Beitr Algebra Geom (2018) 59:549–571
https://doi.org/10.1007/s13366-017-0375-1

ORIGINAL PAPER

The exponential pencil of conics

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Received: 3 October 2017 / Accepted: 13 December 2017 / Published online: 21 December 2017

Abstract The exponential pencil $G_\lambda := G_1(G_0^{-1}G_1)^{\lambda-1}$, generated by two conics G_0, G_1, carries a rich geometric structure: It is closed under conjugation, it is compatible with duality and projective mappings, it is convergent for $\lambda \to \pm\infty$ or periodic, and it is connected in various ways with the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$. The structure of the exponential pencil can be used to characterize the position of G_0 and G_1 relative to each other.

Keywords Pencil of conics · Poncelet's theorem · Conjugate conics

Mathematics Subject Classification 51A05 · 51A10 · 51A20

1 Introduction

The linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$, $\lambda \in \mathbb{R}$, of two circles or conics G_0 and G_1 is an extremely useful tool in the study of the geometry of circles and of conic sections, or, in higher dimensions, of quadrics. The linear pencil has a wide range of applications: for example, the circles of Apollonius (see Coxeter 1989), Gergonne's solution of Apollonius' Problem to construct a circle touching three given circles (see Coolidge 1971), Cayley's characterization of conics which carry Poncelet polygons (see Cayley 1854), or the classification of the relative position of two conics (see Petitjean 2010). But the linear pencil is not only a tool, it is also an interesting object in its own right with

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a rich geometry to study. However, the linear pencil lacks certain desirable properties: For example, it is not compatible with duality, i.e., the linear pencil of the dual of two conics is not the dual of the pencil of the two conics (see Sects. 2.1 and 3), and the linear pencil does, in general, not exist as real conics for all $\lambda \in \mathbb{R}$. In this article, we investigate the exponential pencil $G_\lambda = G_1(G_0^{-1}G_1)^{\lambda-1}$ of two conics G_0 and G_1. It turns out, that this pencil has a remarkable spectrum of geometric properties, which we study in Sect. 3. In Sect. 4 we classify the exponential pencils according to the relative position of the generating conics. But first, we start with some preliminary remarks to set the stage and to fix the notation.

2 Preliminaries

2.1 Matrix powers

Let $f : \mathbb{R} \to \mathbb{C}^{n\times n}$ be analytic such that

(a) $f(0) = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix,
(b) $f(1) = A$,
(c) $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$.

In particular, we have $f(-x) = f(x)^{-1}$ for all $x \in \mathbb{R}$, and therefore A is necessarily regular. Moreover, all matrices $f(x)$, $f(y)$ commute. With the infinitesimal generator $F := f'(0)$, we may write $f(x) = e^{Fx}$. In particular, $A = f(1) = e^F$, i.e., F is a logarithm of A. The logarithm of a matrix is in general not unique. Nonetheless, (a)–(c) determine the values of $f(n)$ for all $n \in \mathbb{Z}$. It is convenient to write $f(x) = A^x$ for a function satisfying (a)–(c). However, we have to keep in mind that two different logarithms of A define different functions $x \mapsto A^x$. In concrete cases, a function A^x can be calculated by the binomial series

$$A^x = (\mathbb{I} + (A - \mathbb{I}))^x = \sum_{k=0}^{\infty} \binom{x}{k} (A - \mathbb{I})^k$$

whenever the series converges.

Let $f(x) = A^x$ be a solution of (a)–(c), and suppose the matrix A is similar to the matrix B, i.e., $B = T^{-1}AT$. Then $g(x) := T^{-1} f(x)T$ is analytic, $g(0) = \mathbb{I}$, $g(1) = T^{-1}AT = B$, and $g(x + y) = g(x) \cdot g(y)$ for arbitrary $x, y \in \mathbb{R}$. Thus, $g(x) = B^x$. In this situation, the infinitesimal generators of f and g are similar: $g'(0) = T^{-1} f'(0)T$.

2.2 Projective plane and conics

We will work in the standard model of the real projective plane, i.e., we consider the set of points $\mathbb{P} = \mathbb{R}^3\backslash\{0\}/\sim$, where $x \sim y \in \mathbb{R}^3\backslash\{0\}$ are equivalent if $x = \lambda y$ for some $\lambda \in \mathbb{R}$. The set of lines is $\mathbb{B} = \mathbb{R}^3\backslash\{0\}/\sim$, where $g \sim h \in \mathbb{R}^3\backslash\{0\}$ are equivalent, if $g = \lambda h$ for some $\lambda \in \mathbb{R}$. We that say a point $[x]$ and a line $[g]$ are incident if $\langle x, g\rangle = 0$, where we denoted equivalence classes by square brackets and the standard inner product in $\mathbb{R}^3$ by $\langle\cdot,\cdot\rangle$.

As usual, a line $[g]$ can be identified with the set of points which are incident with it. Vice versa, a point $[x]$ can be identified with the set of lines which pass through it. The affine plane $\mathbb{R}^2$ is embedded in the present model of the projective plane by the map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \left[\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}\right].$$

The projective general linear group $PGL(3, \mathbb{R})$ consists of equivalence classes $[A]$ of regular matrices $A \in \mathbb{R}^{3\times 3}$ representing maps $\mathbb{P} \to \mathbb{P}$, $[x] \mapsto [Ax]$, where two matrices are equivalent, denoted $A_1 \sim A_2$, if $A_1 = \lambda A_2$ for some $\lambda \in \mathbb{R}$.

A *conic* in this model of the projective plane is an equivalence class of a regular, linear, selfadjoint map $A : \mathbb{R}^3 \to \mathbb{R}^3$ with mixed signature, i.e., A has eigenvalues of both signs. It is convenient to say a matrix A *is* a conic, instead of A is a representative of a conic. We may identify a conic by the set of points $[x]$ such that $\langle x, Ax\rangle = 0$, or by the set of lines $[g]$ for which $\langle A^{-1}g, g\rangle = 0$ (see below). Notice that, in this interpretation, a conic cannot be empty: Since A has positive and negative eigenvalues, there are points $[p]$, $[q]$ with $\langle p, Ap\rangle > 0$ and $\langle q, Aq\rangle < 0$. Hence a continuity argument guarantees the existence of points $[x]$ satisfying $\langle x, Ax\rangle = 0$.

From now on, we will only distinguish in the notation between an equivalence class and a representative if necessary.

Fact 2.1 *Let x be a point on the conic A. Then the line Ax is tangent to the conic A with contact point x.*

Proof We show that the line Ax meets the conic A only in x. Suppose otherwise, that $y \nsim x$ is a point on the conic, i.e., $\langle y, Ay\rangle = 0$, and at the same time on the line Ax, i.e., $\langle y, Ax\rangle = 0$. By assumption, we have $\langle x, Ax\rangle = 0$. Note, that $Ax \nsim Ay$ since A is regular, and $\langle Ay, x\rangle = 0$ since A is selfadjoint. Hence x and y both are perpendicular to the plane spanned by Ax and Ay, which contradicts $y \nsim x$. □

In other words, the set of tangents of a conic A is the image of the points on the conic under the map A. And consequently, a line g is a tangent of the conic *iff* $A^{-1}g$ is a point on the conic, i.e., if and only if $\langle A^{-1}g, g\rangle = 0$.

Definition 2.2 If P is a point, the line AP is called its polar with respect to a conic A. If g is a line, the point $A^{-1}g$ is called its pole with respect to the conic A.

Obviously, the pole of the polar of a point P is again P, and the polar of the pole of a line g is again g. Moreover:

Fact 2.3 *If the polar of a point P with respect to a conic A intersects the conic in a point x, then the tangent in x passes through P.*

Proof For x, we have $\langle x, Ax\rangle = 0$ since x is a point on the conic, and $\langle x, AP\rangle = 0$ since x is a point on the polar of P. The tangent in x is the line Ax, and indeed, P lies on this line, since $\langle P, Ax\rangle = \langle AP, x\rangle = 0$. □

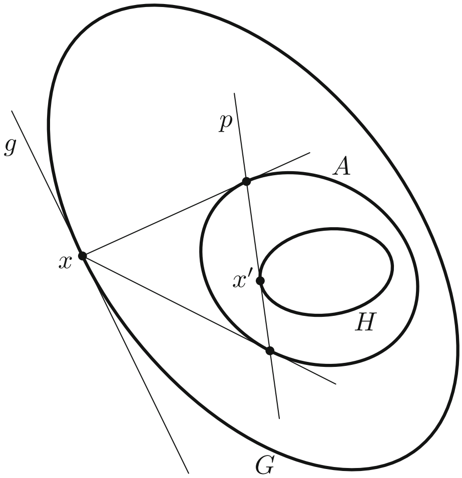
The fundamental theorem in the theory of poles and polars is

Fact 2.4 (La Hire's Theorem) *Let g be a line and P its pole with respect to a conic A. Then, for every point x on g, the polar of x passes through P. And vice versa: Let P be a point and g its polar with respect to a conic A. Then, for every line h through P, the pole of h lies on g.*

Proof We prove the second statement, the first one is similar. The polar of P is the line $g = AP$. A line h through P satisfies $\langle P, h\rangle = 0$ and its pole is $Q = A^{-1}h$. We check, that Q lies on g: Indeed, $\langle Q, g\rangle = \langle A^{-1}h, AP\rangle = \langle AA^{-1}h, P\rangle = \langle h, P\rangle = 0$. □

The next fact can be viewed as a generalization of Fact 2.4:

Theorem 2.5 *Let A and G be conics. Then, for every point x on G, the polar p of x with respect to A is tangent to the conic $H = AG^{-1}A$ in the point $x' = A^{-1}Gx$. Moreover, x' is the pole of the tangent $g = Gx$ in x with respect to A.*



Proof It is clear that $H = AG^{-1}A$ is symmetric and regular, and by Sylvester's law of inertia, H has mixed signature. The point x on G satisfies $\langle x, Gx\rangle = 0$. Its pole with respect to A is the line $g = Ax$. This line is tangent to H *iff* $\langle H^{-1}g, g\rangle = 0$. Indeed, $\langle H^{-1}g, g\rangle = \langle (AG^{-1}A)^{-1}Ax, Ax\rangle = \langle A^{-1}Gx, Ax\rangle = \langle Gx, x\rangle = 0$.

Since $\langle x', Hx'\rangle = \langle A^{-1}Gx, AG^{-1}AA^{-1}Gx\rangle = \langle Gx, x\rangle = 0$, the point $x' = A^{-1}Gx$ lies on H. The tangent to H in x' is $Hx' = AG^{-1}AA^{-1}Gx = Ax$ which is indeed the polar of x with respect to A. The last statement in the theorem follows immediately. □

Definition 2.6 The conic $H = AG^{-1}A$ is called the *conjugate conic* of G with respect to A.

Recall that the *dual of a point* $P \in \mathbb{P}$ is the line $P \in \mathbb{B}$ and the *dual of the line* $g \in \mathbb{B}$ is the point $g \in \mathbb{P}$. In particular, P and g are incident if and only if their duals are incident. The dual lines of all points on a conic A are tangent to the conic A^{-1}, and the dual points of all tangents of a conic A are points on the conic A^{-1}. Therefore, A^{-1} is called the *dual conic of the conic* A. We will denote the dual A^{-1} of a conic A by A'.

The projective space $\mathbb{P} = \mathbb{R}^3\backslash\{0\}/\sim$ can also be represented as the unit sphere $S^2 \subset \mathbb{R}^3$ with antipodal identification of points. Then, this space $\mathbb{S}$, endowed with the natural metric $d([x], [y]) = \arcsin \|x \times y\|$, becomes a complete metric space with bounded metric. The set of closed sets in this space is a complete metric space with respect to the inherited Hausdorff metric. In particular, a conic A given by

$$\{x \in S^2 \mid \langle x, Ax\rangle = 0\}/\sim$$

is a compact set in $\mathbb{S}$. In this sense, we can consider the limit of a sequence of conics.

3 The exponential pencil

The linear pencil of two matrices $g_0, g_1 \in \mathbb{C}^{n\times n}$ is given by

$$g_\lambda := \lambda g_1 + (1-\lambda)g_0, \quad \lambda \in \mathbb{R}.$$

This notation is consistent for the values $\lambda = 0$ and $\lambda = 1$. If g_0 and g_1 commute, exponentiation of the linear pencil gives

$$G_\lambda := e^{g_\lambda} = e^{\lambda g_1+(1-\lambda)g_0} = e^{g_1}(e^{-g_0}e^{g_1})^{\lambda-1} = G_1(G_0^{-1}G_1)^{\lambda-1} \tag{1}$$

where $G_i := e^{g_i}$. The last expression in (1) makes sense also for non-commuting matrices and we may define an *exponential pencil of two matrices* $G_0, G_1 \in \mathbb{C}^{n\times n}$ by

$$G_\lambda := G_1(G_0^{-1}G_1)^{\lambda-1}, \quad \lambda \in \mathbb{R}, \tag{2}$$

provided $(G_0^{-1}G_1)^x$, $x \in \mathbb{R}$, exists in the sense of Sect. 2.1. The notation G_λ in (2) is consistent for the values $\lambda = 0$ and $\lambda = 1$. Notice that for regular matrices G_0, G_1, a unique *discrete* exponential pencil $G_n = G_1(G_0^{-1}G_1)^{n-1}$ for $n \in \mathbb{Z}$ exists. This general concept applies naturally to conics and we define:

Definition 3.1 Let G_0, G_1 be two conics. Then

$$G_\lambda := G_1(G_0^{-1}G_1)^{\lambda-1}, \quad \lambda \in \mathbb{R},$$

is called an exponential pencil generated by G_0 and G_1 provided that all G_λ are symmetric and real.

Remarks

(a) For an exponential pencil to exist, it is necessary and sufficient that $G_0^{-1}G_1$ has a real logarithm F such that G_1F is symmetric.
(b) In Sect. 4 we will see that the existence of an exponential pencil depends on the position of G_0 and G_1 relative to each other, and except for only one case, the exponential pencil is unique.

(c) Each G_λ in an exponential pencil generated by G_0 and G_1 is actually a conic: In contrast to the linear pencil, an exponential pencil of conics does not contain degenerate or complex conics. This is a consequence of the following Lemma.

Lemma 3.2 *If $G_\lambda = G_1(G_0^{-1}G_1)^{\lambda-1}$, $\lambda \in \mathbb{R}$, is an exponential pencil of two conics G_0, G_1, then*

(i) $\det(G_\lambda) = \det(G_1)^\lambda / \det(G_0)^{\lambda-1}$,
(ii) *G_λ has mixed signature for all $\lambda \in \mathbb{R}$.*

Proof (i) Let L be a logarithm of $G_0^{-1}G_1$. Then, we have

$$\begin{aligned}\det G_\lambda &= \det G_1(G_0^{-1}G_1)^{\lambda-1} = \det G_1 \det e^{(\lambda-1)L} = \det G_1 e^{\operatorname{trace}(\lambda-1)L} \\ &= \det G_1 \left(e^{\operatorname{trace} L}\right)^{\lambda-1} = \det G_1 \left(\det e^L\right)^{\lambda-1} = \det G_1 \left(\frac{\det G_1}{\det G_0}\right)^{\lambda-1}.\end{aligned}$$

(ii) Since G_λ is symmetric, it has real eigenvalues which depend continuously on λ. Then, according to (i), the product of the eigenvalues cannot change sign and the signature of G_λ remains constant. □

The next Lemma will have immediate geometric consequences:

Lemma 3.3 *If G_λ, $\lambda \in \mathbb{R}$, is an exponential pencil of G_0, G_1 and $\xi, \mu \in \mathbb{R}$, there holds*

$$G_\mu G_\xi^{-1} G_\mu = G_{2\mu-\xi}.$$

Proof

$$\begin{aligned}G_\mu G_\xi^{-1} G_\mu &= G_1(G_0^{-1}G_1)^{\mu-1}\left(G_1(G_0^{-1}G_1)^{\xi-1}\right)^{-1} G_1(G_0^{-1}G_1)^{\mu-1} \\ &= G_1(G_0^{-1}G_1)^{\mu-1}(G_0^{-1}G_1)^{1-\xi}G_1^{-1}G_1(G_0^{-1}G_1)^{\mu-1} \\ &= G_1\left(G_0^{-1}G_1\right)^{2\mu-\xi-1} = G_{2\mu-\xi}.\end{aligned}$$

□

In view of Theorem 2.5 and Definition 2.6, we get as an immediate consequence of Lemma 3.3:

Theorem 3.4 *An exponential pencil G_λ, $\lambda \in \mathbb{R}$, of two conics is closed under conjugation: The conjugate of G_ξ with respect to G_μ is $G_{2\mu-\xi}$.*

More generally, we have the following:

Lemma 3.5 *If G_{λ_0} and G_{λ_1} belong to a pencil $G_\lambda = G_1(G_0^{-1}G_1)^{\lambda-1}$ generated by G_0, G_1, then G_{λ_0} and G_{λ_1} generate the same exponential pencil as G_0 and G_1. More precisely, we have*

$$G_{\lambda_1}(G_{\lambda_0}^{-1}G_{\lambda_1})^{\lambda-1} = G_1(G_0^{-1}G_1)^{\lambda_0+\lambda(\lambda_1-\lambda_0)-1} = G_{\lambda_0+\lambda(\lambda_1-\lambda_0)}.$$

In particular, the exponential pencil does not depend on the order of the defining conics G_0 and G_1.

Proof Let $f(x) := (G_0^{-1}G_1)^{x(\lambda_1-\lambda_0)}$ for $x \in \mathbb{R}$. Then $f(0) = \mathbb{I}$, and

$$\begin{aligned} f(1) &= (G_0^{-1}G_1)^{\lambda_1-\lambda_0} = (G_0^{-1}G_1)^{1-\lambda_0}G_1^{-1}G_1(G_0^{-1}G_1)^{\lambda_1-1} \\ &= \left(G_1(G_0^{-1}G_1)^{\lambda_0-1}\right)^{-1} G_1(G_0^{-1}G_1)^{\lambda_1-1} = G_{\lambda_0}^{-1}G_{\lambda_1}. \end{aligned}$$

Moreover, $f(x+y) = f(x)f(y)$. Therefore, according to Sect. 2.1, we may write $f(x) = (G_{\lambda_0}^{-1}G_{\lambda_1})^x$. We obtain

$$G_{\lambda_1}(G_{\lambda_0}^{-1}G_{\lambda_1})^{\lambda-1} = G_{\lambda_1}f(\lambda-1) = G_1(G_0^{-1}G_1)^{\lambda_1-1}(G_0^{-1}G_1)^{(\lambda-1)(\lambda_1-\lambda_0)}$$

where we used the original definition of f in the last equality. Now the claim follows immediately. □

It turns out that exponential pencils behave well with respect to duality:

Theorem 3.6 *Let G_0 and G_1 be conics and G_0' and G_1' their duals. Suppose G_0 and G_1 generate an exponential pencil G_λ. Then, the dual of G_λ is an exponential pencil of G_0' and G_1'. More precisely, for all $\lambda \in \mathbb{R}$ we have*

$$G_1'({G_0'}^{-1}G_1')^{\lambda-1} = (G_1(G_0^{-1}G_1)^{\lambda-1})'.$$

Observe that the linear pencil does not enjoy the corresponding property.

Proof Suppose $G_\lambda = G_1(G_0^{-1}G_1)^{\lambda-1}$ is an exponential pencil generated by G_0 and G_1. Then, for $x \in \mathbb{R}$, let $f(x) := G_{1-x}G_1^{-1s} = G_1(G_0^{-1}G_1)^{-x}G_1^{-1}$. Observe that $f(0) = \mathbb{I}$, $f(1) = G_0G_1^{-1}$ and

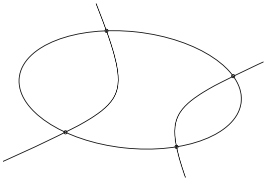
$$\begin{aligned} f(x+y) &= G_1(G_0^{-1}G_1)^{-(x+y)}G_1^{-1} = \left(G_1(G_0^{-1}G_1)^{-x}G_1^{-1}\right)\left(G_1(G_0^{-1}G_1)^{-y}G_1^{-1}\right) \\ &= f(x)f(y) \end{aligned}$$

and therefore, according to Sect. 2.1, we may write $f(x) = (G_0G_1^{-1})^x = ({G_0'}^{-1}G_1')^x$. We obtain

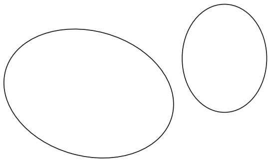
$$(G_1({G_0}^{-1}G_1)^x)^{-1} = G_1^{-1}f(x) = G_1'({G_0'}^{-1}G_1')^x$$

and claim follows by replacing x by $\lambda - 1$. □

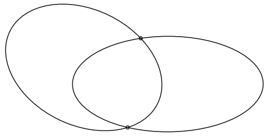
The natural question is now to ask which conics G_0, G_1 generate an exponential pencil. To answer this question, we recall that two conics can lie in 8 different positions relative to each other (see Petitjean 2010):



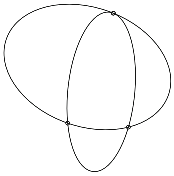
Case 1: four intersections



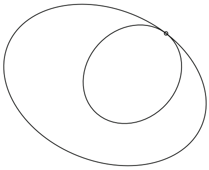
Case 2: no intersections



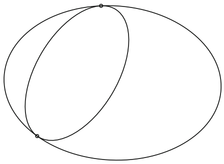
Case 3: two intersections



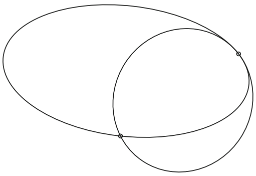
Case 4: two intersections, one 1st order contact



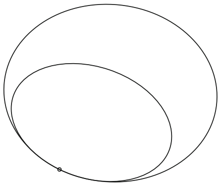
Case 5: one 1st order contact



Case 6: two 1st order contacts



Case 7: one intersection, one 2nd order contact



Case 8: one 3rd order contact

We now go case by case through the list and investigate the existence and the geometric properties of the resulting exponential conics. In particular, it will turn out that the exponential conic and the linear conic are quite closely related. We start with the important observation that the exponential pencil is projectively invariant:

Lemma 3.7 *Let $S \in \mathbb{R}^{n\times n}$ be a regular matrix, inducing a projective map $\mathbb{P} \to \mathbb{P}$, $x \mapsto Sx$. Then the image under S of an exponential pencil $G_\lambda = G_1(G_0^{-1}G_1)^{\lambda-1}$ of two conics G_0, G_1 is an exponential pencil of their images.*

Proof For $T := S^{-1}$, the images of the conics G_0, G_1 under S are $\bar{G}_0 := T^\top G_0 T$ and $\bar{G}_1 := T^\top G_1 T$. We want to show that the image $\bar{G}_\lambda = T^\top G_\lambda T$ is an exponential pencil of $\bar{G}_0$ and $\bar{G}_1$. We start by definig $f(x) := T^{-1}(G_0^{-1}G_1)^x T$ for $x \in \mathbb{R}$. We have $f(0) = \mathbb{I}$, $f(1) = \bar{G}_0^{-1}\bar{G}_1$ and

$$\begin{aligned} f(x+y) &= T^{-1}(G_0^{-1}G_1)^{x+y}T = \big(T^{-1}(G_0^{-1}G_1)^x T\big)\big(T^{-1}(G_0^{-1}G_1)^y T\big) \\ &= f(x)f(y) \end{aligned}$$

and therefore, according to Sect. 2.1, we may write $f(x) = (\bar{G}_0^{-1}\bar{G}_1)^x$. We obtain

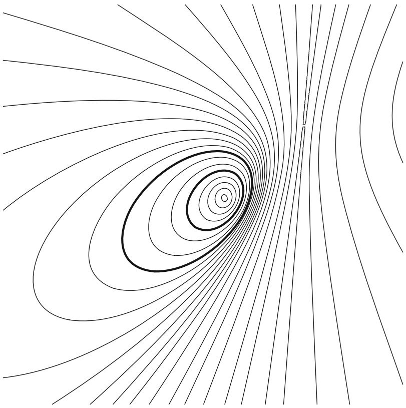
$$T^\top G_{x+1} T = T^\top G_1 (G_0^{-1}G_1)^x T = \bar{G}_1 f(x) = \bar{G}_1(\bar{G}_0^{-1}\bar{G}_1)^x$$

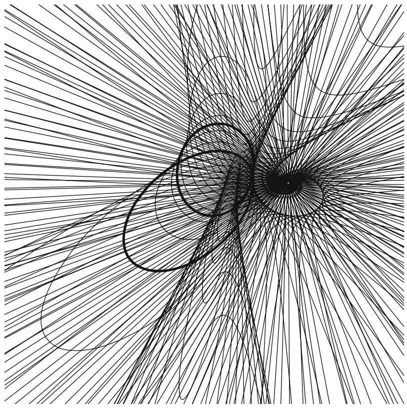
and claim follows by replacing x by $\lambda - 1$. $\square$

The investigation of the exponential pencils in all the Cases 1–8 listed above can now be reduced to a canonical form in each case.

4 Classification of the exponential pencils

The two figures below show the exponential pencil of two conics G_0, G_1 (bold) in two cases. On the left, the geometry seems rather gentle, on the right quite complex.





In this section, we investigate the exponential pencil of two conics in each of the possible cases of their relative position. It turns out that the geometric behavior of the exponential pencil is characteristic for each case.

Theorem 4.1 (Case 1) *Let G_0, G_1 be two conics with four intersection points. Then, they generate an exponential conic $G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1}$ iff the common interior of G_1 and G_0 is connected. In this case, the exponential pencil is unique. G_λ converges for $\lambda \to \pm\infty$ to a line $\ell^\pm$. The family G_λ has an envelope E with asymptotes $\ell^\pm$. Through every exterior point of E (i.e., points with four tangents to E), except for the points on $\ell^\pm$, there pass exactly two members of the exponential pencil G_λ. Each G_λ touches a member of the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$ in two first order contact points.*

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \pm b^2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $a > 1 > b > 0$ or $b > 1 > a > 0$ for the positive sign, and $a > 1$ for the negative sign (see Halbeisen and Hungerbühler 2017). Let

$$A := G_0^{-1}G_1 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \pm b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

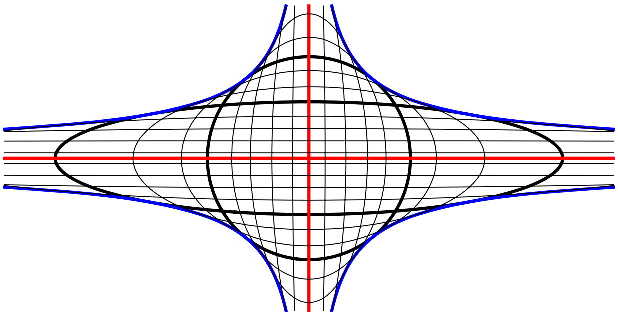
then every solution X of $e^X = A$ leads to an exponential pencil $G_\lambda = G_1 e^{(\lambda-1)X}$ of G_0 and G_1, provided G_λ is a real symmetric matrix for all $\lambda \in \mathbb{R}$. In particular, $h(x) := e^{xX}$ must be real for all $x \in \mathbb{R}$. But then $h'(0) = X$ must be real. We can therefore concentrate on real solutions of $e^X = A$. According to Culver (1966, Theorem 1), such a real solution exists only for the positive sign in A. This corresponds to the case, where the common interior of G_0 and G_1 is connected. Then, the solution of $e^X = A$ is unique, according to Culver (1966, Theorem 2), and we obtain a unique exponential pencil given by

$$G_\lambda = G_1 \left(G_0^{-1} G_1\right)^{\lambda-1} = \begin{pmatrix} a^{2\lambda} & 0 & 0 \\ 0 & b^{2\lambda} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

The envelope E is obtained by eliminating λ from $\frac{\partial}{\partial\lambda}\langle x, G_\lambda x\rangle = 0$ and $\langle x, G_\lambda x\rangle = 0$. One finds

$$(x_1^2)^{\log b}(x_3^2)^{\log a} |\log a|^{\log a} \left|\log \frac{a}{b}\right|^{\log b} = (x_2^2)^{\log a}(x_3^2)^{\log b} |\log b|^{\log b} \left|\log \frac{a}{b}\right|^{\log a}.$$

The figure shows in the affine plane $x_3 = 1$ the pencil generated by the unit circle G_0 and an ellipse G_1 (both bold), together with the asymptotic lines $\ell^\pm$ (red) and the envelope E (blue).



□

Theorem 4.2 (Case 2) *Let G_0, G_1 be two disjoint conics. Then they generate an exponential pencil $G_\lambda = G_1 \left(G_0^{-1} G_1\right)^{\lambda-1}$ iff G_1 is in the interior of G_0 or vice versa, in which case the exponential pencil is unique. G_λ converge for $\lambda \to \pm\infty$ to a point (which coincides with a limit point of the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$), and a line (which contains the second limit point of the linear pencil). Each G_λ touches two members of the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$ in two first order contact points, or, if G_0, G_1 are projectively equivalent to concentric circles, each G_λ belongs to the linear pencil.*

Proof Since G_0, G_1 are disjoint, there exist coordinates for which both conics are diagonal [see for example Pesonen (1956) or Hong et al. (1986)]: W.l.o.g.

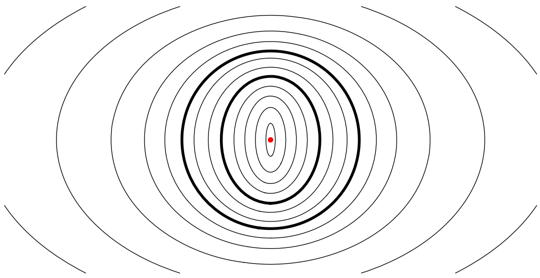
$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad G_1 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \pm b^2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

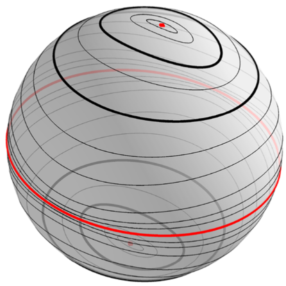
where $1 > a, b > 0$ or $a, b > 1$ in case of the positive sign, and $1 > a > 0, b > 0$ in case of the negative sign. Then,

$$A := G_0^{-1} G_1 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \pm b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As in Case 1, an exponential pencil can only exist for the positive sign in A. This corresponds to the case where G_0 is in the interior of G_1 or vice versa. Now, we have to consider two cases:

Case 2a. $a \neq b$: Then, by the same reasoning as in Case 1, the exponential pencil G_λ is unique and given by (3). The figure on the left shows, in the plane $x_3 = 1$, the exponential pencil generated by the unit circle G_0 and an ellipse G_1 inside of G_0 (both bold). The limit as $\lambda \to \infty$ is the center (red), and as $\lambda \to -\infty$ the ideal line. It is instructive to look at the same configuration on the sphere (figure on the right, limit point and limit ideal line in red).





Case 2b. $a = b$: In this case we have

$$A := G_0^{-1} G_1 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and according to Culver (1966, Theorem 2, and Corollary), there is a continuum of real solutions of $e^{X_\mu} = A$. So, there is a chance that the exponential pencil is not unique. From Gantmacher (1998, §8) we infer that all matrices

$$X_\mu = \begin{pmatrix} \log a^2 & 0 & 0 \\ 0 & \log a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + K \begin{pmatrix} 2n\pi i & 0 & 0 \\ & 2m\pi i & 0 \\ 0 & 0 & 0 \end{pmatrix} K^{-1}, \tag{4}$$

where m, n are integers and K is an arbitrary regular matrix of the form

$$K = \begin{pmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix},$$

are logarithms of A, and there are no other logarithms. Then $e^{(\lambda-1)X_\mu}$ has the same block structure as K. Now, in our case, we need that $G_\lambda = G_1 e^{(\lambda-1)X_\mu}$ is real and symmetric. But this implies that $G_1^{-1}G_\lambda = e^{(\lambda-1)X_\mu}$ is real and symmetric for all λ. Then the derivative of this with respect to λ at $\lambda = 1$ gives that X_μ must be real and symmetric. Then for each $k \in \mathbb{N}$, $e^{X_\mu/2^k}$ is also symmetric and real, and positive definite, because $e^{X_\mu/2^k} = e^{X_\mu/2^{k+1}} e^{X_\mu/2^{k+1}}$. Recall that repeated roots $A^{1/2^k}$ of A which are real, symmetric and positive definite, are unique. This means, that the values of $e^{X_\mu/2^k}$ agree for all integers k. Therefore, the infinitesimal generators X_μ must actually agree. In other words, there is only one real symmetric logarithm X of A, and the exponential pencil is given by (3), i.e. a family of concentric circles. Alternatively, the uniqueness can be seen directly from (4) by imposing symmetry and real valuedness of X_μ. □

Theorem 4.3 (Case 3) *Let G_0, G_1 be two conics with two intersectctions. Then they generate a countable family of exponential pencils $G_\lambda = G_1 \left(G_0^{-1} G_1\right)^{\lambda-1}$. Such a pencil is either periodic with a conic as envelope, or periodically expanding covering the plane infinitely often, with a local envelope which has a singular point S. For integer values of λ, the corresponding conics of all exponential pencils agree.*

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & a^2 - r^2 \end{pmatrix}, \quad a > 0,$$

(see Halbeisen and Hungerbühler 2017). Geometrically, G_1 represents a circle of radius $r > 0$ in the plane $x_3 = 1$ with center in $(a, 0)$ which intersects the unit circle G_0, centered in $(0, 0)$, in two real points. I.e., $-1 < a - r < 1$ and $1 < a + r$, which implies that $\kappa := (1 - a + r)(1 + a - r)(a + r - 1)(a + r + 1) > 0$ because all four factors are strictly positive. We now use a translation T, a swap of axis P, a scaling L, and a rotation R, namely

$$T = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1/\ell & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} c & -\sqrt{1-c^2} & 0 \\ \sqrt{1-c^2} & c & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the following values

$$\tau = \frac{1 + a^2 - r^2}{2a},$$
$$\ell = \frac{\sqrt[4]{\kappa}}{\sqrt{2a}},$$
$$c = \frac{1}{2}\sqrt{2 - \frac{\sqrt{\kappa}}{a}}.$$

Notice that $4a^2 - \kappa = (1 + a^2 - r^2)^2 \geq 0$ and hence the radicand $2 - \frac{\sqrt{\kappa}}{a} \geq 0$ in c. For $U = TPLR$ this leads to the following representation of the conics:

$$U^\top G_0 U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^\top G_1 U = \begin{pmatrix} (a^2 - r^2 - 1)/2 & \sqrt{\kappa}/2 & 0 \\ \sqrt{\kappa}/2 & (r^2 - a^2 + 1)/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the plane $x_3 = 1$ these are rotated hyperbolas centered at (0, 0, 1), and we denote them again by G_0 and G_1. Then, $G_0^{-1}G_1$ has the form

$$A := G_0^{-1}G_1 = \begin{pmatrix} r\cos\phi_k & -r\sin\phi_k & 0 \\ r\sin\phi_k & r\cos\phi_k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $\phi_k = 2k\pi + \arccos\frac{1-a^2+r^2}{2r}$, where k is an arbitrary integer. Notice, that $-2r < 1 - a^2 + r^2 < 2r$, again because the factors of κ are strictly positive, and hence the values ϕ_k are real. Here, according to Gantmacher (1998, §8), we find the following solutions X of $A = e^{X_k}$:

$$X_k = \begin{pmatrix} \log r & -\phi_k & 0 \\ \phi_k & \log r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

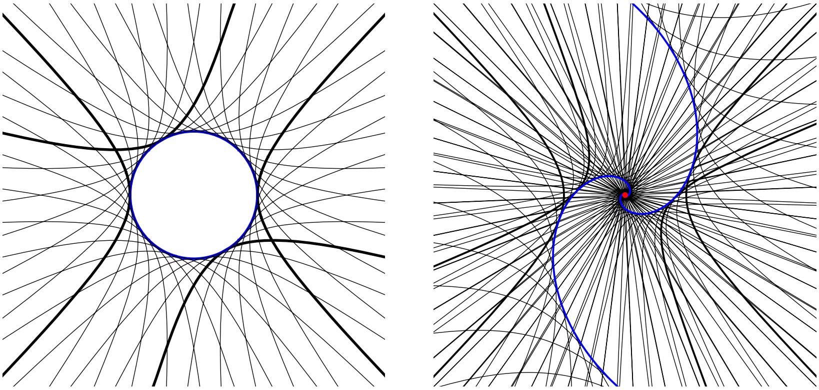
Therefore, we get

$$e^{(\lambda-1)X_k} = (G_0^{-1}G_1)^{\lambda-1} = \begin{pmatrix} r^{\lambda-1}\cos(\lambda-1)\phi_k & -r^{\lambda-1}\sin(\lambda-1)\phi_k & 0 \\ r^{\lambda-1}\sin(\lambda-1)\phi_k & r^{\lambda-1}\cos(\lambda-1)\phi_k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and finally

$$G_\lambda = G_1(G_0^{-1}G_1)^{\lambda-1} = \begin{pmatrix} -r^\lambda\cos\lambda\phi_k & r^\lambda\sin\lambda\phi_k & 0 \\ r^\lambda\sin\lambda\phi_k & r^\lambda\cos\lambda\phi_k & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $r = 1$ (and only in this case), the resulting exponential pencil is periodic with period $2\pi/\phi_k$. Hence, in the plane $x_3 = 1$, G_λ are rectangular hyperbolas, rotating around the origin with constant angular velocity ϕ_k. For $r \neq 1$, the rectangular hyperbolas are rotating with constant angular velocity ϕ_k and at the same time exponentially shrinking ($r > 1$) or expanding ($0 < r < 1$) with factor r^λ. The figures below show the two cases: G_0 and G_1 are bold, the envelope is blue, the singular point S is red.



□

Remark The case when $r = 1$ (i.e., when the resulting exponential pencil is periodic), was studied with respect to Poncelet's Theorem in Halbeisen and Hungerbühler (2016) and Halbeisen and Hungerbühler (2017).

Theorem 4.4 (Case 4) *Let G_0, G_1 be two conics with two intersections and one first order contact. Then they generate an exponential pencil $G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1}$ iff the contact point of G_1 and G_0 lies on the boundary of their common interior. Then the exponential pencil is unique. Each G_λ touches a member of the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$ in two first order contact points. For $\lambda \to \pm\infty$, G_λ converges to the tangent in the contact point, and to a line trough the contact point, respectively. The family G_λ has an envelope E.*

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} \mu+1 & 0 & -\mu \\ 0 & 1-\mu & 0 \\ -\mu & 0 & \mu-1 \end{pmatrix}, \quad \mu \neq 1, \mu \neq 0,$$

(see Halbeisen and Hungerbühler (2017)). Then,

$$A := G_0^{-1}G_1 = \begin{pmatrix} \mu+1 & 0 & -\mu \\ 0 & 1-\mu & 0 \\ \mu & 0 & 1-\mu \end{pmatrix}.$$

With

$$T = \begin{pmatrix} 1 & 1/\mu & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\mu \end{pmatrix} = \mathbb{I} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mu \end{pmatrix}}_{=:\alpha}$$

we get $G_0^{-1}G_1 = \mathbb{I} + T\alpha T^{-1}$. As in the proof of Case 2b, we are only interested in real logarithms of A. By Culver (1966, Theorem 1), the real logarithm of A exists *iff* $\mu < 1$. This corresponds to the situation where the contact point sits on the boundary of the common interior of G_0 and G_1. By Culver (1966, Theorem 2), the real logarithm is unique. By the binomic series we get

$$\begin{aligned}(G_0^{-1}G_1)^x &= (\mathbb{I} + T\alpha T^{-1})^x = T\sum_{k=0}^{\infty}\binom{x}{k}\alpha^k T^{-1} \\ &= \begin{pmatrix} 1+\mu x & 0 & -\mu x \\ 0 & (1-\mu)^x & 0 \\ \mu x & 0 & 1-\mu x \end{pmatrix},\end{aligned}$$

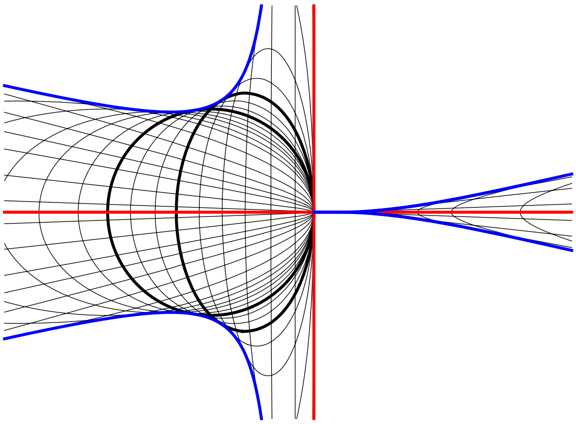
and finally the exponential pencil

$$G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1} = \begin{pmatrix} 1+\lambda\mu & 0 & -\lambda\mu \\ 0 & (1-\mu)^\lambda & 0 \\ -\lambda\mu & 0 & \lambda\mu-1 \end{pmatrix}.$$

Notice that the binomial series converges only for $|\mu| < 1$. But the expression we got for $(G_0^{-1}G_1)^x$ satisfies the properties of Sect. 2.1 and therefore the result for G_λ is correct for arbitrary $\mu < 1$, $\mu \neq 0$. The conics G_λ are symmetric to the line $(0, 1, 0)^\top$ and touch G_0, G_1 in their contact point. The envelope E is obtained by eliminating λ from $\frac{\partial}{\partial\lambda}\langle x, G_\lambda x\rangle = 0$ and $\langle x, G_\lambda x\rangle = 0$. In the plane $x_3 = 1$ one finds

$$(1+x_1)\ln(1-\mu) = (1-x_1)\mu\left(\ln\left(-\frac{\mu(x_1-1)^2}{x_2^2\ln(1-\mu)}\right) - 1\right).$$

The figure shows, in the plane $x_3 = 1$, the pencil generated by the unit circle G_0 and an ellipse G_1 (both bold) together with the limiting lines (red) and the envelope E (blue).



$\square$

Theorem 4.5 (Case 5) *Let G_0, G_1 be two conics with one first order contact point C. Then, they generate an exponential pencil $G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1}$ iff G_1 lies inside of G_0 or vice versa. This exponential pencil is unique. The family G_λ together with the tangent in C forms a foliation of $\mathbb{P}\backslash\{C\}$. Each G_λ touches a member of the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$ in two first order contact points. If G_1 is inside of G_0, then G_λ converges to C for $\lambda \to \infty$, and to the tangent in C for $\lambda \to -\infty$. If G_0 lies inside of G_1 it is the other way round.*

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & 2a-1 \end{pmatrix}, \quad a \neq 1, a \neq 0,$$

(see Halbeisen and Hungerbühler 2017), i.e., G_0 is a unit circle centered in $(0, 0, 1)^\top$ and G_1 a circle with center $(a, 0, 1)^\top$ which touches G_0 in $(1, 0, 1)^\top$. Then,

$$A := G_0^{-1}G_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ a & 0 & 1-2a \end{pmatrix}.$$

With

$$T = \begin{pmatrix} 0 & 0 & 1/a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & 1 \\ 0 & 0 & 1-a \end{pmatrix} = \mathbb{I} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 1 \\ 0 & 0 & -a \end{pmatrix}}_{=:\alpha}$$

we get $G_0^{-1}G_1 = \mathbb{I} + T\alpha T^{-1}$. As in Case 4, the real logarithm of A exists, and is unique, *iff* $1 > a$. This corresponds to the case where G_0 is inside G_1 or vice versa.

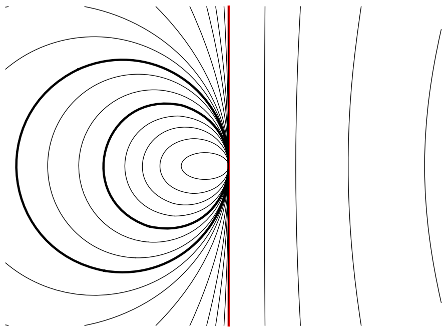
Then, by the binomic series, we get

$$
\begin{aligned}
(G_0^{-1}G_1)^x &= (\mathbb{I} + T\alpha T^{-1})^x = T\sum_{k=0}^{\infty}\binom{x}{k}\alpha^k T^{-1} \\
&= \begin{pmatrix} (1-a)^{x-1}(1+a(x-1)) & 0 & -(1-a)^{x-1}ax \\ \mu & 1 & 0 \\ (1-a)^{x-1}ax & 0 & (1-a)^{x-1}(1-a(x+1)) \end{pmatrix},
\end{aligned}
$$

and finally the exponential pencil

$$
G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1} = \begin{pmatrix} (1-a)^{\lambda-1}(1+a(\lambda-1)) & 0 & -(1-a)^{\lambda-1}a\lambda \\ 0 & 1 & 0 \\ -(1-a)^{\lambda-1}a\lambda & 0 & (1-a)^{\lambda-1}(a(\lambda+1)-1) \end{pmatrix}.
$$

Notice that the binomial series converges only for $|a| < 1$. However, the expression we obtained for $(G_0^{-1}G_1)^x$ satisfies the properties of Sect. 2.1 and therefore, the result for G_λ is correct for arbitrary $a < 1$, $a \neq 0$. The conics G_λ are symmetric to the line $(0, 1, 0)^\top$ and touch G_0, G_1 in C. The figure shows, in the plane $x_3 = 1$, the pencil generated by the unit circle G_0 and a circle G_1 inside of G_0 (both bold), together with the tangent in the contact point (red).



□

Theorem 4.6 (Case 6) *Let G_0, G_1 be two conics with two first order contact points C_0, C_1. Then they generate an exponential pencil $G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1}$ iff G_0 lies inside of G_1 or vice versa. This exponential pencil is unique, and each conic G_λ is a member of the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$. If G_1 is inside of G_0, then G_λ and g_λ have the same limit for $\lambda \to \infty$, and for $\lambda \to -\infty$ the limit of G_λ consists of the tangents in C_0 and C_1. If G_0 is inside of G_1 it is the other way round.*

The proof will actually give some more information.

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\mu & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mu \neq 0,\ \mu \neq 1,$$

(see Halbeisen and Hungerbühler 2017). Then,

$$A := G_0^{-1} G_1 = \begin{pmatrix} 1 & 0 & 0 \\ & 1-\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

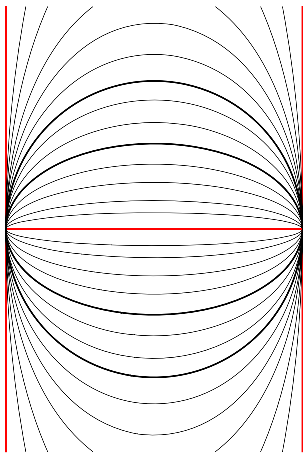
Like in Case 2, A has only one symmetric, real logarithm if $\mu < 1$. This inequality is equivalent to the fact that one conic lies inside the other, and we get

$$(G_0^{-1} G_1)^x = \begin{pmatrix} 1 & 0 & 0 \\ & (1-\mu)^x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, we obtain as exponential pencil

$$G_1 (G_0^{-1} G_1)^{\lambda - 1} = \begin{pmatrix} 1 & 0 & 0 \\ & (1-\mu)^\lambda & 0 \\ 0 & 0 & -1 \end{pmatrix} = g_{(1-(1-\mu)^\lambda)/\mu}.$$

The figure shows the pencil generated by the unit circle G_0 and an ellipse G_1 (both bold) together with the limits (red).



□

Theorem 4.7 (Case 7) *Let G_0, G_1 be two conics with one intersection and one second order contact. Then, they generate a unique exponential pencil G_λ =*

$G_1\left(G_0^{-1}G_1\right)^{\lambda-1}$. *The family* G_λ *has a conic* E *as envelope.* E *belongs to the linear pencil of* $G_2 - 3G_0 - 6G_1$ *and the double line joining the intersection point and the second order contact point of* G_0 *and* G_1*. Through every exterior point of* E*, except for the tangent in the contact point of* G_0 *and* G_1*, there pass exactly two members of the exponential pencil* G_λ*.*

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 1 & -\mu & 0 \\ -\mu & 1 & \mu \\ 0 & \mu & -1 \end{pmatrix}, \quad \mu \neq 0$$

(see Halbeisen and Hungerbühler 2017). Then,

$$A := G_0^{-1}G_1 = \begin{pmatrix} 1 & -\mu & 0 \\ -\mu & 1 & \mu \\ 0 & -\mu & 1 \end{pmatrix}.$$

With

$$T = \begin{pmatrix} 1 & 0 & 1/\mu^2 \\ 0 & -1/\mu & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{=:\alpha}$$

we get $G_0^{-1}G_1 = \mathbb{I} + T\alpha T^{-1}$. By Culver (1966, Theorem 2), A has a unique real logarithm, and we can use the binomic series (which, in this case, consists of only three terms), to obtain

$$\begin{aligned}(G_0^{-1}G_1)^x &= (\mathbb{I} + T\alpha T^{-1})^x = T\sum_{k=0}^{\infty}\binom{x}{k}\alpha^k T^{-1} \\ &= \begin{pmatrix} 1 + x(x-1)\mu^2/2 & -x\mu & x(1-x)\mu^2/2 \\ -x\mu & 1 & x\mu \\ x(x-1)\mu^2/2 & -x\mu & 1 + x(1-x)\mu^2/2 \end{pmatrix},\end{aligned}$$

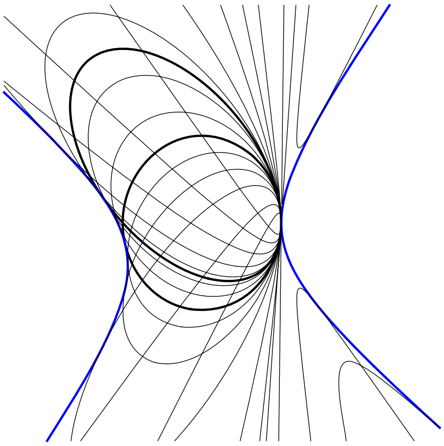
and finally the exponential pencil

$$G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1} = \begin{pmatrix} 1 + \lambda(\lambda-1)\mu^2/2 & -\lambda\mu & \lambda(1-\lambda)\mu^2/2 \\ -\lambda\mu & 1 & \mu\lambda \\ \lambda(1-\lambda)\mu^2/2 & \mu\lambda & \mu^2\lambda(\lambda-1)/2 - 1 \end{pmatrix}.$$

The envelope E is obtained by eliminating λ from $\frac{\partial}{\partial\lambda}\langle x, G_\lambda x\rangle = 0$ and $\langle x, G_\lambda x\rangle = 0$. One finds the conic

$$E = \begin{pmatrix} \mu^2-8 & 4\mu & -\mu^2 \\ 4\mu & 8 & -4\mu \\ -\mu^2 & -4\mu & 8+\mu^2 \end{pmatrix} = G_2 - 3G_0 - 6G_1 + 16\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is then a simple calculation to check, that $\langle x, G_\lambda x\rangle = 0$ has exactly two solutions λ whenever x is in the interior of E and away from the tangent in the contact point of G_0 and G_1. The figure shows in the plane $x_3 = 1$ the pencil generated by the unit circle G_0 and an ellipse G_1 (both bold) together with the envelope E (blue).



□

Theorem 4.8 (Case 8) *Let G_0, G_2 be two conics with one third order contact point C. Then they generate a unique exponential pencil $G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1}$ which coincides with the linear pencil $g_\lambda = \lambda G_1 + (1-\lambda)G_0$. The pencil G_λ together with the tangent t in C yields a foliation of the projective space outside C. For $\lambda \to \pm\infty$, G_λ converges to t and C respectively.*

Proof After applying a suitable projective map, we may assume that

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu+1 & -\mu \\ 0 & -\mu & \mu-1 \end{pmatrix}, \quad \mu \neq 0$$

(see Halbeisen and Hungerbühler 2017). Then,

$$A := G_0^{-1}G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu+1 & -\mu \\ 0 & \mu & 1-\mu \end{pmatrix}.$$

With

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1/\mu & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:\alpha}$$

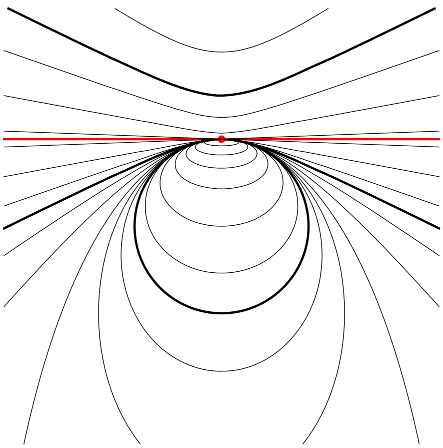
we get $G_0^{-1}G_1 = \mathbb{I} + T\alpha T^{-1}$. Again, we have a unique real logarithm of A and therefore, by the binomic series (which, in this case, consists of only two terms), we get

$$(G_0^{-1}G_1)^x = (\mathbb{I} + T\alpha T^{-1})^x = T\sum_{k=0}^{\infty}\binom{x}{k}\alpha^k T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x\mu & -x\mu \\ 0 & x\mu & 1-x\mu \end{pmatrix},$$

and finally the exponential pencil

$$G_\lambda = G_1\left(G_0^{-1}G_1\right)^{\lambda-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu\lambda+1 & -\mu\lambda \\ 0 & -\mu\lambda & \mu\lambda-1 \end{pmatrix} = g_\lambda.$$

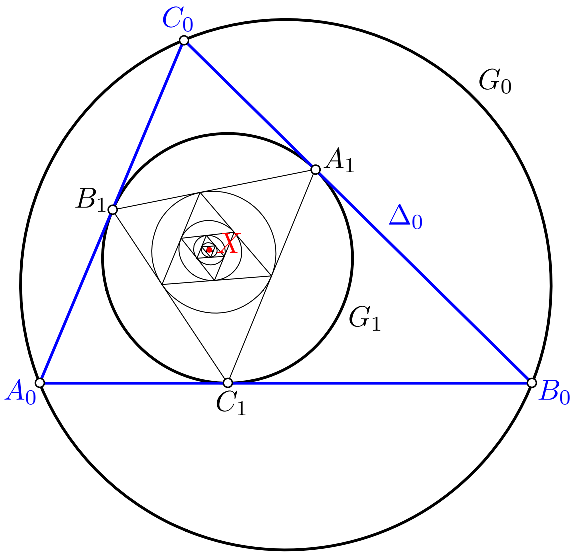
It is easy to check, that for every point $P \notin t$ there is exacly one λ such that $\langle P, G_\lambda P\rangle = 0$ The figure shows in the plane $x_3 = 1$ the pencil generated by the unit circle G_0 and a hyperbola G_1 (both bold) and the limits (red).



5 A triangle center

Starting with the circumcircle G_0 and the incircle G_1 of a triangle $\Delta_0 = A_0B_0C_0$, we obtain a discrete chain of conjugate conics $G_n = G_1(G_0^{-1}G_1)^{n-1}$, for $n = 0, 1, 2, \ldots$. Because of Theorem 2.5, the triangle Δ_1 joining the contact points A_1, B_1, C_1 of the incircle of Δ_0 is tangent to G_2. Iteration of this construction yields a sequence of triangles Δ_n (see figure below) having vertices on G_n and sides tangent to G_{n+1}. The corresponding contact points on G_{n+1} are the vertices of Δ_{n+1}. This is a chain of dual Poncelet triangles in the sense of Halbeisen and Hungerbühler (2016).

According to Theorem 4.2, the linear and the exponental pencil of G_0 and G_1 have the same limit point. Hence, the sequence of triangles Δ_n converges together with the G_n for $n \to \infty$ to the dilation center X of Δ_0: This is Triangle Center $X(3513)$ in the *Encyclopedia of Triangle Centers* [5]. This center has hereby a new interpretation. The figure shows the situation for a triangle Δ_0 (blue) and G_0, G_1 (bold) with the limit point X (red).



Since Δ_0 is a Poncelet triangle for G_0, G_1, any other point A_0' on G_0 defines a triangle Δ_0' with vertices A_0', B_0', C_0' on G_0 with incircle G_1. Each such triangle Δ_0' generates a chain of dual Poncelet triangles with the same center X.

Acknowledgements We would like to thank the referee for his careful reading and the valuable suggestions which greatly helped to improve the article.

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