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# Remarks on Bases of Banach Spaces

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**Abstract**

We consider Hamel bases and $\omega$-bases of Banach spaces. An $\omega$-base is essentially the same as a Galerkin base, but it is allowed to be uncountable. Like Hamel bases, all $\omega$-bases of a Banach space share the same cardinality. First we show that in a infinite dimensional Banach space, every Hamel base has the cardinality of the Banach space, which is at least the cardinality of the continuum. For $\omega$-bases, this is not necessarily the case, even when they are uncountable. For this, we give an example of a Banach space which contains an uncountable $\omega$-base of cardinality less than the continuum. The construction of an $\omega$-base in a given Banach space is not straightforward, and it is not known, whether every Banach space has an $\omega$-base. Nevertheless, there are natural examples of Banach spaces which have an uncountable $\omega$-base.

# 1 Introduction

With the aid of the axiom of choice, one can prove that every vector space has a Hamel base (cf. [Ha] and [Hd, p. 295]). Furthermore, the axiom of choice is necessary for the existence of Hamel bases of vector spaces (cf. [La]). The proof of the existence of a Hamel base is not constructive, but since the axiom of choice is consistent with the other axioms of set theory (cf. [Gö]), it is consistent to assume the existence of a Hamel base in every vector space. If the continuum hypothesis (see Section 2) holds—which is by [Gö] consistent to assume—we get, as a consequence of Baire's Category Theorem, that every Hamel base of an infinite dimensional Banach space has at least the cardinality of the continuum. But it is also consistent to assume that the continuum hypothesis fails (cf. [Co]), and in this case, Baire's Category Theorem only implies that a Hamel base of an infinite dimensional

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Banach space must be uncountable. The continuum hypothesis can also be charachterized by Hamel bases; namely, if one considers $\mathbb{R}$ as a Banach space over $\mathbb{Q}$, then the continuum hypothesis is equivalent to the statement, that $\mathbb{R}$ can be covered by countably many Hamel bases (see [Si] and [EK]). These examples show, that properties of vector spaces are closely related to the axioms of set theory and conversely, that set theory has implications to functional analysis on a very fundamental level. In this article we investigate with set theoretical notions, like independent families and trees, bases in complete topological vector spaces (for different notions of bases in topological vector spaces, see [Kh]).

## 2 Some set theory

In this section, we summarize some set theoretical notations and definitions. All the notations and definitions are standard and are in accordance with [Je] or [Ku].

A set $x$ is *transitive* if every element of $x$ is a subset of $x$. A relation $R$ *well-orders* a set $x$, or $\langle R, x\rangle$ is a *well-ordering*, if $\langle R, x\rangle$ is a total ordering and every non-empty subset of $x$ has an $R$-least element. The axiom of choice is equivalent to the statement, that every set can be well-ordered. A set $x$ is an *ordinal*, if $x$ is transitive and well-ordered by $\in$. The axiom of choice is also equivalent to the statement, that for every set $x$ there exists an ordinal $\alpha$ and a bijection $f : \alpha \to x$. The class of all ordinals is transitive and well-ordered by $\in$. The set of all natural numbers is equal to the set of all finite ordinals and is denoted by $\omega$. (A natural number $n$ is the set of all natural numbers which are smaller than $n$, e.g. $0 = \emptyset$.)

For a set $x$, the *cardinality* of $x$, denoted by $|x|$ is the least ordinal $\alpha$ for which there exists a bijection $f : \alpha \to x$. A set $x$ is called finite, if $|x| \in \omega$, otherwise it is called infinite. Further it is called countable, if $|x| \leq |\omega| =: \aleph_0$. For a set $x$, $\mathcal{P}(x)$ denotes the power set of $x$. There exists a bijection between $\mathbb{R}$ and $\mathcal{P}(\omega)$, hence $|\mathbb{R}| = |\mathcal{P}(\omega)|$, and we denote this cardinality by $\mathfrak{c}$. The continuum hypothesis states $\mathfrak{c} = |\omega_1| =: \aleph_1$, where $\omega_1$ denotes the least ordinal which is not countable. Finaly let $[x]^\omega := \{y \in \mathcal{P}(x) : |y| = \aleph_0\}$ and $[x]^{<\omega} := \{y \in \mathcal{P}(x) : |y| < \aleph_0\}$. If $x$ is infinite, then $|[x]^{<\omega}| = |x|$. We use the same symbol for a set $y \in \mathcal{P}(x)$ and for its characteristic function, i.e. we write $y(z) = 1$ if $z \in y$ and $y(z) = 0$ otherwise.

# 3 The cardinality of Hamel bases of Banach spaces

By the axiom of choice, every vector space $E$ over a field $K$ possesses an algebraic base, i.e. a Hamel base. A Hamel base is hence a set $H$ of vectors such that

(i) $H$ spans $E$, i.e. $E = \langle H \rangle$ (which denotes the set of all finite $K$-linear combinations of vectors of $H$) and

(ii) $H$ is finitely linearly independent over $K$, i.e. finitely many vectors in $H$ are linearly independent over $K$.

This is equivalent to say, that $H$ is a minimal set with property (i) or that $H$ is a maximal set with property (ii).

By Baire's Category Theorem it is easy to see that a Hamel base in an infinite dimensional real or complex Banach space $E$ cannot be countable. In this section, we will show that if $E$ is an infinite dimensional Banach space over a field $K$, where $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$, then every Hamel base of $E$ has the cardinality $|E|$, which is at least the cardinality of the continuum. We start by recalling the well-known

**FACT 3.1** *If $E$ is an infinite dimensional vector space over a field $K$ and $H_1$ and $H_2$ are two Hamel bases of $E$, then $|H_1| = |H_2|$.*

**PROOF:** For every $g \in H_1$ there exist $h_0, \ldots, h_n \in H_2$ and $s_0, \ldots, s_n \in K \setminus \{0\}$ with $g = \sum_{i=0}^{n} s_i h_i$. This defines a map $\Phi : H_1 \to [H_2]^{<\omega}, g \mapsto \{h_0, \ldots, h_n\}$. Now, every vector $v \in E$ can be written as a finite linear combination of vectors in $H_1$ and hence as a finite linear combination of vectors in $H := \bigcup \Phi(H_1) \subset H_2$. Since $H_2$ is minimal spanning, it follows that $H = H_2$. On the other hand, $|H| \leq |H_1|$ by construction. The reverse inequality follows by exchanging the rôle of $H_1$ and $H_2$. $\dashv$

The next lemma summarizes a few simple facts which will be useful later.

**LEMMA 3.2** *(a) If $E$ is at the same time a vector space over a field $K_1$ and over a field $K_2 \subseteq K_1$ and if $H_i$ is a Hamel base of $E$ with respect to $K_i$ ($i = 1, 2$), then $|H_1| \leq |H_2|$.*

*(b) If $E$ is a Banach space over a field $K$ containing $\mathbb{N}$, then $|E| \geq \mathfrak{c}$.*

*(c) If $E$ is a vector space over an infinite field $K$ and if $H$ is a Hamel base of $E$, then $|E| = \max\{|K|, |H|\}$.*

*(d) If $K_1$ and $K_2$ are two fields with $\mathbb{N} \subseteq K_1 \subseteq K_2 \subseteq \mathbb{C}$ such that $K_1$ is dense in $K_2$, and if $E$ is a Banach space over $K_1$, then there exists a $K_1$-linear homeomorphism from $E$ to a Banach space $E' = E$ over $K_2$.*

**Proof:** (a) The vectors in $H_1$ are finitely linearly independent with respect to $K_1$, hence also with respect to $K_2$. Thus $H_1$ can be extended to a Hamel base $H_1' \supseteq H_1$ with respect to $K_2$. Hence, using Fact 3.1, we conclude $|H_2| = |H_1'| \geq |H_1|$.

(b) If $|K| = \mathfrak{c}$, we have a natural bijection between $K$ and a one dimensional subspace $\{kX : k \in K\}$, $X \in E \setminus \{0\}$, hence $|E| \geq |K| \geq \mathfrak{c}$.

If $|K| < \mathfrak{c}$, then $I := \mathbb{R} \setminus K$ has cardinality $\mathfrak{c}$. We choose $X \in E \setminus \{0\}$. Then, for each $r \in I$ there exist $q_0, q_1, \ldots \in \mathbb{Q} \subseteq K$ with $\lim_{n\to\infty} q_n = r$, and $\{q_n X\}_{n\in\omega}$ is a Cauchy sequence in $E$ with $\lim_{n\to\infty} q_n X =: \psi(r) \in E$, independent of the choice of the sequence $\{q_n\}_{n\in\omega}$. Notice that

$$\psi : I \to E, \quad r \mapsto \psi(r)$$

is injective, and hence $|E| \geq |I| \geq \mathfrak{c}$.

(c) For each $Y \in E$ there exist finitely many uniquely determined $X_{\iota_0}, \ldots, X_{\iota_{n(Y)}} \in H$, $\iota_i < \iota_{i+1}$, and $s_0, \ldots, s_{n(Y)} \in K \setminus \{0\}$ such that $Y = \sum_{k=0}^{n(Y)} s_k X_{\iota_k}$, and the function

$$\begin{aligned} \varphi : E &\to [K]^{<\omega} \times [H]^{<\omega} \\ Y &\mapsto \langle \langle s_0, \ldots, s_{n(Y)} \rangle, \langle X_{\iota_0}, \ldots, X_{\iota_{n(Y)}} \rangle \rangle \end{aligned}$$

is a bijection. Because $K$ is an infinite set, we have $|[K]^{<\omega}| = |K|$. If $H$ is infinite, then $|[H]^{<\omega}| = |H|$, and hence we get $|E| = |[K]^{<\omega} \times [H]^{<\omega}| = \max\{|K|, |H|\}$. If $H$ is finite, then $|[H]^{<\omega}| \leq |K|$, and the formula remains true.

(d) We define on the set $E' := E$, equipped with the same additive group and the same norm as on $E$, the following multiplication

$$(K_2, E) \to E, \quad (\lambda, x) \mapsto \lambda x := \lim_{K_1 \ni \lambda_i \to \lambda} \lambda_i x.$$

It is easy to check, that $(K_2, E)$ is a normed vector space and that $\psi : (K_1, E) \to (K_2, E), x \mapsto x$, is a $K_1$-linear homeomorphism. $\dashv$

REMARK: The Banach spaces in Lemma 3.2(d) need not have the same dimension. A drastic example is, that $\mathbb{R}$ is an infinite dimensional Banach space over $\mathbb{Q}$, but a one dimensional Banach space over $\mathbb{R}$.

To prove the next lemma, we first have to discuss the following combinatorial construction. Moreover we will recall a lemma of Mazur.

Let $\omega^{<\omega}$ denote the set of all finite sequences $\sigma = \langle s_0, \ldots, s_n \rangle$ of $\omega$. For $\sigma = \langle s_0, \ldots, s_n \rangle \in \omega^{<\omega}$ let $\sigma^\frown m := \langle s_0, \ldots, s_n, m \rangle$. We say $\sigma = \langle s_0, \ldots, s_n \rangle$ is an *initial segment* of $\tau = \langle t_0, \ldots, t_k \rangle$ and write $\sigma \preccurlyeq \tau$ if $n \leq k$ and $\forall i \leq n\ (s_i = t_i)$. A set $T \subseteq \omega^{<\omega}$ is called a *tree* if we have $\forall \tau \in T\ (\sigma \preccurlyeq \tau \rightarrow \sigma \in T)$. For $x = \langle x_0, x_1, \ldots, x_n, \ldots \rangle$, the finite sequence $\langle x_0, \ldots, x_{k-1} \rangle$ is denoted by $x|_k$ and $\bar{x} := \{x_i : i \in \omega\}$. An infinite sequence $x \in \omega^\omega$ is called a *branch* of $T$ if $\forall k \in \omega\ (x|_k \in T)$. Finally, let $T(\omega)$ be the set of all branches of $T$.

Now we define the tree $\mathfrak{T}$ as follows:

(a) $\forall \sigma \in \mathfrak{T}\ (|\{m : \sigma^\frown m \in \mathfrak{T}\}| = 2)$;

(b) $\langle 0 \rangle, \langle 1 \rangle \in \mathfrak{T}$;

(c) if $\sigma^\frown n^\frown m \in \mathfrak{T}$, then $m = 2n + 2$ or $m = 2n + 3$.

Because there is a canonical bijection between the set of all infinite 0-1-sequences and $\mathfrak{T}(\omega)$, the cardinality of the set $\mathfrak{T}(\omega)$ equals $\mathfrak{c}$.

Let $\mathcal{A} := \{\bar{x} : x \in \mathfrak{T}(\omega)\}$, then for any two different elements $\bar{x}$ and $\bar{y}$ of $\mathcal{A}$ we have $|\bar{x} \cap \bar{y}| < \aleph_0$ (this is because for any two different branches $x$ and $y$ of $\mathfrak{T}$, we find an $m \in \omega$ such that $(\bar{x} \setminus m) \cap (\bar{y} \setminus m) = \emptyset$).

The following lemma is a slight generalization of [LT, Lemma 1.a.6].

**LEMMA 3.3** *Let $E$ be an infinite dimensional Banach space over $K$. Let $F \subseteq E$ be a finite dimensional subspace and let $\varepsilon > 0$. Then there is an $x \in E$ with $\|x\| = 1$ so that $\|y\| \leq (1+\varepsilon)\|y + \lambda x\|$ for every $y \in F$ and every scalar $\lambda \in K$*

**PROOF:** The proof carries over from [LT, Lemma 1.a.6], where the assertion is proved for $K = \mathbb{R}$ and $K = \mathbb{C}$.

⊣

Now we are ready to prove

**LEMMA 3.4** *If $K \subseteq \mathbb{C}$ is a field containing $\mathbb{N}$ and $E$ is a Banach space over $K$ such that $\dim(E) = \infty$, then every Hamel base of $E$ has at least cardinality $\mathfrak{c}$.*

**PROOF:** We consider two cases. In the first case $|K| < \mathfrak{c}$ and in the second case $|K| = \mathfrak{c}$. For both cases, let $H = \{X_\iota : \iota < \kappa < \mathfrak{c}\} \subseteq E$ be a family (of cardinality $\kappa < \mathfrak{c}$) of vectors of $E$. We will show that $H$ is not a Hamel base of $E$.

*1. case:* Assume $|K| < \mathfrak{c}$ and that $H$ is a Hamel base of $E$. By Lemma 3.2 we have $\mathfrak{c} \leq |E| = \max\{|K|, |H|\} < \mathfrak{c}$, which is a contradiction.

*2. case:* Assume $|K| = \mathfrak{c}$ and that $H$ is a Hamel base of $E$. Lemma 3.3 is usually used in order to construct a subspace of $E$ which possesses a Schauder base. Since we did not assume that $K$ is complete, the construction does not lead to a complete subspace, nevertheless, the resulting sequence is sufficient for our purposes: We start with a unit vector $x_0 \in E$. Then we construct iteratively the sequence $\{x_i\}_{i \in \omega}$ such that

$$||y|| \leq (1 + \varepsilon_n)||y + \lambda x_{n+1}||$$

for all $y \in \langle x_0, \ldots, x_n \rangle$ and all $\lambda \in K$. Here, we choose the sequence of positive numbers $\{\varepsilon_n\}_{n \in \omega}$ such that $\prod_{n=0}^{\infty}(1 + \varepsilon_n) \leq 1 + \varepsilon$ for some $\varepsilon > 0$. Now, we claim that $\sum_{n=0}^{\infty} \lambda_n x_n = 0$ implies that $\lambda_n = 0$ for all $n \in \omega$. If not, we find a first index $i$ with $\lambda_i \neq 0$. Then we have

$$\begin{aligned} ||\lambda_i x_i|| &\leq (1 + \varepsilon_i)||\lambda_i x_i + \lambda_{i+1} x_{i+1}|| \\ &\leq (1 + \varepsilon_i)(1 + \varepsilon_{i+1})||\lambda_i x_i + \lambda_{i+1} x_{i+1} + \lambda_{i+2} x_{i+2}|| \\ &\leq \prod_{k=i}^{n}(1 + \varepsilon_k)||\sum_{k=i}^{n} \lambda_k x_k|| \end{aligned}$$

Since the first factor is uniformly bounded in $n$ and the second factor converges to 0 as $n \to \infty$, we obtain $||\lambda_i x_i|| = 0$ which contradicts $\lambda_i \neq 0$.

Let us consider the injective map

$$\zeta : \mathcal{A} \to E, \quad \bar{y} \mapsto \sum_{i=0}^{\infty} \bar{y}(i) 2^{-i} x_i.$$

We recall the notation $\bar{y}(i) = 1$ if $i \in \bar{y}$ and $\bar{y}(i) = 0$ otherwise. Notice that, by construction, the vectors in $\{\zeta(\bar{y}) : \bar{y} \in \mathcal{A}\}$ are finitely linearly independent over $K$: In fact, if we take distinct $\bar{y}_1, \ldots, \bar{y}_m \in \mathcal{A}$, then there exists a number $k \in \omega$ such that for all $k' > k$, if $\bar{y}_i(k') = 1$ then $\bar{y}_j(k') = 0$ for $j \neq i$, hence, the vectors $\zeta(\bar{y}_1), \ldots, \zeta(\bar{y}_m)$ are linearly independent over $K$.

Then, the composed function $\varphi \circ \zeta : \mathcal{A} \to [K]^{<\omega} \times [H]^{<\omega}$ is injective ($\varphi$ is defined as in the proof of Lemma 3.2(c)). On the other hand, since $|\mathcal{A}| = \mathfrak{c}$ and $|[H]^{<\omega}| = |H| < \mathfrak{c}$, we find by the pigeonhole principle (see [Je, p. 321]) a set $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = \mathfrak{c}$ such that $\mathrm{pr}_2 \circ \varphi \circ \zeta : \mathcal{B} \to [H]^{<\omega}$ is constant ($\mathrm{pr}_2$ denotes the projection $\mathrm{pr}_2(\langle a, b\rangle) := b$). So, let $H_0 = \langle \mathrm{pr}_2 \circ \varphi \circ \zeta(\mathcal{B})\rangle$ denote the corresponding finite dimensional subspace. Since $\zeta$ is injective, $\zeta(\mathcal{B}) \subset H_0$ is a set of cardinality $\mathfrak{c}$ and consists of linearly independent vectors which is a contradiction. ⊣

Now we can give the the main result of this section.

**THEOREM 3.5** *If $K \subseteq \mathbb{C}$ is a field containing $\mathbb{N}$ and $E$ is a Banach space over $K$ such that $\dim(E) = \infty$, then every Hamel base of $E$ has cardinality $|E|$.*

**PROOF:** Let $E$ be a Banach space over $K$ such that $\dim(E) = \infty$ and let $H$ be a Hamel base of $E$. By Lemma 3.2 we have $|E| = \max\{|K|, |H|\}$. By Lemma 3.4 we have $|H| \geq \mathfrak{c}$, and because $|K| \leq \mathfrak{c}$, we get $|E| = |H|$. ⊣

REMARK: It is worth mentioning, that the previous result for $F$-spaces follows directly from Martin's Axiom (the definition and some consequences can be found in [Ku, Ch. II]): Let $E$ be an $F$-space, i.e. a topological vector space whose topology is induced by a complete invariant metric $d$. If $H_0 \subseteq H \subseteq E$ with $|H_0| = \aleph_0$, $|H| < \mathfrak{c}$, we may consider the countable set $A := \langle H_0\rangle_{\mathbb{Q}}$, the set of all finite rational linear combinations of vectors of $H_0$. Let $P$ be the set $\{B_{1/n}(a_i) : a_i \in A, n \in \omega\}$, where $B_{1/n}(a_i) := \{x \in E : d(x, a_i) < \frac{1}{n}\}$. Let $\mathbb{P} = \langle P, \subseteq\rangle$, then $\mathbb{P}$ is a partially ordered set in which every anti-chain is countable. A set $D \subseteq P$ is called dense, if for every $p \in P$ there exists a $q \in D$ such that $q \subseteq p$. For every finite dimensional $K$-linear subspace $V \subseteq E$, the set $D_V := \{p \in P : p \cap V = \emptyset\}$ is dense. Since $|H| < \mathfrak{c}$ we have strictly less than $\mathfrak{c}$ many dense sets of this form and Martin's Axion gives a descending chain in $P$ such that for every dense set $D_V$ we find an element in this chain, which is contained in $D_V$. Since $E$ is a complete space, this chain converges to a point which does not belong to any of the finite dimensional subspaces spanned by $H$. Hence, $H$ is not a Hamel base.

As a corollary, we obtain a slightly stronger version of a theorem in [Ja, Chapter 9]:

**COROLLARY 3.6** *The set $E^f$ of all linear functions $E \to \mathbb{R}$ on an infinite dimensional Banach space $E$ has cardinality $2^{|E|}$.*

**PROOF:** It is easy to see that $|E^f| = |\mathfrak{c}^H| = 2^{|H|}$, where $H$ is a Hamel base of $E$, and therefore $|E^f| = 2^{|E|}$. $\dashv$

## 4 $\omega$-bases of subspaces of $\ell^\infty$

It was a long standing question, whether every separable Banach space has a Schauder base. Enflo solved this problem in [En] in the negative. It is a natural question, whether the answer could be positive for a weaker notion of base, e.g. if one allows the set of base vectors to be uncountable. In this section we want to make some steps towards this problem.

¿From now on, let $E$ be a Banach space over $\mathbb{R}$. For $T \subseteq E$, $\langle T \rangle$ continues to denote the vector space which is generated by the set $T$ and $\overline{T}$ is the closure of $T$ in the norm-topology.

A set $S \subseteq E$ is called *$\omega$-spanning* in $E$ *iff* $\overline{\langle S \rangle} = E$ and it is called *$\omega$-independent*, *iff* $Y \in S$ implies $Y \notin \overline{\langle S \setminus \{Y\} \rangle}$. An $\omega$-spanning set of a Banach space $E$ which is $\omega$-independent is called an *$\omega$-base* of $E$.

REMARK: An $\omega$-spanning set in a Banach space $E$ which is countable, is sometimes called a *Galerkin base*. In our case, an $\omega$-spanning set need not be countable. The "$\omega$" means, that we allow countable linear combinations (instead of finite linear combination as in Hamel bases) to represent a vector.

**LEMMA 4.1** *Let $E$ be an infinite dimensional Banach space. Then the following are equivalent:*

*(a) $S \subseteq E$ is an $\omega$-base of $E$.*

*(b) $S$ is a maximal $\omega$-independent family in $E$.*

*(c) $S$ is minimal $\omega$-spanning in $E$.*

**PROOF:** We prove $(a) \iff (b)$ ($(a) \iff (c)$ is similar): First, suppose that $S$ is an $\omega$-base but not maximal $\omega$-independent. Thus there exists $Y \in E$ such that $T = S \cup \{Y\}$ is still $\omega$-independent. In particular, $Y \notin \overline{\langle T \setminus \{Y\} \rangle} = \overline{\langle S \rangle}$. But this contradicts the fact that $S$ is $\omega$-spanning.

Second, suppose that $S$ is maximal $\omega$-independent but not an $\omega$-base, i.e. not $\omega$-spanning. Then there exists $Y \in E$ with $Y \notin \overline{\langle S \rangle}$. But then $S \cup \{Y\}$ would be $\omega$-independent contradicting the fact that $S$ was maximal with this property. ⊣

REMARK: Although the previous characterisation is similar to that of a Hamel base, we cannot conclude that every Banach space possesses an $\omega$-base in the same way as in the case of a Hamel base.

**FACT 4.2** *If $S_1$ and $S_2$ are two $\omega$-bases of an infinite dimensional Banach space $E$, then $|S_1| = |S_2|$. Therefore, all $\omega$-bases of $E$ have the same cardinality.*

**PROOF:** Because $S_2$ is an $\omega$-base of $E$, for every $Y \in S_1$ and every $n \in \omega$ we find $X_0, \dots, X_{k_n} \in S_2$ and $r_0, \dots, r_{k_n} \in \mathbb{R}$ such that $||Y - \sum_{i \leq k_n} r_i X_i|| < \frac{1}{n}$. So, to approximate the vector $Y \in S_1$, we need only countably many elements of $S_2$, say $\{X_Y^0, X_Y^1, \dots\}$. Let

$$\begin{array}{rccl} g: & S_1 & \longrightarrow & S_2^\omega \\ & Y & \longmapsto & \{X_Y^0, X_Y^1, \dots\}. \end{array}$$

Then $S' := \bigcup\{g(Y) : Y \in S_1\} \subseteq S_2$ has the same cardinality as $S_1$ (this is because $|\omega \times \kappa| = |\kappa|$, for any infinite cardinal $\kappa$). ¿From the fact that $\overline{\langle S_1 \rangle} = E$ it easily follows that $\overline{\langle S' \rangle} = E$. Hence, by Lemma 4.1, we have $S' = S_2$ and hence $|S_1| = |S_2|$. ⊣

Let $\mathbb{R}^\omega$ denote the set of all infinite sequences $\langle r_0, r_1, \dots \rangle$ of $\mathbb{R}$. For $\bar{r} = \langle r_0, \dots, r_n, \dots \rangle \in \mathbb{R}^\omega$, let $\bar{r}(n) := r_n$.

Let $\ell^\infty$ be the set of all sequences $\bar{r} = \langle r_0, r_1, \dots, r_n, \dots \rangle$ of $\mathbb{R}$ such that $\sup\{|r_i| : i \in \omega\} =: ||\bar{r}|| < \infty$. This set is a complete normed vector space, hence a Banach space.

We will show that there is always an $\omega$-independent family in $\ell^\infty$ of cardinality $\mathfrak{c}$.

Let $\mathcal{I} \subseteq [\omega]^\omega$, then $\mathcal{I}$ is called an *independent family* (i.f.) *iff* whenever $m, n \in$

$\omega$ and $x_0, \ldots, x_m, y_0, \ldots, y_n$ are distinct members of $\mathcal{I}$, then

$$|x_0 \cap \cdots \cap x_m \cap (\omega \setminus y_0) \cap \cdots \cap (\omega \setminus y_n)| = \aleph_0 .$$

Notice that this is equivalent to

$$|\bigcap_{i \leq m} x_i \setminus \bigcup_{j \leq n} y_j| = \aleph_0 .$$

There is always an i.f. of cardinality $\mathfrak{c}$ (cf. [Ku, Ex. A6]) which can be constructed even without using the axiom of choice. Now we fix an i.f. $I \subseteq [\omega]^\omega$ of cardinality $\mathfrak{c}$. For $x \in I$, we let $l(x) = \langle a_0, \ldots, a_n, \ldots \rangle$ be such that

$$a_i = \begin{cases} 1 & \text{if } i \in x, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $l(x) \in \ell^\infty$. Let $J = \{l(x) : x \in I\}$.

**THEOREM 4.3** *The set $J \subseteq \ell^\infty$ constructed above is an $\omega$-independent family of $\ell^\infty$ of cardinality $\mathfrak{c}$.*

**PROOF:** It is sufficient to show that for any $Y \in J$, for every finite set $\{X_0, X_1, \ldots,$
$X_{n-1}\} \subseteq J \setminus \{Y\}$, and for every finite sequence $\langle r_0, \ldots r_{n-1} \rangle$ of $\mathbb{R}$, we find a $k \in \omega$ such that $|Y(k) - \sum_{i<n} r_i X_i(k)| = 1$, which implies that $||Y - \sum_{i<n} r_i X_i|| \geq 1$.

For $X_i$, let $x_i \in I$ be such that $l(x_i) = X_i$ and let $y \in I$ be such that $l(y) = Y$. Because $I$ is an i.f., for every finite 0-1-sequence $\langle a_0, \ldots, a_k \rangle$, we have

$$|\{n \in \omega : n \notin y \wedge \forall i \leq k (n \in x_i \leftrightarrow a_i = 1)\}| =$$
$$|\{n \in \omega : n \in y \wedge \forall i \leq k (n \in x_i \leftrightarrow a_i = 1)\}| = \aleph_0 . \quad (*)$$

To see this, let $A_1 := \{i \leq k : a_i = 1\}$ and $A_0 := \{i \leq k : a_i = 0\}$ and consider the sets

$$(y \cap \bigcap_{i \in A_1} x_i) \setminus \bigcup_{i \in A_0} x_i$$

and

$$\bigcap_{i \in A_1} x_i \setminus (y \cup \bigcup_{i \in A_0} x_i),$$

which are both infinite.

Now we can find a $k \in \omega$ such that $Y(k) = 1$ and $X_i(k) = 0$ (for all $i < n$), which completes the proof. ⊣

As a corollary we get

**COROLLARY 4.4** *For every $\kappa \leq \mathfrak{c}$ there exists a Banach space $E \subseteq \ell^\infty$ and a set $J \subseteq \ell^\infty$ of cardinality $\kappa$ such that $J$ is an $\omega$-base of $E$.*

**PROOF:** Let $E := \overline{\langle J \rangle}$ where $J$ is constructed as in the proof of Theorem 4.3 from an i.f. of cardinality $\kappa$.

⊣

Using the axiom of choice one can show that every set $S \subseteq E$ with $\overline{\langle S \rangle} = E$ contains a subset $S' \subseteq S$ such that $S'$ is finitely linearly independent and $\overline{\langle S' \rangle} = E$. If we replace "finitely linearly independent" by "$\omega$-independent", this is no longer true, even if $E$ is separable, as the following example shows.

Let $\{z_i : i \in \omega\} \subseteq [\omega]^\omega$ be an i.f. and let $f : [\omega]^{<\omega} \to \omega$ be an injective function. With the function $f$ we define the set

$$I := \{z_{f(s)} : s \in [\omega]^{<\omega}\},$$

where $[\omega]^{<\omega}$ is the set of strictly increasing, finite sequences of $\omega$. For a non-empty, strictly increasing, finite sequence $s = \langle a_0, \dots, a_n \rangle \in [\omega]^{<\omega}$, let

$$X_s(k) := \sum_{i \leq n} 2^{-a_i} z_{f(\langle a_0, \dots, a_i \rangle)}(k).$$

Now let $X_s := \langle X_s(0), X_s(1), \dots \rangle$, then for every $s \in [\omega]^{<\omega}$, $X_s \in \ell^\infty$.

With similar arguments as above it follows that $J := \{X_s : s \in [\omega]^{<\omega}\}$ is finitely linearly independent, but $J$ is obviously not $\omega$-independent. Moreover, for $E_J := \overline{\langle J \rangle}$ we have the following proposition, which makes clear why it is more difficult to construct an $\omega$-base than to find a Hamel base. In fact, it is not known whether every Banach space has an $\omega$-base.

**PROPOSITION 4.5** *There is a Banach $E$ space which has an $\omega$-spanning subset $S$ such that no subset of $S$ is an $\omega$-base of $E$.*

**PROOF:** Let $E := E_J$ and $S := J$ and notice that for every $s = \langle a_0, \dots, a_n \rangle \in [\omega]^{<\omega}$, the vector $z_{f(s)}$ belongs to $E_J$ (this is because $z_{f(\langle a_0, \dots, a_n \rangle)} =$

$2^{a_n}(X_{\langle a_0,\dots,a_n\rangle} - X_{\langle a_0,\dots,a_{n-1}\rangle})$. If $S' \subseteq S$ is $\omega$-independent and $X_s \in S'$, then we find an $\varepsilon > 0$ such that for every finite sequence $\langle X_{s_0},\dots,X_{s_n}\rangle$ of $S' \setminus \{X_s\}$ and every finite sequence $\langle r_0,\dots,r_n\rangle$ of $\mathbb{R}$ we have $\|X_s - \sum_{i<n} r_i X_{s_i}\| > \varepsilon$. Let $k$ be such that $2^{-k} < \varepsilon$, then no vector $X_t$—where $t = \langle a_0,\dots,a_n,a_{n+1},\dots,a_l\rangle$ and $a_{n+1} > k$—is in $S'$, because $\|X_s - X_t\| < \varepsilon$. Let us choose $t = s^\frown(k+1)$ and $X_{\sigma_i} \in S' \setminus \{X_s\}$, $i = 0,\dots,m$. We claim, that $\|\lambda X_s + \sum_{i=0}^m \lambda_i X_{\sigma_i} - X_t\| \geq 2^{-(k+1)}$ for every choice of $\lambda, \lambda_i \in \mathbb{R}$: To see this, we first rewrite $\lambda X_s + \sum_{i=0}^m \lambda_i X_{\sigma_i} = \sum_{j\in\Lambda} \lambda'_j z_j$ for a finite index set $\Lambda$. Assume that $z_j = z_{f(t)}$ for a $j \in \Lambda$, then there exists a $\sigma_i$ such that $\sigma_i = t^\frown t'$. But this is impossible, since otherwise we would have $\|X_s - X_{\sigma_i}\| < \varepsilon$, which is not true for vectors $X_{\sigma_i} \in S' \setminus \{X_s\}$. Now, since $I$ is an i.f. we find an $h \in \omega$ such that $z_{f(t)}(h) = 1$ and $z_j(h) = 0$ for all $j \in \Lambda$. Then $\|\lambda X_s + \sum_{i=0}^m \lambda_i X_{\sigma_i} - X_t\| \geq |\lambda X_s(h) + \sum_{i=0}^m \lambda_i X_{\sigma_i}(h) - X_t(h)| = 2^{-(k+1)}$.

Therefore, we have that $E_J \ni X_t \notin \overline{\langle S'\rangle}$, which completes the proof. ⊣

We conclude this section by giving an example of an $\omega$-base $S$ of a proper subspace of $\ell^\infty(\kappa)$ (where $\kappa$ is a cardinal) such that $|S| = 2^\kappa$.

The set $\ell^\infty(\kappa)$ is the set of all $\kappa$-sequences $\bar{r} = \langle r_0,\dots,r_\alpha,\dots\rangle_\kappa$ of $\mathbb{R}$ such that $\sup\{|r_\alpha| : \alpha \in \kappa\} < \infty$. With the supremum norm, $\ell^\infty(\kappa)$ is a Banach space.

Now we can prove the following

**PROPOSITION 4.6** *In $\ell^\infty(\kappa)$ we find an $\omega$-independent set $S$ of cardinality $2^\kappa$, such that $\overline{\langle S\rangle} \neq \ell^\infty(\kappa)$.*

**PROOF:** The proof is similar to the proof of Theorem 4.3, starting with an i.f. of cardinality $2^\kappa$ (see [Ku, Ex. A6]). ⊣

Note that if $\mathfrak{c} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$, then the $S$ constructed above (with $\kappa = \aleph_1$) has cardinality $\aleph_3 > \mathfrak{c}$.

# 5 $\omega$-bases of $\ell^p(\kappa)$

In this section we will show that there exist natural examples of Banach spaces containing an $\omega$-base. For this, we fix a real $p \in [1,\infty[$.

For a cardinal number $\kappa$, let $\ell^p(\kappa)$ be the set of all $\kappa$-sequences $\bar{r} = \langle r_0,\dots,$

$r_\alpha, \dots\rangle_\kappa$ of $\mathbb{R}$, such that $\sum_{\alpha<\kappa} |r_\alpha|^p < \infty$. With $\left(\sum_{\alpha<\kappa} |r_\alpha|^p\right)^{1/p}$ as the norm, $\ell^p(\kappa)$ is a Banach space, and for $p = 2$ even a Hilbert space.

An easy counting argument shows that if $\bar{r} = \langle r_0, r_1, \dots\rangle_\kappa \in \ell^p(\kappa)$, then the support of $\bar{r}$, denoted by $\operatorname{spt}(\bar{r}) := \{\alpha < \kappa : r_\alpha \neq 0\}$, must be countable.

We can now prove the following

**PROPOSITION 5.1** *For every cardinal number $\kappa$, the Banach space $\ell^p(\kappa)$ contains an $\omega$-base of cardinality $\kappa$.*

**PROOF:** Let $e_\alpha = \langle r_0, \dots\rangle_\kappa \in \ell^p(\kappa)$ be such that $\operatorname{spt}(e_\alpha) = \{\alpha\}$ and $r_\alpha = 1$. Further let $S := \{e_\alpha : \alpha < \kappa\}$. It clear that $S$ has cardinality $\kappa$ and it is easy to see that $S$ forms an $\omega$-base of $\ell^p(\kappa)$. $\dashv$

Unlike in the case of Hamel bases, it is possible that $\ell^p(\kappa)$ contains an $\omega$-base $S$ of cardinality less than $|\ell^p(\kappa)|$, even if $\mathfrak{c} < |S|$.

**THEOREM 5.2** *Let $\kappa$ be an infinite cardinal and let $S$ be an $\omega$-base of $\ell^p(\kappa)$; then each of the following two cases are possible (this means, both cases are consistent with ZFC):*

1. $|S| < |\ell^p(\kappa)| = \mathfrak{c}$,
2. $\mathfrak{c} < |S| < |\ell^p(\kappa)|$.

**PROOF:** For both cases let $\kappa := \aleph_\omega$, where $\aleph_\omega = \bigcup_{n\in\omega} \aleph_n$.

For the case 1 assume $\mathfrak{c} > \aleph_\omega$ (e.g. $\mathfrak{c} = \aleph_{\omega+1}$). Then by the properties of the gimel function (cf. [Je, p. 51]) we get $|[\aleph_\omega]^\omega| == \aleph_\omega^{\aleph_0} = \mathfrak{c}$.

For the case 2 assume $\mathfrak{c} < \aleph_\omega$. Again by the properties of the gimel function we get $\aleph_\omega^{\aleph_0} > \aleph_\omega$. $\dashv$

REMARK: For any infinite cardinal number $\kappa$ we have $\ell^1(\kappa)^* = \ell^\infty(\kappa)$ and therefore $|\ell^1(\kappa)^*| = 2^\kappa$ (the star $^*$ denotes the dual of a vector space). In fact, $|\ell^\infty(\kappa)| = |\mathfrak{c}^\kappa| = 2^\kappa$. In particular, if $\kappa = \aleph_1 = \mathfrak{c}$, then $|\ell^1(\kappa)^*| = |\ell^1(\kappa)^f| = 2^{\mathfrak{c}}$; and if $\kappa = \aleph_\omega$ and the generalized continuum hypothesis holds, then (by the properties of the gimel function) we have $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1} = 2^{\aleph_\omega}$, which implies that $|\ell^1(\kappa)^*| = 2^{\aleph_\omega} = \aleph_{\omega+1} < 2^{\aleph_{\omega+1}} = \aleph_{\omega+2} = |\ell^1(\kappa)^f|$.

# 6 Some problems and suggestions for further research

In order to define a "topological basis" $S \subseteq V$ in a vector space $V$, one basically has to define in which way approximating sequences are allowed to be built and to fix the topology. More precisely, one has to give a rule in form of a subset $R$ of allowed sequences in $V^\omega$. Examples are:

The Galerkin rule:

$$R_{\mathrm{G}}(S) := \{\{x_n\}_{n\in\omega} : x_n = \sum_{i=0}^{n} r_i(n) X_i, r_i(n) \in \mathbb{R}, \{X_i\}_{i\in\omega} \in S^\omega\}$$

The Schauder rule:

$$R_{\mathrm{S}}(S) := \{\{x_n\}_{n\in\omega} : x_n = \sum_{i=0}^{n} r_i X_i, \{r_i\}_{i\in\omega} \in \mathbb{R}^\omega, \{X_i\}_{i\in\omega} \in S^\omega\}$$

The Hamel rule:

$$R_{\mathrm{H}}(S) :=$$

$$\{\{x_n\}_{n\in\omega} : x_n = \sum_{i=0}^{N} r_i X_i, N \in \omega, \{r_i\}_{i\leq N} \in \mathbb{R}^{<\omega}, \{X_i\}_{i\leq N} \in S^{<\omega}\}$$

Further examples occur if only absolutely or unconditionally convergent series are allowed, or if one only allows to choose the vectors $X_i$ in a certain given order.

Typical examples for the topology are the norm-topology $\tau_n$ in normed spaces, or the weak topology $\tau_w$ or weak* topology $\tau_{w^*}$.

One can define that $S$ is $(R,\tau)$-spanning, if $V = \{\tau\text{-}\lim x_n : \{x_n\}_{n\in\omega} \in R\}$ and that $S$ is $(R,\tau)$-independent, if for all $X \in S$ there holds $X \notin \{\tau\text{-}\lim x_n : \{x_n\}_{n\in\omega} \in R(S \setminus \{X\})\}$. Then, $S$ is an $(R,\tau)$-base if $S$ is $(R,\tau)$-spanning and $(R,\tau)$-independent. In this terminology, our $\omega$-bases are $(R_{\mathrm{G}}, \tau_n)$-bases. The $\omega$-bases of $\ell^p(\kappa)$ introduced in Section 5 are actually $(R_{\mathrm{S}}, \tau_n)$-bases.

As an example, we consider the space $\ell^p(\kappa)$ with $1 \leq p < \infty$ and $\kappa = \aleph_0$. As we have seen, the set $S = \{e_\alpha : \alpha < \kappa\}$ (see Section 5) is an $(R_{\mathrm{S}}, \tau_n)$-base.

It is easy to see, that the dual set, i.e. $S^* := \{\varphi_\alpha : \alpha < \kappa\}$ defined by $\varphi_\alpha(e_\beta) = \delta_{\alpha\beta}$, is an $(R_S, \tau_{w^*})$-base of $(\ell^p(\kappa))^* = \ell^{p'}(\kappa)$ (where $p'$ denotes the conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). Of course, $\tau_w = \tau_{w^*}$ if $p > 1$. Note however, that $S^*$ is only $(R_S, \tau_{w^*})$-independent in $\ell^{p'}(\kappa)$ if $\kappa > \aleph_0$, but no longer $(R_S, \tau_{w^*})$-spanning, and that therefore an $(R_S, \tau_{w^*})$-base has to have a cardinality strictly larger than $\kappa$.

The *general problems* are:

- Decide, whether a given vector space has a certein $(R, \tau)$-base.
- Characterize vector spaces, which have a certain $(R, \tau)$-base.
- Investigate, whether all $(R, \tau)$-bases of a given vector space have the same cardinality, and if so to compute this cardinality.

Of particular interest would be to investigate whether the answer to any of these questions can be proved with the usual axioms of set theory (ZFC), and if the axiom of choice is essential; or whether the answer can be proved to be independent of ZFC. In the latter case, it might be possible to find new cardinal invariants in form of the cardinality of certain bases (cf. [Va]).

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