The American Mathematical Monthly

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/uamm20

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To cite this article: Lorenz Halbeisen, Norbert Hungerbühler & Vanessa Loureiro (2023) The Hidden Twin of Morley's Five Circles Theorem, The American Mathematical Monthly, 130:10, 879-891, DOI: 10.1080/00029890.2023.2254179

To link to this article: https://doi.org/10.1080/00029890.2023.2254179

Published online: 20 Oct 2023.

The Hidden Twin of Morley's Five Circles Theorem

Lorenz Halbeisen, Norbert Hungerbühler, and Vanessa Loureiro

Abstract. We give an algebraic proof of a slightly extended version of Morley's Five Circles Theorem. The theorem holds in all Miquelian Möbius planes obtained from a separable quadratic field extension, in particular in the classical real Möbius plane. Moreover, the calculations bring to light a hidden twin of the Five Circles Theorem that seems to have been overlooked until now.

1. INTRODUCTION. The classical Five Circles Theorem is due to Frank Morley [**8**], [**9**, p. 265]. We quote it here in the version of Tobias Dantzig who provided a proof based on elementary properties of the Euclidean plane in [**3**].

Theorem 1. *If a ring of five circles be formed, the center of each upon a fixed circle and each circle of the ring intersecting the next on this fixed circle, the five other intersections when joined in succession will form a pentacle whose vertices lie one upon each of the five circles (see Figure 1).*

This theorem should not be confounded with similar incidence results like Miquel's Pentagon Theorem [**7**, Théorème III] (see [**6**] for a computer assisted algebraic proof, and the gray box on page 246 in [**10**] for a comment) or the five circle incidence theorem in [**5**]. The aim of this article is to set up a simple algebraic proof of a slightly extended version of Morley's Five Circles Theorem which can be carried out by hand and which works for all Miquelian Möbius planes obtained from a separable quadratic field extension. This shows in particular that the theorem rests on a lesser axiomatic foundation than Euclidean geometry. Moreover, the careful analysis will bring to light a hidden twin of the Five Circles Theorem that seems to have been overlooked until now.

Let us restate Theorem 1 in a more formal way: A circle K carries the five centers $Z_1, \dots, Z_5$ of five circles $K_1, \dots, K_5$. K_{i-2} and K_{i+2} intersect on K in the point P_i (indices read cyclically) and in Q_i. The line l_i passes through Q_{i-2} and Q_{i+2}. Then Theorem 1 claims that the intersection R_i of l_{i-1} and l_{i+1} lies on K_i. Figure 1 illustrates the situation.

It is not necessary, that the centers $Z_1, \dots, Z_5$ sit in cyclic order on K, as Figure 2 suggests.

This looks quite convincing, however, in Figure 3 the theorem seems to fail even though combinatorially the conditions are satisfied. So, why does the theorem work in one case but not in the other? Or more precisely: What is the exact formulation of the conditions so that the vertices R_i of the pentagon lie on the circles K_i?

doi.org/10.1080/00029890.2023.2254179
MSC: 51B10; 05B25; 51E30

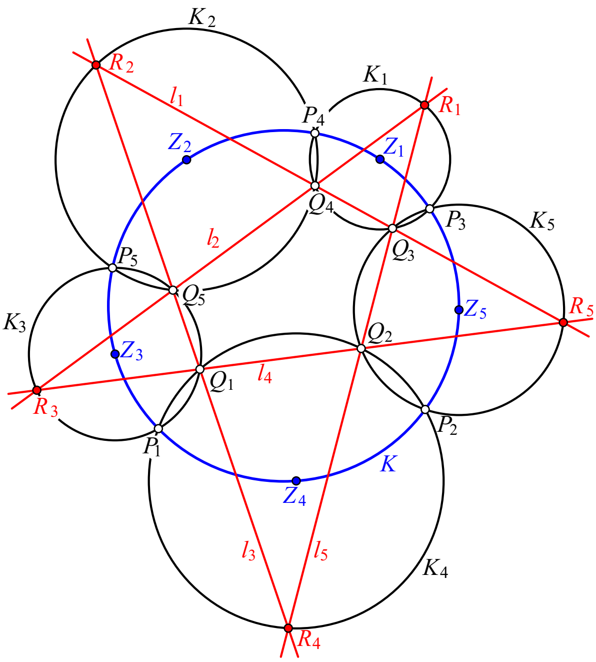

Figure 1. The classical Five Circles Theorem.

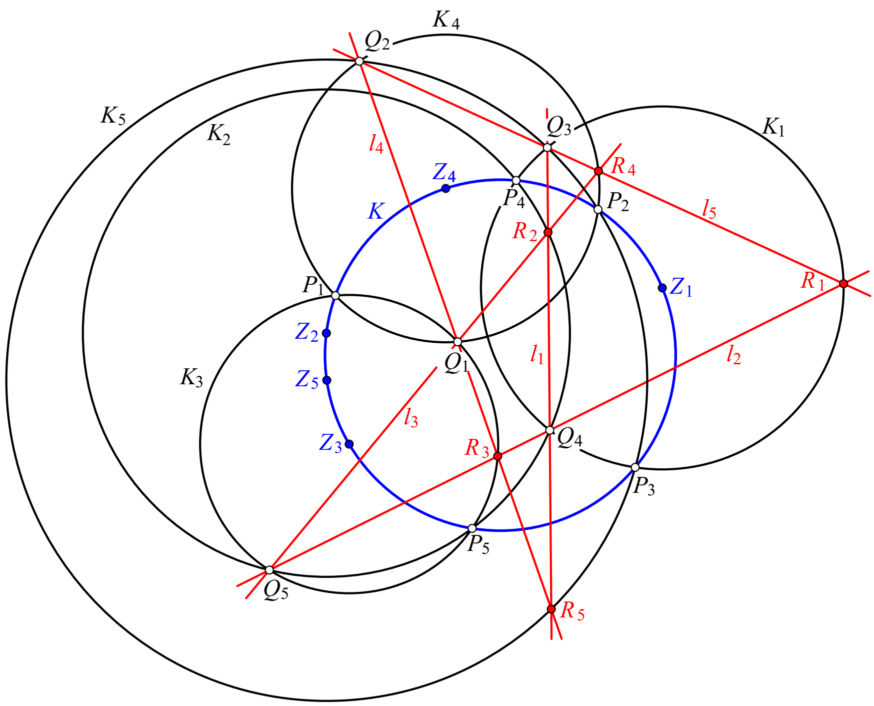

Figure 2. Theorem 1 with centers $Z_1, \ldots, Z_5$ not in cyclic order.

We first analyze this question in Section 2 in the classical model of the Möbius plane. In Section 3 we will generalize the results to Miquelian Möbius planes obtained from a separable quadratic field extension.

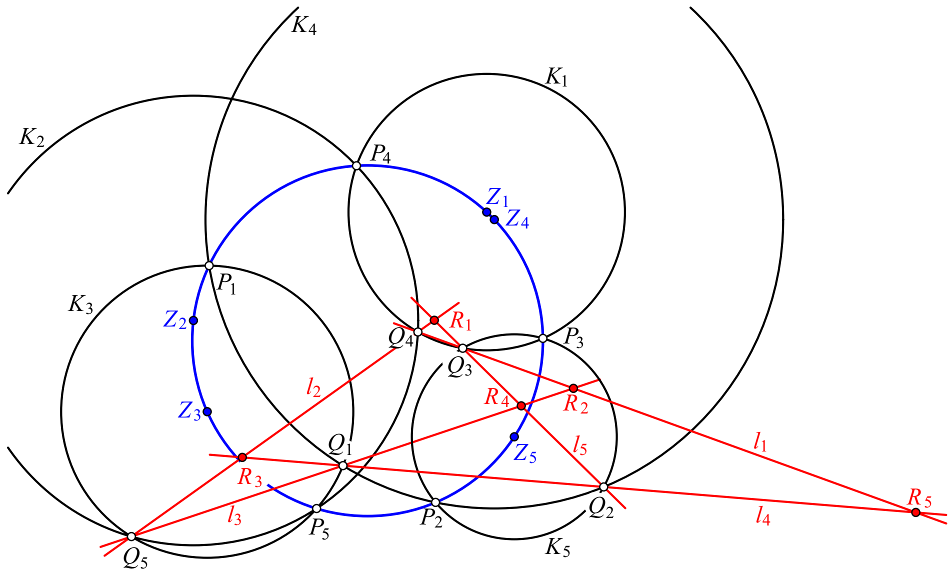

Figure 3. Why does Theorem 1 not hold here? We will see in Theorem 5 that in this case other incidences apply instead.

2. THE FIVE CIRCLES THEOREM IN THE CLASSICAL MÖBIUS PLANE.

The classical model of the Möbius plane is the Riemann sphere, which we interpret conveniently as $\mathbb{C} \cup \{\infty\}$. We use the standard notions of Möbius geometry: Circles (or blocks of the first type) are the complex solutions z of the equation

$$\mathcal{B}^1_{c,r}\ :\ (z-c)(\bar{z}-\bar{c}) = r \tag{1}$$

for $c \in \mathbb{C}$ and $0 < r \in \mathbb{R}$. The center of the circle is c, and $\sqrt{r}$ is its radius. Lines (or blocks of the second type) are the complex solutions z of the equation

$$\mathcal{B}^2_{c,r}\ :\ \bar{c}z + c\bar{z} = r \tag{2}$$

for $c \in \mathbb{C}\backslash\{0\}$ and $r \in \mathbb{R}$, together with ∞. In the sequel we will use $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ for the set of points, and $\mathbb{B}$ for the set of blocks.

Let $K \in \mathbb{B}$ be a circle, and let $Z_1, \ldots, Z_5 \in \mathbb{P}$ be five different points on K. There exists a Möbius transformation $z \mapsto az + b$ that maps the circle K to the unit circle $\mathcal{B}^1_{0,1}$ with the equation $z\bar{z} = 1$. This transformation maps circles to circles and lines to lines. In particular, the image of the center of a circle is the center of the image of the circle. Therefore we may assume without loss of generality that K is the unit circle. Our first goal is to identify every family consisting of five circles $K_1, \ldots, K_5$ with the property that $K_{i-2} \cap K_{i+2} = \{P_i, Q_i\}$ for $P_i \in K$ and $Q_i \in \mathbb{P}$, and such that Z_i is the center of K_i for each $i \in \{1, \ldots, 5\}$. To achieve this, we introduce the anti-Möbius transformation

$$\varphi_Z : \mathbb{P} \to \mathbb{P}, \quad z \mapsto \varphi_Z(z) := Z^2\bar{z},$$

for $Z \in K$. It is easy to see that φ_Z has the following properties:

- If $z \in K$, then $\varphi_Z(z) \in K$.
- φ_Z is an involution.
- $Z, -Z$ and 0 are fixed points of φ_Z.

Hence φ_Z is a reflection with respect to the line containing the points $Z, -Z$ and 0.

Lemma 2. *Let $Z_1, \dots, Z_5 \in K = \mathcal{B}^1_{0,1}$ be five different points. Then there are two quintuples (I) and (II) of circles $K_1, \dots, K_5$ having the following properties: For each $i \in \{1, \dots, 5\}$ the circle K_i has center Z_i, and K_{i-2} and K_{i+2} intersect at $P_i \in K$ and Q_i. The points P_i for the two quintuples are*

$$\text{(I) } P_i = -\frac{Z_i Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}} \qquad \text{and} \qquad \text{(II) } P_i = \frac{Z_i Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}}, \tag{3}$$

respectively.

Proof. Suppose that there exist circles $K_1, \dots, K_5$ such that for each $i \in \{1, \dots, 5\}$ the circle K_i has center Z_i, and K_{i-2} and K_{i+2} intersect at $P_i \in K$. The idea is now to use the map φ_{Z_i} which maps the point P_{i+2} to P_{i-2}. This corresponds to the system of equations

$$P_4 = \varphi_{Z_1}(P_3) = Z_1^2 \overline{P}_3 \tag{4}$$

$$P_5 = \varphi_{Z_2}(P_4) = Z_2^2 \overline{P}_4 = \overline{Z_1^2} Z_2^2 P_3 \tag{5}$$

$$P_1 = \varphi_{Z_3}(P_5) = Z_3^2 \overline{P}_5 = Z_1^2 \overline{Z_2^2} Z_3^2 \overline{P}_3 \tag{6}$$

$$P_2 = \varphi_{Z_4}(P_1) = Z_4^2 \overline{P}_1 = \overline{Z_1^2} Z_2^2 \overline{Z_3^2} Z_4^2 P_3 \tag{7}$$

$$P_3 = \varphi_{Z_5}(P_2) = Z_5^2 \overline{P}_2 = Z_1^2 \overline{Z_2^2} Z_3^2 \overline{Z_4^2} Z_5^2 \overline{P}_3. \tag{8}$$

We multiply (8) by P_3 and obtain

$$P_3^2 = Z_1^2 \overline{Z_2^2} Z_3^2 \overline{Z_4^2} Z_5^2$$

and hence

$$P_3 = Z_1 \overline{Z}_2 Z_3 \overline{Z}_4 Z_5 \quad \text{or} \quad P_3 = -Z_1 \overline{Z}_2 Z_3 \overline{Z}_4 Z_5.$$

Replacing P_3 in (4)–(7) by these expressions we obtain the formulas (3) by using $\overline{Z}_i = 1/Z_i$.

Conversely, it is obvious that these points satisfy $P_i \overline{P}_i = 1$, and hence lie on K. So, let K_i be the circle with center Z_i through P_{i-2}. It remains to verify that P_{i+2} also belongs to K_i. Indeed we have

$$\begin{aligned}(P_{i-2} - Z_i)(\overline{P}_{i-2} - \overline{Z}_i) &= (Z_i^2 \overline{P}_{i+2} - Z_i)(\overline{Z_i^2} P_{i+2} - \overline{Z}_i) \\ &= (Z_i \overline{P}_{i+2} - 1)(\overline{Z}_i P_{i+2} - 1) \\ &= (P_{i+2} - Z_i)(\overline{P}_{i+2} - \overline{Z}_i),\end{aligned}$$

and the claim follows. ■

Now we want to express the points Q_i in terms of the centers Z_i.

Lemma 3. *Let $K_1, \dots, K_5$ be a quintuple of circles with centers $Z_1, \dots, Z_5$ on the circle $K = \mathcal{B}^1_{0,1}$. Suppose that for each $i \in \{1, \dots, 5\}$, $K_{i-2} \cap K_{i+2} = \{P_i, Q_i\}$ with $P_i \in K$. Then we have*

$$Q_i = Z_{i-2} + Z_{i+2} - \overline{P}_i Z_{i-2} Z_{i+2}. \tag{9}$$

Proof. The claim can be checked by showing that the point Q_i given by (9) satisfies the equations of both circles K_{i-2} and K_{i+2}. Indeed, for K_{i-2}, we have

$$(Q_i - Z_{i-2})(\overline{Q}_i - \overline{Z}_{i-2}) = (Z_{i+2} - \overline{P}_i Z_{i-2} Z_{i+2})(\overline{Z}_{i+2} - P_i \overline{Z}_{i-2} \overline{Z}_{i+2})$$

$$= (1 - \overline{P}_i Z_{i-2})(1 - P_i \overline{Z}_{i-2})$$
$$= (P_i - Z_{i-2})(\overline{P}_i - \overline{Z}_{i-2}).$$

A similar calculation shows that Q_i lies on K_{i+2}. ■

With these preparations we are now ready to investigate the incidence relations in the two quintuples (I) and (II) of circles in Lemma 2.

The quintuple (I) and the classical Five Circles Theorem. Let us consider the five circles $K_1, \ldots, K_5$ having centers $Z_1, \ldots, Z_5$ on $K = \mathcal{B}^1_{0,1}$, and $K_{i-2} \cap K_{i+2} = \{Q_i, P_i\}$ with

$$P_i = -\frac{Z_i Z_{i+2} Z_{i-2}}{Z_{i+1} Z_{i-1}}, \qquad Q_i = Z_{i-2} + Z_{i+2} - \overline{P}_i Z_{i-2} Z_{i+2}.$$

Inserting the expression for P_i in the expression for Q_i yields that

$$Q_i = Z_{i-2} + Z_{i+2} + \frac{Z_{i+1} Z_{i-1}}{Z_i}.$$

Let l_i denote the line through the points Q_{i-2}, Q_{i+2}, and ∞. Moreover, we will consider the lines h_i through the points Z_i, P_i, and ∞. These lines are

$$l_i = \{z : (Q_{i+2} - z)(\overline{Q}_{i-2} - \bar{z}) = (\overline{Q}_{i+2} - \bar{z})(Q_{i-2} - z)\} \cup \{\infty\}, \tag{10}$$
$$h_i = \{z : (Z_i - z)(\overline{P}_i - \bar{z}) = (\overline{Z}_i - \bar{z})(P_i - z)\} \cup \{\infty\}. \tag{11}$$

Now we claim that the lines l_{i-1}, l_{i+1}, and h_i meet in the point

$$R_i = Z_i \left(\frac{Z_{i+2}}{Z_{i+1}} + \frac{Z_{i-2}}{Z_{i-1}} + 1 \right). \tag{12}$$

Let us check that R_i belongs to l_{i-1}. If we use the expression (12) for R_i in place of z in (10) we obtain for the bracket factors

$$Q_{i+1} - R_i = \frac{(Z_{i-1} - Z_i)(Z_{i-2} + Z_{i-1})}{Z_{i-1}}$$
$$\overline{Q}_{i+2} - \overline{R}_i = \frac{(Z_i Z_{i+2} - Z_{i+1} Z_{i-1})(Z_{i+1} Z_{i-2} + Z_{i+2} Z_{i-1})}{Z_i Z_{i+1} Z_{i+2} Z_{i-2} Z_{i-1}}$$
$$\overline{Q}_{i+1} - \overline{R}_i = \frac{(Z_i - Z_{i-1})(Z_{i-2} + Z_{i-1})}{Z_i Z_{i-2} Z_{i-1}}$$
$$Q_{i+2} - R_i = \frac{(Z_{i+1} Z_{i-1} - Z_i Z_{i+2})(Z_{i+1} Z_{i-2} + Z_{i+2} Z_{i-1})}{Z_{i+1} Z_{i+2} Z_{i-1}}.$$

Indeed the product of the first two expressions agrees with the product of the last two. Hence $z = R_i$ satisfies the equation of the line l_{i-1}. Similar calculations show that R_i also lies on the lines l_{i+1} and h_i.

In order to prove the original version of the Five Circles Theorem, we need to show that R_i belongs to the circle K_i with center Z_i through the points P_{i+2}, P_{i-2}, Q_{i-2}, Q_{i+2} which is given by the equation

$$(Z_i - z)(\overline{Z}_i - \bar{z}) = (Z_i - P_{i+2})(\overline{Z}_i - \overline{P}_{i+2}). \tag{13}$$

If we use the expression (12) for R_i in place of z in (13) we obtain for the bracket factors

$$\begin{aligned} Z_i - R_i &= -\frac{Z_i\,(Z_{i+1}Z_{i-2} + Z_{i+2}Z_{i-1})}{Z_{i+1}Z_{i-1}} \\ \overline{Z}_i - \overline{R}_i &= -\frac{Z_{i+1}Z_{i-2} + Z_{i+2}Z_{i-1}}{Z_i Z_{i+2} Z_{i-2}} \\ Z_i - P_{i+2} &= \frac{Z_i\,(Z_{i+1}Z_{i-2} + Z_{i+2}Z_{i-1})}{Z_{i+1}Z_{i-2}} \\ \overline{Z}_i - \overline{P}_{i+2} &= \frac{Z_{i+1}Z_{i-2} + Z_{i+2}Z_{i-1}}{Z_i Z_{i+2} Z_{i-1}}. \end{aligned}$$

Indeed the product of the first two expressions agrees with the product of the last two. Hence $R_i \in K_i$ as claimed.

Observe that we actually proved a slightly enhanced version of the Five Circles Theorem since we showed that the lines h_i also pass through the points R_i (see Figure 4). In fact, there is yet another incidence to be discovered in this configuration: Let us reflect the point Z_i in the perpendicular bisector of Z_{i-1} and Z_{i+1}. The mirrored point is $C_i = \frac{Z_{i+1}Z_{i-1}}{Z_i} \in K$. Using the expressions we found for the points P_i, Q_i, and R_i it is then easy to verify that

$$\begin{aligned} (C_i - P_{i-1})(\overline{C}_i - \overline{P}_{i-1}) &= (C_i - P_{i+1})(\overline{C}_i - \overline{P}_{i+1}) \\ &= (C_i - R_{i-1})(\overline{C}_i - \overline{R}_{i-1}) \\ &= (C_i - R_{i+1})(\overline{C}_i - \overline{R}_{i+1}) \\ &= (C_i - Q_i)(\overline{C}_i - \overline{Q}_i) = \frac{(Z_{i-2} + Z_{i+2})^2}{Z_{i-2}Z_{i+2}}. \end{aligned}$$

Hence the five points P_{i-1}, P_{i+1}, R_{i-1}, R_{i+1}, and Q_i lie on a circle with center C_i and radius $|Z_{i-2} + Z_{i+2}|$.

Notice that if

$$Z_{i-2}Z_{i+1} = -Z_{i+2}Z_{i-1} \tag{14}$$

then

$$P_{i-2} = P_{i+2} = Q_{i-2} = Q_{i+2} = R_i = Z_i,$$

which means that the circle K_i degenerates to a point. Vice versa, $P_{i-2} = P_{i+2} = Z_i$ implies (14).

Before we formulate our results as a theorem, we turn our attention to the second quintuple of circles which we identified in Lemma 2. It will turn out that these circles carry a twin of the original Five Circles Theorem.

The quintuple (II) and the Twin of the Five Circles Theorem. Let us consider the five circles $K_1, \ldots, K_5$ having centers $Z_1, \ldots, Z_5$ on $K = \mathcal{B}^1_{0,1}$, and $K_{i-2} \cap K_{i+2} = \{P_i, Q_i\}$ with

$$P_i = \frac{Z_i Z_{i+2} Z_{i-2}}{Z_{i+1}Z_{i-1}}, \quad Q_i = Z_{i-2} + Z_{i+2} - \overline{P}_i Z_{i-2} Z_{i+2} = Z_{i-2} + Z_{i+2} - \frac{Z_{i+1}Z_{i-1}}{Z_i}.$$

Let, as before, l_i denote the line through Q_{i-2}, Q_{i+2}, ∞ given by (10), and h_i the line through Z_i, P_i, ∞ given by (11). Then the points

$$S_{i,i-1} = Z_i \left(\frac{Z_{i-2}}{Z_{i-1}} - \frac{Z_{i+2}}{Z_{i+1}} + 1 \right) \quad \text{and} \quad S_{i,i+1} = Z_i \left(\frac{Z_{i+2}}{Z_{i+1}} - \frac{Z_{i-2}}{Z_{i-1}} + 1 \right)$$

are the intersections of h_i with l_{i-1} and l_{i+1}, respectively, which can easily be checked by inserting these expressions in (10) and (11). We now claim that $S_{i,i-1}$ and $S_{i,i+1}$ are points on K_i. Indeed, if we insert $S_{i,i-1}$ for z in (13) we obtain for the bracket factors

$$\begin{aligned} Z_i - S_{i,i-1} &= \frac{Z_i\,(Z_{i+1}Z_{i-2} - Z_{i+2}Z_{i-1})}{Z_{i+1}Z_{i-1}} \\ \overline{Z}_i - \overline{S}_{i,i-1} &= -\frac{Z_{i+1}Z_{i-2} - Z_{i+2}Z_{i-1}}{Z_i Z_{i+2} Z_{i-2}} \\ Z_i - P_{i+2} &= \frac{Z_i\,(Z_{i+1}Z_{i-2} - Z_{i+2}Z_{i-1})}{Z_{i+1}Z_{i-2}} \\ \overline{Z}_i - \overline{P}_{i+2} &= -\frac{Z_{i+1}Z_{i-2} - Z_{i+2}Z_{i-1}}{Z_i Z_{i+2} Z_{i-1}} \end{aligned}$$

and we see that the product of the first two and the product of the last two expressions agree. A similar calculation shows that $S_{i,i+1}$ also lies on K_i. Notice that in the classical Five Circles Theorem carried by the quintuple (I) the intersection of h_i and l_{i-1} agrees with the intersection of h_i and l_{i+1}. For the quintuple (II) of circles this is no longer the case. Indeed, we have:

Lemma 4. *For the quintuple (II) there holds $S_{i,i-1} \neq S_{i,i+1}$ for all $i \in \{1, \ldots, 5\}$, unless K_i degenerates to a point.*

Proof. Assume by contradiction that $S_{i,i-1} = S_{i,i+1}$, i.e.,

$$S_{i,i-1} - S_{i,i+1} = 2Z_i \left(\frac{Z_{i-2}}{Z_{i-1}} - \frac{Z_{i+2}}{Z_{i+1}} \right) = 0.$$

This is equivalent to

$$Z_i = \frac{Z_i Z_{i+2} Z_{i-1}}{Z_{i+1} Z_{i-2}} = P_{i+2},$$

where we have used (3) for the last equality. But this would mean that K_i degenerates to a point. ■

Let us again consider the points $C_i = \frac{Z_{i+1}Z_{i-1}}{Z_i}$ which we obtain by reflecting Z_i in the perpendicular bisector of Z_{i-1} and Z_{i+1}. Then, using as usual that $\overline{Z}_i = 1/Z_i$, it is easy to check that

$$\begin{aligned} (C_i - P_{i-1})(\overline{C}_i - \overline{P}_{i-1}) &= (C_i - P_{i+1})(\overline{C}_i - \overline{P}_{i+1}) \\ &= (C_i - S_{i+1,i})(\overline{C}_i - \overline{S}_{i+1i}) \\ &= (C_i - S_{i-1,i})(\overline{C}_i - \overline{S}_{i-1,i}) = 2 - \frac{Z_{i-2}}{Z_{i+2}} - \frac{Z_{i+2}}{Z_{i-2}} \neq 0, \end{aligned}$$

since $Z_{i-2} \neq Z_{i+2}$. Thus the four points P_{i-1}, P_{i+1}, $S_{i+1,i}$, and $S_{i-1,i}$ lie on a circle with center $C_i \in K$.

Similarly, for the points $D_i = -C_i$ we have

$$\begin{aligned}(D_i - P_{i-1})(\overline{D}_i - \overline{P}_{i-1}) &= (D_i - P_{i+1})(\overline{D}_i - \overline{P}_{i+1}) \\ &= (D_i - S_{i-1,i-2})(\overline{D}_i - \overline{S}_{i-1,i-2}) \\ &= (D_i - S_{i+1,i+2})(\overline{D}_i - \overline{S}_{i+1,i+2}) \\ &= (D_i - Q_i)(\overline{D}_i - \overline{Q}_i) = \frac{(Z_{i-2} + Z_{i+2})^2}{Z_{i-2}Z_{i+2}}.\end{aligned}$$

Hence the five points P_{i-1}, P_{i+1}, $S_{i-1,i-2}$, $S_{i+1,i+2}$, and Q_i lie on a circle with center $D_i \in K$ with radius $|Z_{i-2} + Z_{i+2}|$. It follows that the line through the points C_i, D_i, ∞ is the perpendicular bisector of the points P_{i-1}, P_{i+1}.

Notice that if

$$Z_{i-2}Z_{i+1} = Z_{i+2}Z_{i-1} \tag{15}$$

then

$$P_{i+2} = P_{i-2} = Q_{i-2} = Q_{i+2} = S_{i,i-1} = S_{i,i+1} = Z_i,$$

which means that the circle K_i degenerates to a point. Vice versa, $P_{i+2} = P_{i-2} = Z_i$ implies (15).

We can now combine our findings and formulate the following theorem which contains the classical Five Circles Theorem and its twin. Since our calculations carry over to Miquelian Möbius planes obtained from a separable quadratic field extension, the theorem is valid in this more general framework (see Section 3).

Theorem 5. *Let $Z_1, \ldots, Z_5$ be five different points on the circle K given by the equation $z\bar{z} = 1$, and let $C_i = \frac{Z_{i-1}Z_{i+1}}{Z_i} \in K$. Then there are two families of five circles $K_1, \ldots, K_5$, where Z_i is the center of K_i and such that K_{i-2} and K_{i+2} intersect at $P_i \in K$ and at Q_i for each $i \in \{1, \ldots, 5\}$. Let l_i denote the line through Q_{i-2}, Q_{i+2}, ∞, and h_i the line through Z_i, P_i, ∞. Then,*

- *in one family the three lines h_i, l_{i+1}, l_{i-1}, meet in a point $R_i \in K_i$ and the five points P_{i-1}, P_{i+1}, R_{i-1}, R_{i+1}, and Q_i lie on a circle with center C_i (see Figure 4).*
- *In the other family the lines h_i and l_{i-1} meet in $S_{i,i-1} \in K_i$ and the lines h_i and l_{i+1} meet in $S_{i,i+1} \in K_i$. Moreover, the four points P_{i-1}, P_{i+1}, $S_{i+1,i}$, and $S_{i-1,i}$ lie on a circle with center C_i and the five points P_{i-1}, P_{i+1}, $S_{i-1,i-2}$, $S_{i+1,i+2}$, and Q_i lie on a circle with center $D_i = -C_i$. (see Figure 5).*

Recall that the point C_i is geometrically obtained by reflecting Z_i in the perpendicular bisector of Z_{i-1} and Z_{i+1}, and D_i is the antipode of C_i on K.

3. GENERALIZATION TO MIQUELIAN MÖBIUS PLANES. A Möbius plane is an incidence structure consisting of points $\mathbb{P}$ and blocks $\mathbb{B}$ which satisfies the following axioms (see, e.g., [**4**, Chapter 6] or [**1**]):

(M1) For any three points P, Q, R, $P \neq Q$, $P \neq R$ and $Q \neq R$, there exists a unique block C with $P \in C$, $Q \in C$ and $R \in C$.

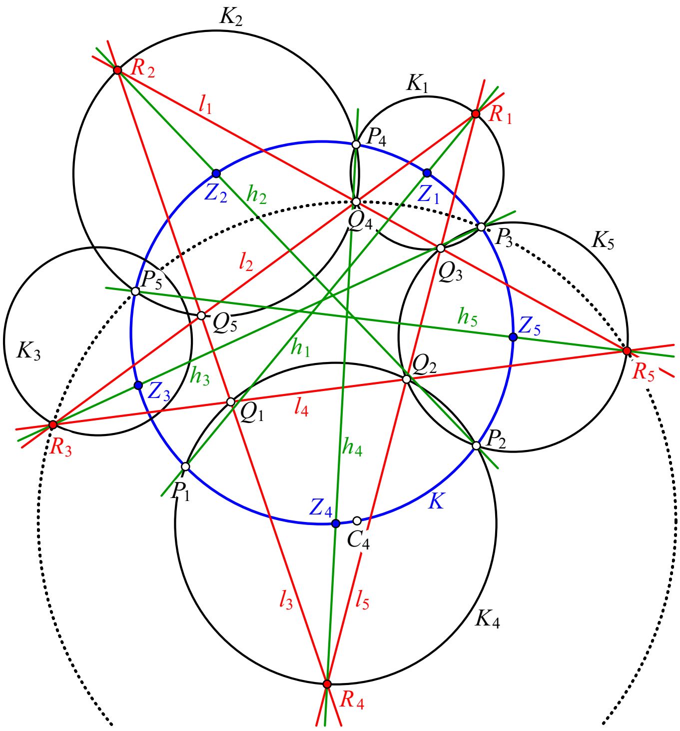

Figure 4. The enhanced Five Circles Theorem. In order not to overload the figure, from the five additional circles only the dotted one with center C_4 is drawn.

(M2) For any block C, and points P, Q with $P \in C$ and $Q \notin C$, there exists a unique block D such that $P \in D$ and $Q \in D$, but for all points R with $R \in C$, $P \neq R$, we have $R \notin D$.

(M3) There are four points P_1, P_2, P_3, P_4 such that for all blocks C, we have $P_i \notin C$ for at least one $i \in \{1, 2, 3, 4\}$. Moreover, for all blocks C there exists a point P with $P \in C$.

The blocks generalize the lines and circles of the classical Möbius plane. Note, however, that the term "center of a circle" does not appear in the axioms.

A Möbius plane is called Miquelian if in addition the Six Circles Theorem of Miquel [7, Théorème I] holds:

Theorem 6 (Miquel). *If one can assign 8 points $P_1, \ldots, P_8$ to the corners of a cube in such a way that the points assigned to five of its faces each lie on a circle, then this is also the case for the points assigned to the 6th face (see Figure 6).*

A famous result by Chen [2] states that a Miquelian Möbius plane is isomorphic to a Möbius plane $\mathfrak{M}(K, q)$ over a field K where $q(z) = z^2 + a_0 z + b_0$ is an irreducible polynomial with $a_0, b_0 \in K$. Here, the set of points in $\mathfrak{M}(K, q)$ is

$$\mathbb{P} := K^2 \cup \{\infty\},$$

where $\infty \notin K$, and the set of blocks $\mathbb{B}$ consists of

- lines, i.e., the sets of solutions (x_1, x_2) of the equations $ux_1 + vx_2 + w = 0$ for $u, v, w \in K$, $(u, v) \neq (0, 0)$, and the element ∞, and

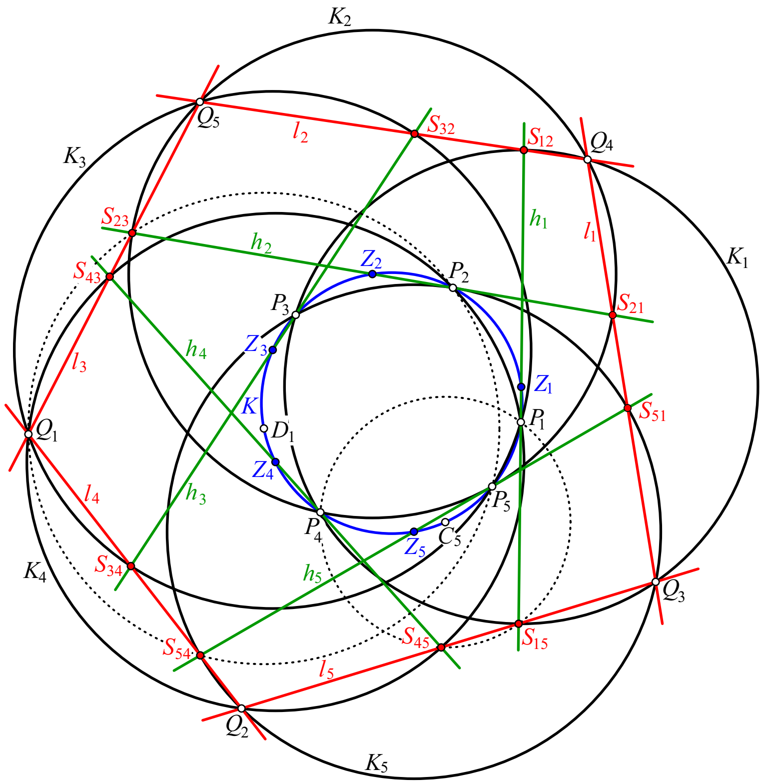

Figure 5. The twin of the Five Circles Theorem. In order not to overload the figure, from the ten additional circles only the dotted ones with the centers C_5 and D_1 are drawn.

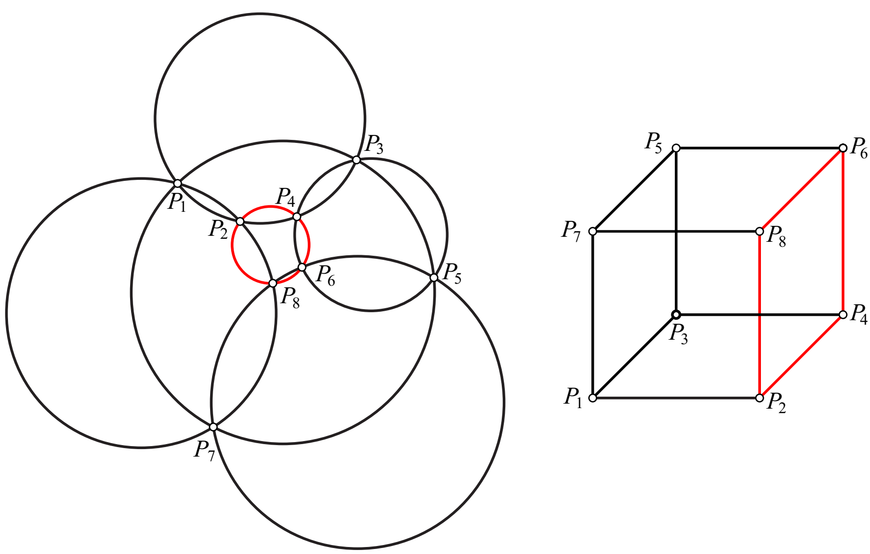

Figure 6. The Six Circles Theorem of Miquel.

- circles, i.e., the sets of solutions (x_1, x_2) of the equations $x_1^2 + a_0x_1x_2 + b_0x_2^2 + ux_1 + vx_2 + w = 0$ for $u, v, w \in K$, if this set of solutions consists of more than one point.

A point is incident with a block, if it is an element of the block. Let E be the splitting field of q. Hence there are $\alpha_1, \alpha_2 \in E$ such that $q(z) = (z + \alpha_1)(z + \alpha_2)$, and E is a two dimensional vector space over K with basis $\{1, \alpha_1\}$ or $\{1, \alpha_2\}$. Since every point $(x_1, x_2) \in K^2$ can be represented by $z = x_1 + \alpha_1 x_2$ or $z = x_1 + \alpha_2 x_2$, we can identify K^2 with E. If q is separable, i.e., $\alpha_1 \neq \alpha_2$, then the mapping

$$\bar{} : E \to E, \quad z = x_1 + \alpha_1 x_2 \mapsto \bar{z} = x_1 + \alpha_2 x_2 = x_1 + a_0 x_2 - \alpha_1 x_2$$

is an involutorial automorphism of E (observe that $\alpha_1 + \alpha_2 = a_0$). Hence we have

$$x_1 = \frac{\alpha_1 \bar{z} - \alpha_2 z}{\alpha_1 - \alpha_2}, \quad x_2 = \frac{z - \bar{z}}{\alpha_1 - \alpha_2},$$

and the equation of a line $ux_1 + vx_2 + w = 0$ can be written in the form $\bar{c}z + c\bar{z} = r$ with $c \in E \setminus \{0\}$ and $r \in K$. Similarly, the equation of a circle $x_1^2 + a_0 x_1 x_2 + b_0 x_2^2 + ux_1 + vx_2 + w = 0$ can be written as a quadratic equation of the form $(z - c)(\bar{z} - \bar{c}) = r$ for $r \in K \setminus \{0\}$, and $c \in E$; use $x_1^2 + a_0 x_1 x_2 + b_0 x_2^2 = z\bar{z}$ for $z = x_1 + \alpha_1 x_2$. Hence in this case the center c can be assigned to the circle. For $K = \mathbb{R}$ and $q(z) = z^2 + 1$ we have $E = \mathbb{C}$ and we are in the situation of the classical model of the Möbius plane as described in the previous section. Another example is the Galois field $K = GF(t)$ for an odd prime power $t = p^n$, and $q(z) = z^2 - \alpha$ for a non-square $\alpha \in GF(t)$. Then, $GF(t)(\alpha) \cong GF(t^2)$ and the conjugation is given by the Frobenius automorphism $z \mapsto \bar{z} = z^t$.

If q is separable, the proofs of the previous section carry over verbatim to the Möbius plane $\mathfrak{M}(K, q)$. Notice also, that every finite extension of a finite field is separable. Hence Theorem 5 is valid in each Miquelian Möbius plane $\mathfrak{M}(K, q)$ if q is separable, and in particular in every finite Miquelian Möbius plane. Notice however that by (3), over a field K of characteristic 2, the two families of five circles in Theorem 5 coincide: The twin is identical to the classical statement in this case.

4. SOME CLOSING REMARKS. The reader may notice that there are other incidences hidden in the Five Circles configuration:

- Each one of the quadruples $Z_i, P_{i+1}, Q_{i+2}, \infty$ and $Z_i, P_{i-1}, Q_{i-2}, \infty$ are contained in a block.
- The line l_i and the line l_i' through the points Z_{i-1}, Z_{i+1}, ∞ are touching at ∞ (i.e., the lines are parallel).

However, both observations are general properties of Miquelian Möbius planes obtained from a separable quadratic field extension and not limited to the Five Circles configuration as we show in the following two propositions.

Proposition 7. *Let K be a circle, $P, X, Y \in K$, K_X the circle with center X through P, and K_Y the circle with center Y through P. Let $Q \neq P$ be the second intersection of K_X and K_Y, and $P_X \neq P$ and $P_Y \neq P$ the second intersections of K with K_X and K_Y, respectively. Then the points X, Q, P_Y, ∞ and the points Y, Q, P_X, ∞ each lie on a line (see Figure 7, left).*

Proof. We may again assume that K is the unit circle with the equation $z\bar{z} = 1$. Using the maps φ_X and φ_Y, we find $P_X = X^2\overline{P}$ and $P_Y = Y^2\overline{P}$. It is then elementary to check that

$$Q' = X + Y - XY\overline{P}$$

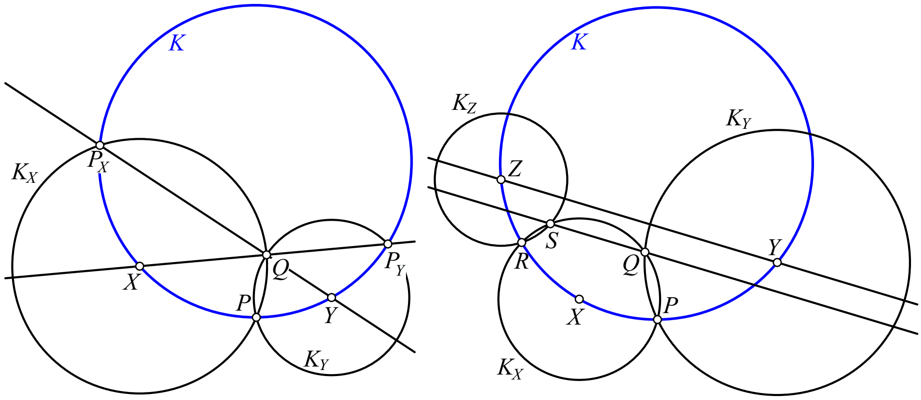

Figure 7. Two general incidence relations in Miquelian Möbius planes.

is the intersection of the lines through X, P_Y, ∞ and Y, P_X, ∞. If we use that $\overline{X} = 1/X$, $\overline{Y} = 1/Y$, and $\overline{P} = 1/P$, a short calculation shows that $z = Q'$ solves the equations

$$(X - z)(\overline{X} - \overline{z}) = (X - P)(\overline{X} - \overline{P})$$
$$(Y - z)(\overline{Y} - \overline{z}) = (Y - P)(\overline{Y} - \overline{P})$$

of K_X and K_Y, respectively. Hence it follows that $Q' = Q$. ■

Proposition 8. *Let K be a circle, $X, Y, Z \in K$, K_X a circle with center X, K_Y a circle with center Y, and K_Z a circle with center Z, such that $K_X \cap K_Y = \{P, Q\}$ with $P \in K$ and $K_X \cap K_Z = \{R, S\}$ with $R \in K$. Then the lines through Y, Z, ∞ and through Q, S, ∞ are touching at ∞ (see Figure 7, right).*

Proof. We assume again that K is the unit circle $z\overline{z} = 1$. By the map φ_X we find that $R = X^2\overline{P}$. Then we infer from the proof of Proposition 7 that $Q = X + Y - XY\overline{P}$ and $S = X + Z - \overline{X}ZP$. Thus the lines through R, S, ∞ and through Y, Z, ∞ are given by the equations

$$(Q - z)(\overline{S} - \overline{z}) = (\overline{Q} - \overline{z})(S - z),$$
$$(Y - z)(\overline{Z} - \overline{z}) = (\overline{Y} - \overline{z})(Z - z).$$

It is now easy to check that $(Q - S)(\overline{Z} - \overline{Y}) - (\overline{Q} - \overline{S})(Z - Y) = 0$, and hence the two lines are indeed touching at ∞. ■

ACKNOWLEDGMENT. The authors thank the anonymous referees and the editors of the *American Mathematical Monthly* for their valuable comments and suggestions which helped to improve this article.

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On Erdős's Proof of the Existence of Cages

We consider the problem of the existence of an r-regular graph with girth (length of shortest cycle) g; such a graph of minimum order is an *(r, g)-cage*. For example, the Petersen graph is the unique $(3, 5)$-cage; see [2] for more details. Proofs of the existence of cages have been given by Sachs [3] and Erdős, in [1]. Here we give a simplification of Erdős's argument with a slightly weaker bound.

Theorem. *For all integers $r \geq 2$ and $g \geq 3$, there is an r-regular graph with girth g and order*

$$n = 4 + 2(r-1) + 2(r-1)^2 + \cdots + 2(r-1)^{g-2} + (r-1)^{g-1}.$$

Proof. The disjoint union of a g-cycle and a $(g+1)$-cycle satisfies the theorem for $r = 2$, so assume that $r \geq 3$. Choose G to be a graph of order n with maximum degree at most r, girth g, and the maximum number of edges subject to these constraints. Such a graph must exist because the disjoint union of a g-cycle with $n - g$ isolated vertices has maximum degree $2 \leq r$ and girth g.

Call a vertex v of G *deficient* if $\deg_G v < r$. We may assume that G has a deficient vertex, as otherwise we are done. Let u and v be deficient vertices, and choose them to be distinct if possible (otherwise we allow $u = v$). Since u and v are deficient, for each nonnegative integer i, there are at most $(r-1)^i$ vertices at distance i from each of them. Our choice of G ensures that it has enough vertices that we can find some vertex x with $d_G(u, x) \geq g - 1$ and $d_G(v, x) \geq g$.

If x is itself deficient, then the graph $G + ux$ has maximum degree at most r, girth g (because $d_G(u, x) \geq g - 1$), and one more edge than G, a contradiction to our choice of G. Thus we may assume that x has degree $r \geq 3$. This implies that G has some edge xy whose removal does not destroy all of its cycles of length g, so the girth of $G - xy$ is still g.

Because $d_G(v, x) \geq g$, we have $d_G(v, y) \geq g - 1$. This implies that the graph $G - xy + ux + vy$ has girth g and one more edge than G. If the maximum degree of $G - xy + ux + vy$ is at most r, this contradicts our choice of G. The only way this could fail is if $u = v$ and $\deg_G u = r - 1$. In that case, the other $n - 1$ vertices are not deficient (we chose $v \neq u$ if it was possible). Thus the sum of the degrees of all the vertices of G—which is always twice the number of edges—is $nr - 1$. However, n is even when r is odd, so $nr - 1$ is always odd, and thus this situation cannot occur. ■

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—Submitted by Vincent Vatter, University of Florida

doi.org/10.1080/00029890.2023.2258742

MSC: Primary 05C35

This article has been corrected with minor changes. These changes do not impact the academic content of the article.