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# NOTES

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## Proof of a Conjecture of Lewis Carroll

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**Introduction** Charles Lutwidge Dodgson, who is better known as Lewis Carroll, the author of “Alice’s Adventures in Wonderland” (1865), was also a many-sided mathematician. His work treats—among many other subjects—investigations on right triangles with sides of integer length. In *Life and Letters of Lewis Carroll* ([1, p. 343]) we find the diary entry

Dec. 19 (Sun).—Sat up last night till 4 a.m., over a tempting problem, sent me from New York, “to find 3 equal rational-sided rt.-angled  $\Delta$ ’s.” I found *two*, whose sides are 20, 21, 29; 12, 35, 37; but could not find *three*.

Thus Carroll’s search was for triples of integer-sided right triangles, all having the same area. Despite his failure, Carroll conjectured that infinitely many such triples exist. (Two triples are identified if they are equal up to scaling by a rational factor.) Apparently, Carroll was not aware of Fermat’s observation [4] that if  $z$  is the hypotenuse and  $b$  and  $d$  are the legs of a rational right triangle, then we obtain a new rational right triangle of the same area with sides

$$z' = \frac{z^4 + 4b^2d^2}{2z(b^2 - d^2)}; \quad b' = \frac{z^4 - 4b^2d^2}{2z(b^2 - d^2)}; \quad d' = \frac{4z^2bd}{2z(b^2 - d^2)}.$$

It is not clear, however, whether iterating this procedure actually produces at least three non-congruent triangles. Nor is it obvious whether we can find infinitely many *different triples* by this procedure. Problems of this type have been treated by several authors (see, e.g., [2], [3] and [7]), but I am not aware of any explicit answer to Carroll’s conjecture.

We prove in this Note that Carroll’s conjecture is true.

**Preliminaries** It is well known (see, e.g., [6]) that the integer equation

$$x^2 + y^2 = z^2$$

has the general solution

$$x = 2\lambda mn; \quad y = \lambda(m^2 - n^2); \quad z = \lambda(m^2 + n^2) \quad (1)$$

with  $\lambda, m, n \in \mathbb{N}$ . If we restrict  $m$  and  $n$  to be relatively prime and not both odd, and set  $\lambda = 1$ , we obtain an infinite family of primitive triangles (i.e.,  $\gcd(x, y, z) = 1$ ).

Since  $x$  and  $y$  are the legs the triangle has area

$$A = \frac{1}{2}xy = mn(m^2 - n^2). \quad (2)$$

Thus Carroll's conjecture is an assertion on the solutions of the third-order equation (2).

**Proof of the conjecture** The following theorem answers Carroll's conjecture.

**THEOREM.** *There exists an infinite family of triples of integer-sided right triangles with the same area. In particular, if  $m$  is a prime number of the form  $m = 6N + 1$ ,  $N \in \mathbb{N}$ , then there exist positive integer solutions  $n$  and  $l$  of the equation  $m^2 = n^2 + nl + l^2$ , and a primitive triple of integer-sided right triangles of area  $A = mnl(n + l)$ . Different values of  $m$  lead to different primitive triples.*

*Proof.* The theory of diophantine quadratic forms (see, e.g., [6]) asserts that the equation

$$m^2 = n^2 + nl + l^2 \quad (3)$$

has nonzero integer solutions if  $m$  has the form  $m = \prod_{i=1}^r p_i$ , where the  $p_i$  are primes of the form  $p_i = 6N_i + 1$ . In particular, solutions  $n$  and  $l$  exist if  $m$  itself is prime of the form  $m = 6N + 1$ . Now Dirichlet's theorem (see, e.g., [5]) says that every arithmetic sequence  $\{a_N\}_{N \in \mathbb{N}}$ ,  $a_N = \alpha N + \beta$ , where  $\alpha$  and  $\beta$  are relatively prime, includes infinitely many prime numbers. In particular, there are infinitely many prime numbers of the form  $6N + 1$ .

The solutions  $n$  and  $l$  of (3) trivially satisfy  $n \neq \pm l$ . Multiplying (3) by  $n - l$  and rearranging terms gives

$$m^2 n - n^3 = m^2 l - l^3 = \frac{A}{m}, \quad (4)$$

where the last equality defines the number  $A$ . We may assume that  $A > 0$  (otherwise, replace  $(n, l)$  with  $(-n, -l)$ ). We may interpret  $n$  and  $l$  as two roots of the cubic polynomial  $p(x) = x^3 - m^2 x + A/m$ . If  $i$  is the third root, then Viëta's theorem (see, e.g., [5]) implies that

$$n + l + i = 0 \quad (5)$$

and

$$nli = -\frac{A}{m}.$$

Hence, all the roots  $n$ ,  $l$  and  $i$  are integers, with  $\pm n \neq i \neq \pm l$ . Since  $A/m > 0$ , two roots (say  $n$  and  $l$ ) are positive and one is negative. Thus

$$mn(m^2 - n^2) = A; \quad ml(m^2 - l^2) = A; \quad -im(i^2 - m^2) = A.$$

In other words the three integer-sided right triangles  $D_1, D_2, D_3$ , with sides

$$\begin{array}{lll} x_1 = 2mn & y_1 = m^2 - n^2 & z_1 = m^2 + n^2 \\ x_2 = 2ml & y_2 = m^2 - l^2 & z_2 = m^2 + l^2 \\ x_3 = -2mi & y_3 = i^2 - m^2 & z_3 = i^2 + m^2 \end{array} \quad (6)$$

have common area  $A = -mnl$ . Since the  $z_i$  are all different, so are the triangles.

Notice that (5) implies that at least one of  $n$ ,  $l$  and  $i$  is even. It is easy to see that none of  $n$ ,  $l$  and  $i$  is a multiple of  $m$ . Since  $m$  is prime, it follows that at least one of  $D_1$ ,  $D_2$  and  $D_3$  is primitive. Hence the triple  $D_1, D_2, D_3$  is primitive.

Now, it follows from  $m < A$  that we obtain infinitely many primitive triples in this way. To see that different values of  $m$  lead to different primitive triples, choose the triangle  $D$  of a triple that has maximum hypotenuse  $z$ ; denote its legs by  $x$  and  $y$ . Since  $m$  is prime, it follows from (6) that

$$\text{either } m = \sqrt{\frac{z-x}{2}} \in \mathbb{N} \text{ or } m = \sqrt{\frac{z-y}{2}} \in \mathbb{N}.$$

Hence  $m$  is determined by the triple and this completes the proof.

**Remark** The method above does not produce all primitive triples. The primitive triple with minimum area which is of a different type is

$$\begin{array}{lll} x_1 = 4080 & y_1 = 1001 & z_1 = 4201 \\ x_2 = 1430 & y_2 = 2856 & z_2 = 3194 \\ x_3 = 528 & y_3 = 7735 & z_3 = 7753 \end{array}$$

It is generated by (1), using the numbers

$$(m_1, n_1) = (51, 40); \quad (m_2, n_2) = (55, 13); \quad (m_3, n_3) = (88, 3).$$

The common area is  $A = 2042040$ . (This example was found by a *Mathematica* program.)

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