by the observation above, if, for example, the conic  $C_3$  is replaced by a conic  $\tilde{C}_3$  in the two-dimensional pencil of  $C_1, C_2, C_3$  such that  $C_1, C_2, \tilde{C}_3$  satisfy condition (C), then the conics  $C_1, C_2$  and  $\tilde{C}_3$  are the polar conics of  $K$  with respect to the poles  $P_1, P_2$  and some point  $Q$ , where the three points  $P_1, P_2, Q$  are not collinear.

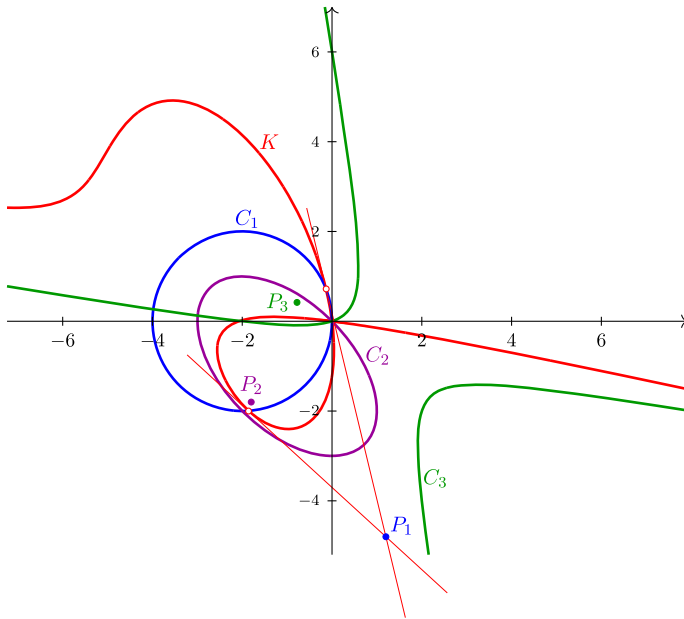
**Example** In our example,  $\tilde{K}$  in the affine plane is given by

$$\begin{aligned} -2192 - 2919x + 264x^2 + 122x^3 - 1557y + 3384xy + 198x^2y + 3726y^2 \\ - 81xy^2 - 81y^3 = 0, \end{aligned}$$

and finally, the sought cubic  $K$  is

$$-13x^3 - 66x^2y - 27x^2 - 216xy - 39xy^2 - 27y^2 - 22y^3 = 0.$$

Figure 3 shows the cubic  $K$  together with the three polar conics  $C_i$  with respect to their three poles  $P_i$ . Recall that the lines connecting  $P_i$  and the points of intersection of  $K$  with the polar curve  $C_i$  are tangent to  $K$ .



**Fig. 3** The cubic  $K$  together with the three poles  $P_1, P_2, P_3$  and the three polar conics  $C_1, C_2, C_3$  of the example. The tangents from  $P_1$  to  $K$  are also displayed

**Remark 2** We close this discussion by considering the situation when condition (C) is violated for three given conics  $C_1, C_2, C_3$ . Suppose that  $K$  is a cubic such that  $C_j$  is the polar conic with respect to some pole  $P_j$  for  $j = 1, 2, 3$ . Then,  $\det(T_1 P_1, T_2 P_1, T_3 P_1) = 0$  in condition (C) for  $P_1 \in \ker(T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3)$  means that the polar lines  $g_1 = T_1 P_1, g_2 = T_2 P_1 = T_1 P_2, g_3 = T_3 P_1 = T_1 P_3$  of the conics  $C_1, C_2, C_3$  with respect to the poles  $P_1, P_2, P_3$  are concurrent, which in turn means that  $P_1, P_2, P_3$  are collinear. Hence,  $C_1, C_2, C_3$  are identical or span only a one-dimensional pencil. This shows that for the three conics in Remark 1(c), there is no cubic  $K$  with the property that  $C_1, C_2, C_3$  are conic sections with respect to three poles. This means that (C) is a necessary condition in Theorem 2.4. On the other hand, if condition (C) is violated and  $C_1, C_2, C_3$  span only a one-dimensional pencil, then a cubic  $K$  with the required properties exists, but this cubic is no longer unique: Just take an arbitrary conic  $\tilde{C}_3$  such that  $C_1, C_2, \tilde{C}_3$  satisfy condition (C) and apply Theorem 2.4 in order to obtain a cubic  $\tilde{K}$  with respect to  $C_1, C_2$  and  $\tilde{C}_3$ . Then there is a point  $P_3$  on the line through  $P_1, P_2$  and such that the polar conic of  $\tilde{K}$  with respect to  $P_3$  is  $C_3$ .

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## References

1. Bobillier, E.: Géométrie de situation Théorèmes sur les polaires successives. *Ann. Math. Pures Appl.* **19**, 302–307 (1828)
2. Dolgachev, I.V.: *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, Cambridge (2012)
3. Halbeisen, L., Hungerbühler, N.: Generalized pencils of conics derived from cubics. *Beitr. Algebra Geom.* **61**(4), 681–693 (2020)
4. Joachimsthal, F.: Remarques sur la condition de l'égalité de deux racines d'une équation algébrique; et sur quelques théorèmes de Géométrie, qui en suivent. *J. Reine Angew. Math.* **33**, 371–376 (1846)
5. Monge, G.: *Application de l'analyse à la géométrie*. Mad. Ve. Bernard, Libraire de l'Ecole Impériale Polytechnique, quatrième édition, Paris (1809)
6. Sendra-Arranz, J.: The Hessian correspondence of hypersurfaces of degree 3 and 4. <http://arxiv.org/abs/2307.10415> (2023)
7. von Staudt, K.G.C.: *Geometrie der Lage*. Bauer und Raspe (1847)
8. Wieleitner, H.: *Algebraische Kurven. II. Allgemeine Eigenschaften*. Sammlung Göschen, vol. 436. Walter de Gruyter, Berlin (1939)

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