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PAPPUS PORISMS ON A SET OF LINES

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ABSTRACT. The Theorem of Pappus and the Scissors Theorem can be interpreted as porisms on two lines. We are concerned with the question whether corresponding closing figures can also be realized for more than two lines and any number of reversion points. It turns out that this is indeed the case, both for concurrent and for non-concurrent lines. We explicitly determine which conditions the reversion points must satisfy. All resulting porisms can be constructed with ruler alone. Along the way we show that the Theorem of Pappus and the Scissors Theorem are equivalent.

1. INTRODUCTION AND NOTATION

Let P be a point in the projective plane and ℓ_1, ℓ_2 be two lines not incident with P. We will always denote the intersection of ℓ_1 and ℓ_2 as O. We consider the reversion map $\ell_1 \to \ell_2, Q \mapsto R$, where P, Q, R are collinear (see Figure 1). P is called reversion point.

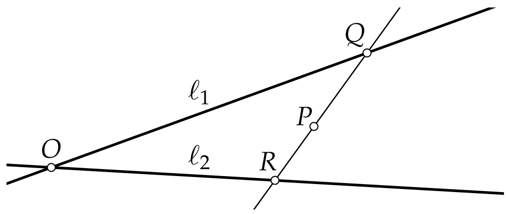

Figure 1. The reversion map.

Throughout this paper we will use the notation

$$Q \xrightarrow[\ell_1 \quad \ell_2]{P} R$$

for the situation shown in Figure 1. If it is clear from the context which lines are involved, we will omit them in the notation. For completeness we remark that the reversion map from ℓ_1 to ℓ_2 through the point P can be extended to an involutive projective map of the whole plane with fixed point P and a fixed point line ℓ, where $(\ell_1, \ell_2, OP, \ell)$ is a harmonic pencil of lines.

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The Hexagon Theorem of Pappus can be formulated as a porism in the projective plane:

Theorem 1 (Pappus Porism)**.** *Let $A_1, A_2, \ldots, A_6$ be a Pappus hexagon on the lines ℓ_1, ℓ_2 with intersection points P_1, P_2, P_3 on the Pappus line ℓ, noted as*

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} A_4 \xrightarrow{P_1} A_5 \xrightarrow{P_2} A_6 \xrightarrow{P_3} A_1$$

(see Figure 2). Then there exists a Pappus hexagon $A'_1, A'_2, \ldots, A'_6$ on ℓ_1, ℓ_2 with the same intersection points P_1, P_2, P_3 for any point A'_1 on ℓ_1:

$$A'_1 \xrightarrow{P_1} A'_2 \xrightarrow{P_2} A'_3 \xrightarrow{P_3} A'_4 \xrightarrow{P_1} A'_5 \xrightarrow{P_2} A'_6 \xrightarrow{P_3} A'_1. \tag{1.1}$$

The cases when A'_1 is the intersection of ℓ_1 with ℓ or ℓ_2 are considered as degenerate situations.

Note that we can also make the hexagon start at a point A'_1 on ℓ_2 instead of ℓ_1, and it closes in the same way as indicated in (1.1): Indeed, if $A'_1, \ldots, A'_6$ is a closing hexagon with starting point A'_1 on ℓ_1, we can renumber the points A'_i cyclically by taking A'_4 as new starting point A'_1 on ℓ_2.

Proof of Theorem 1. By the Theorem of Pappus, applied to the hexagon $A_1, \ldots, A_6$, the points P_1, P_2, P_3 are collinear. Then the Braikenridge-Maclaurin Theorem for degenerate conics (see, e.g., [2, p. 76]) applied to the points $A'_1, A'_2, \ldots, A'_5$ and the points P_1, P_2, P_3 implies that A'_6 lies on ℓ_2. □

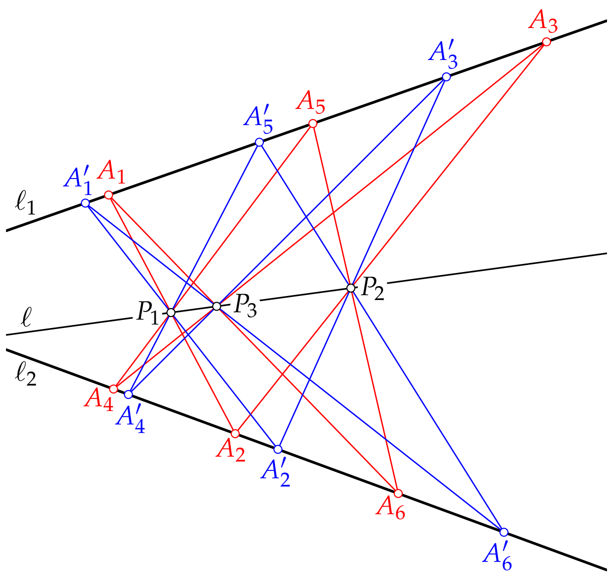

Figure 2. Pappus hexagons $A_1, A_2, \ldots, A_6$ and $A'_1, A'_2, \ldots, A'_6$.

The Scissors Theorem is also a porism:

Theorem 2 (Scissors Theorem)**.** *Let A_1, A_2, A_3, A_4 be a Scissors quadrilateral on the lines ℓ_1, ℓ_2 with intersection points P_1, P_2, P_3, P_4 on a line ℓ, i.e.,*

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} A_4 \xrightarrow{P_4} A_1 \tag{1.2}$$

(see Figure 3). Then there exists a Scissors quadrilateral A_1', A_2', A_3', A_4' on ℓ_1, ℓ_2 with the same intersection points P_1, P_2, P_3, P_4 for any point A_1' on ℓ_1 and ℓ_2:

$$A_1' \xrightarrow{P_1} A_2' \xrightarrow{P_2} A_3' \xrightarrow{P_3} A_4' \xrightarrow{P_4} A_1'. \tag{1.3}$$

Note that, in contrast to Theorem 1, it makes a difference whether we start with A_1' on ℓ_1 or on ℓ_2: A closing quadrilateral $A_1', \ldots, A_4'$ with starting point A_1' on ℓ_1 cannot be renumbered cyclically with A_2' as new A_1' on ℓ_2 such that (1.3) results. Stated differently: The green and blue closing quadrilaterals in Figure 3 cannot be made to coincide by sliding the starting point.

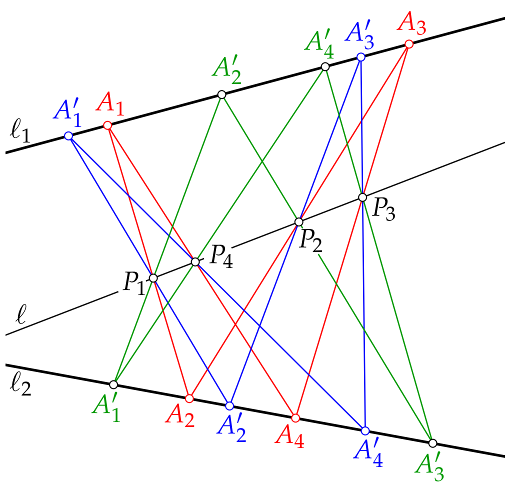

Figure 3. Scissors quadrilaterals $A_1, A_2, \ldots, A_4$ and $A_1', A_2', \ldots, A_4'$.

Also here, the cases when A_1' is the intersection of ℓ_1 with ℓ or ℓ_2 are included in the theorem as degenerate situations. The Hexagon Theorem of Pappus and the Scissors Theorem are closely related. In fact we have:

Proposition 3. *The Scissors Theorem is equivalent to the Hexagon Theorem of Pappus.*

Proof. Let $ABCDEF$ be a hexagon with A, C, E on a line g and B, D, F on a line h (see Figure 4). Let P be the intersection of AB and DE, Q the intersection of BC and EF, and R the intersection of CD and PQ. Finally, let S be the intersection of PQ and AD. Consider the quadrilateral $ABCD$ which has the intersection points P, Q, R, S with the line PQ:

$$A \xrightarrow{P} B \xrightarrow{Q} C \xrightarrow{R} D \xrightarrow{S} A.$$

Hence, according to Theorem 2, we have the closing quadrilateral $DEFA$:

$$D \xrightarrow{P} E \xrightarrow{Q} F \xrightarrow{R} A \xrightarrow{S} D.$$

In particular, all pairs of opposite sides (AB and DE, BC and EF, CD and FA) in the hexagon $ABCDEF$ meet on PQ.

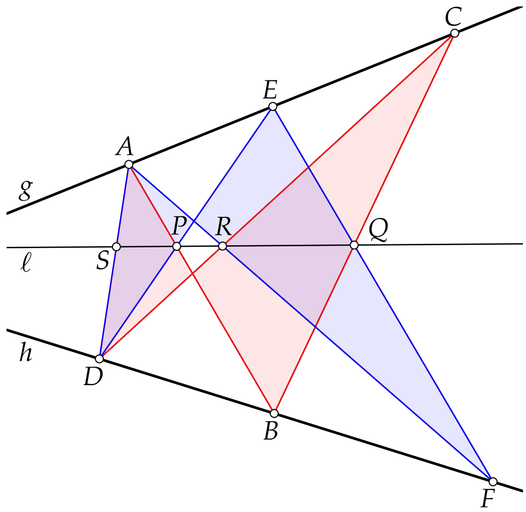

Figure 4. The Scissors Theorem implies the Theorem of Pappus.

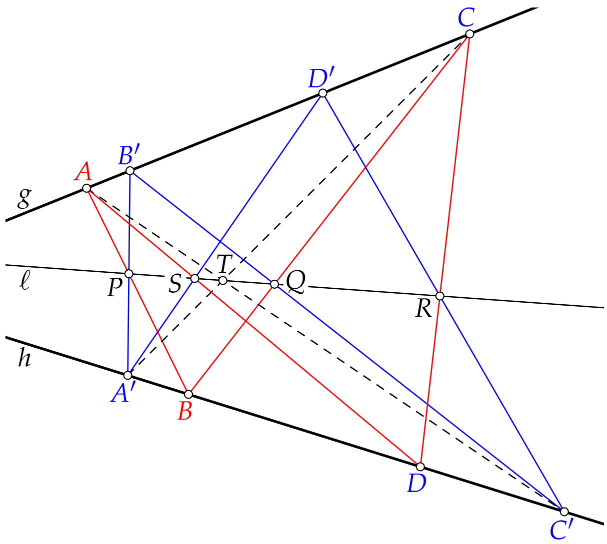

Figure 5. The Theorem of Pappus implies the Scissors Theorem.

For the reverse implication, consider the quadrilateral $ABCD$ with A, C on a line g and B, D on a line h. Its intersection points with a line ℓ are denoted by P, Q, R, S (see Figure 5):

$$A \xrightarrow{P} B \xrightarrow{Q} C \xrightarrow{R} D \xrightarrow{S} A.$$

Let $A'B'C'D'$ be another quadrilateral with A', C' on h and $B'D'$ on g such that $A'B'$ passes through P, $B'C'$ passes through Q and $C'D'$ passes through R. We want to show that $D'A'$ passes through S. To see this, we apply the Theorem of Pappus to the hexagon $ABCA'B'C'$ and find that the intersection point T of $A'C$ and AC' belongs to ℓ. Then, by applying the Theorem of Pappus to the hexagon $ADCA'D'C'$ it follows that, indeed, S belongs to $D'A'$. We conclude that every quadrilateral $A'B'C'D'$ through P, Q, R, S and starting point A' on h closes:

$$A' \xrightarrow{P} B' \xrightarrow{Q} C' \xrightarrow{R} D' \xrightarrow{S} A'.$$

By exchanging the role of $ABCD$ and $A'B'C'D'$ we find that also every quadrilateral $ABCD$ with starting point A on g closes in the same way. □

The purpose of this paper is to investigate whether porisms like Pappus's or the Scissors Theorem are also possible for an arbitrary number of reversion points in a position as general as possible, and whether variants with more than two support lines also exist.

In the next section we will show that the position of the points P_i in the Theorems 1 and 2 can be characterised and expressed by the cross ratio. At the same time we will generalise the porisms to an arbitrary number of points. We should add here, that the porisms which we will develop are also inspired by the Butterfly porism and its relatives on conics: See [4, 6, 7, 8, 9, 5] and recent generalisations in [3].

2. Generalization and quantification for two lines

In the following, we denote by (A, B, C, D) the cross ratio of collinear points A, B, C, D.

Theorem 4. *Let ℓ_1, ℓ_2 be two lines and ℓ a third line intersecting ℓ_1 in S_1 and ℓ_2 in $S_2 \neq S_1$. Let $A_1, A_2, \dots, A_{2n}$ be points that lie alternately on ℓ_1 and ℓ_2 with intersection points P_i of ℓ and A_iA_{i+1} (indices read cyclically, see Figure 6). Then we have*

$$\prod_{i=1}^{n} (S_1, S_2, P_{2i}, P_{2i+1}) = 1. \tag{2.1}$$

Vice versa, if (2.1) is valid for points $S_1, S_2, P_1, \dots, P_{2n}$ and ℓ_1 is a line through S_1 and ℓ_2 a line through $S_2 \neq S_1$, then the porism

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1 \tag{2.2}$$

is valid for every A_1 on ℓ_1 and on ℓ_2.

Proof. Let O denote the intersection of ℓ_1 and ℓ_2 (see Figure 7). We define $d(X, Y) := \log(O, S_2, X, Y)$ as a signed projective distance on ℓ_2 with the properties

$$d(X, Y) = -d(Y, X) \text{ and } d(X, Y) + d(Y, Z) = d(X, Z).$$

Observe that $(O, S_2, X, Y) = (S_1, S_2, X', Y')$ for an arbitrary point $P \notin \{O, S_1\}$ on ℓ_1.

It follows that

$$\log \prod_{i=1}^{n} (S_1, S_2, P_{2i}, P_{2i+1}) = \sum_{i=1}^{n} d(A_{2i}, A_{2i+2}) = 0$$

which proves (2.1).

On the other hand, suppose that (2.1) holds. Then there is a closing $2n$-gon

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{\bar{P}_{2n}} A_1$$

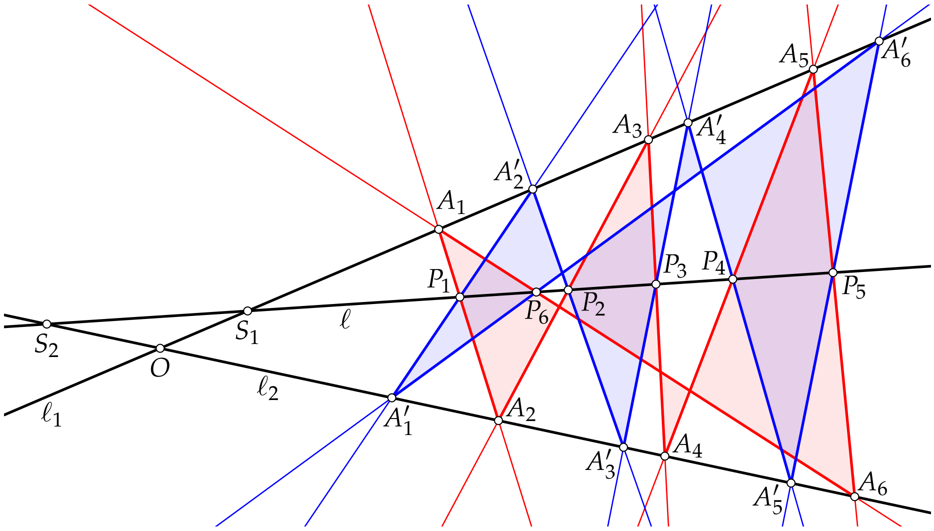

Figure 6. Illustration for Theorem 4 with $n = 3$.

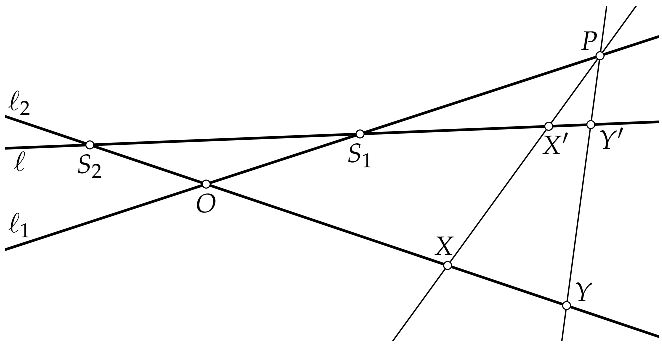

Figure 7. Projective distance on ℓ_2.

for which (2.1) holds with a point $\bar{P}_{2n}$ on ℓ in place of P_{2n}. It follows that $\bar{P}_{2n} = P_{2n}$ and we are done. □

Remark 5. Notice that condition (2.1) is automatic for $n = 2k + 1$ if $P_i = P_{i+n}$ for $i = 1, \dots, n$. In particular, the original Pappus porism 1 follows immediately with $n = 3$.

Now we address the case when ℓ, ℓ_1 and ℓ_2 are concurrent.

Theorem 6. *Let ℓ, ℓ_1, ℓ_2 be lines through a point S. Let $A_1, A_2, \dots, A_{2n}$ be points that lie alternately on ℓ_1 and ℓ_2 with intersection points P_i of ℓ and A_iA_{i+1} (indices read cyclically). Then we have*

$$\sum_{i=1}^{2n} \frac{(-1)^i}{OP_i} = 0, \tag{2.3}$$

where OP_i denotes the oriented euclidean distance between O and P_i. Vice versa, if (2.3) is valid for points $S, P_1, \dots, P_{2n}$ and ℓ_1, ℓ_2 are lines through S then the porism

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1 \tag{2.4}$$

is valid for every A_1 on ℓ_1 and on ℓ_2.

Proof. Using a projective map we may assume that in projective coordinates $O = (0, 0, 1)$, $A_{2i+1} = (c_{2i+1}, c_{2i+1}, 1)$, $A_{2i} = (c_{2i}, -c_{2i}, 1)$, and $\ell = (0, 1, 0)$. It is then easy to compute $P_i = (\frac{2c_i c_{i+1}}{c_i + c_{i+1}}, 0, 1)$ and (2.3) follows immediately. Observe that condition (2.3) is invariant under projective transformation and hence the claim follows. □

Remark 7. Notice that condition (2.3) is automatic for $n = 2k + 1$ if $P_i = P_{i+n}$ for $i = 1, \dots, n$. In particular, also in this case the original Pappus porism 1 follows with $n = 3$.

3. Generalization to non-collinear reversion points

It turns out that Pappus-like porisms also exist for points P_i which are not collinear.

Theorem 8. *Let ℓ_1, ℓ_2 be lines and $P_1, \dots, P_{2n-1}$ points not incident with ℓ_1 and ℓ_2. Then there exists a unique point P_{2n} such that the porism*

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1 \tag{3.1}$$

is valid for every A_1 on ℓ_1.

Proof. Consider three points A_1, A_1', A_1'' on ℓ_1 and their cross ratio with the intesection point O of ℓ_1 and ℓ_2. Then we define the points

$$A_i \xrightarrow{P_i} A_{i+1}, \quad A_i' \xrightarrow{P_i} A_{i+1}', \quad A_i'' \xrightarrow{P_i} A_{i+1}''$$

for $i = 1, 2, \dots, 2n - 1$. Observe that

$$(O, A_1, A_1', A_1'') = (O, A_2, A_2', A_2'') = \dots = (O, A_{2n}, A_{2n}', A_{2n}'').$$

Hence the lines $A_1 A_{2n}$, $A_1' A_{2n}'$ and $A_1'' A_{2n}''$ are concurrent in a point P_{2n}. Since this point does not depend on the position of A_1'', we are done. □

Observe that the proof shows how the point P_{2n} can easily be constructed with ruler alone. In particular, we have a second formulation of Theorem 8:

Corollary 9. *Let ℓ_1, ℓ_2 be two lines. If*

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1 \tag{3.2}$$

holds for two different points A_1 and A_1' on ℓ_1, then this porism holds for all A_1 on ℓ_1.

Notice, that the chain in (3.2) will not close in general if we start with a point A_1 on ℓ_2 instead on ℓ_1.

We now want to give a quantitative version of Theorem 8:

Theorem 10. *Let ℓ_1, ℓ_2 be lines intersecting in O, and $P_1, \ldots, P_{2n}$ be points not incident with ℓ_1 and ℓ_2 such that the porism*

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1 \tag{3.3}$$

is valid for every A_1 on ℓ_1. Then, for the lines $g_i = OP_i$ we have

$$\prod_{i=1}^{n} (\ell_1, \ell_2, g_{2i-1}, g_{2i}) = 1. \tag{3.4}$$

Proof. We start with the case $n = 2$: Choose A_1 as intersection of the lines P_1P_2 and P_3P_4, and A'_1 as intersection of P_1P_4 and ℓ_1: See Figure 8. Then the points $P_1, P_2, P_3, P_4, A_1, A'_2$ form a complete quadrilateral. We consider the pairs g_1, g_3 and g_2, g_4 as pairs of conjugate lines. This defines a line involution with respect to which $\ell_1 = OA_1$ and $\ell_2 = OA'_2$ are conjugate lines (see Chasles [1, Note X, § 34, (28), p. 317]). In particular, we have

$$(\ell_1, \ell_2, g_1, g_2) = (\ell_2, \ell_1, g_3, g_4)$$

which implies $(\ell_1, \ell_2, g_1, g_2)(\ell_1, \ell_2, g_3, g_4) = 1$.

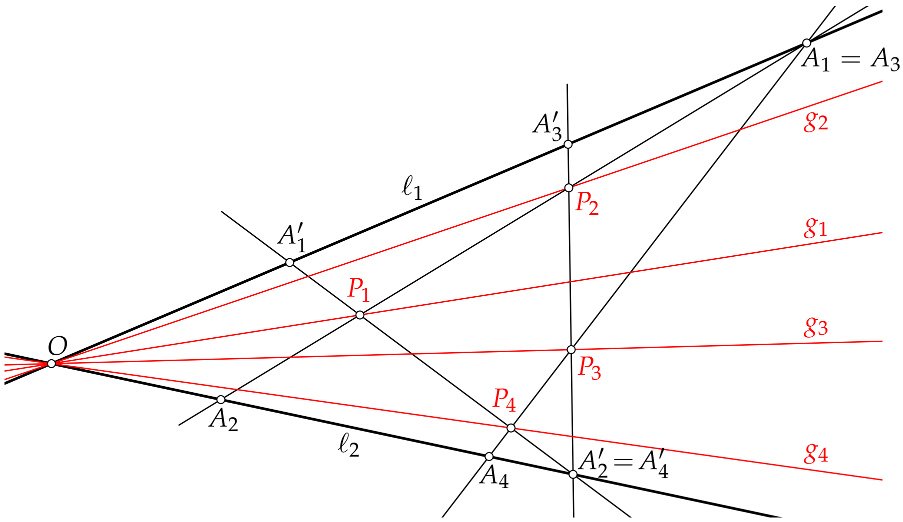

Figure 8. Illustration for Theorem 10 with four reversion points.

Now we proceed by induction: Assume that (3.4) is established for $n-1$. According to Theorem 8, there exists a point $\tilde{P}_{2n-2}$ such that the porism

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{2n-2}} A_{2n-3} \xrightarrow{P_{2n-3}} A_{2n-2} \xrightarrow{\tilde{P}_{2n-2}} A_1$$

is valid. Denote $\tilde{g}_{2n-2} = O\tilde{P}_{2n-2}$. By the induction hypothesis, we have

$$(\ell_1, \ell_2, g_1, g_2)(\ell_1, \ell_2, g_3, g_4) \ldots (\ell_1, \ell_2, g_{2n-3}, \tilde{g}_{2n-2}) = 1. \tag{3.5}$$

On the other hand we have

$$A_1 \xrightarrow{\tilde{P}_{2n-2}} A_{2n-2} \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1$$

with

$$(\ell_1, \ell_2, \tilde{g}_{2n-2}, g_{2n-2})(\ell_1, \ell_2, g_{2n-1}, g_{2n}) = 1 \tag{3.6}$$

from the base case. In the product of (3.5) and (3.6) the terms involving $\tilde{g}_{2n-2}$ cancel out and we obtain (3.4). □

For later use we retain the following consequence of Theorem 10:

Remark 11. If the porism

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}^{(1)}} A_1$$

is valid for all points A_1 on ℓ_1, and if the porism

$$A_1' \xrightarrow{P_1} A_2' \xrightarrow{P_2} A_3' \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1}' \xrightarrow{P_{2n-1}} A_{2n}' \xrightarrow{P_{2n}^{(2)}} A_1'$$

is valid for all points A_1' on ℓ_2 then the points $O, P_{2n}^{(1)}$ and $P_{2n}^{(2)}$ are collinear.

The next theorem explains, in which situation the porism in (3.1) is valid for A_1 on ℓ_1 *and* on ℓ_2. The first two cases, (I) and (II), are degenerate cases, the interesting generic case is (III):

Theorem 12. *Let $P_1, \dots, P_{2n-2}$ be given points not incident with two lines ℓ_1, ℓ_2. Then the following is true:*

(I) If $P_1, \dots, P_{n-2}$ have the closing property

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-4}} A_{2n-3} \xrightarrow{P_{2n-3}} A_{2n-2} \xrightarrow{P_{2n-2}} A_1 \tag{3.7}$$

for all A_1 on ℓ_1 and on ℓ_2, then $P_1, \dots, P_{2n-2}, P_{2n-1}, P_{2n}$ have the closing property

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{P_{2n}} A_1 \tag{3.8}$$

for all A_1 on ℓ_1 and on ℓ_2 if and only if $P_{2n-1} = P_{2n}$ is an arbitrary point in the plane not incident with ℓ_1 and ℓ_2.

(II) If $P_1, \dots, P_{2n-2}$ have the closing property (3.7) for all A_1 on ℓ_1 but not for all A_1 on ℓ_2, then the porism (3.8) cannot hold for all A_1 on ℓ_2.

(III) If $P_1, \dots, P_{2n-2}$ do not have the closing property (3.7) both for A_1 on ℓ_2 and for A_1 on ℓ_2, then there is a line ℓ with the property that for an arbitrary point P_{2n-1} on ℓ not incident with ℓ_1 and ℓ_2 there is a unique point P_{2n} on ℓ such that $P_1, \dots, P_{2n-2}, P_{2n-1}, P_{2n}$ have the closing property (3.8) for all A_1 on ℓ_1 and on ℓ_2. No other choice for P_{2n-1} and P_{2n} is possible.

Proof. (I) Clearly, if (3.7) holds for all A_1 on ℓ_1, then (3.8) holds for all A_1 on ℓ_1 if and only if $P_{2n-1} = P_{2n}$. And the same is true for ℓ_2 in place of ℓ_1.

(II): As in case (I), if (3.7) holds for all A_1 on ℓ_1, then (3.8) holds for all A_1 on ℓ_1 if and only if $P_{2n-1} = P_{2n}$. But if $P_{2n-1} = P_{2n}$, then (3.8) cannot hold for an A_1 on ℓ_2 for which (3.7) fails.

(III): Suppose first that the porism (3.8) is valid for all A_1 on ℓ_1 and ℓ_2. Consider three $2n$-gons $A_1, \ldots, A_{2n}$, $A'_1, \ldots, A'_{2n}$, and $A''_1, \ldots, A''_{2n}$ as in Figure 9. Then, by the Pappus Theorem applied to the hexagons

$$H_1 = A_1 A'_{2n-1} A'_{2n} A'_1 A_{2n-1} A_{2n} \text{ and } H_2 = A_1 A''_{2n-1} A''_{2n} A''_{2n-1} A_{2n}$$

it follows that P_{2n-1} and P_{2n} must lie on the common Pappus line ℓ of the two hexagons. The lines $A_1 A'_{2n-1}$ and $A'_1 A_{2n-1}$ determine the point X on ℓ, and the lines $A_1 A''_{2n-1}$ and $A''_1 A_{2n-1}$ determine the point $Y \neq X$ on ℓ. Thus, ℓ is determined by $P_1, \ldots, P_{2n-2}$ by the construction above. Hence P_{2n-1} must be chosen on ℓ and once P_{2n-1} is fixed, the location of P_{2n} on ℓ follows.

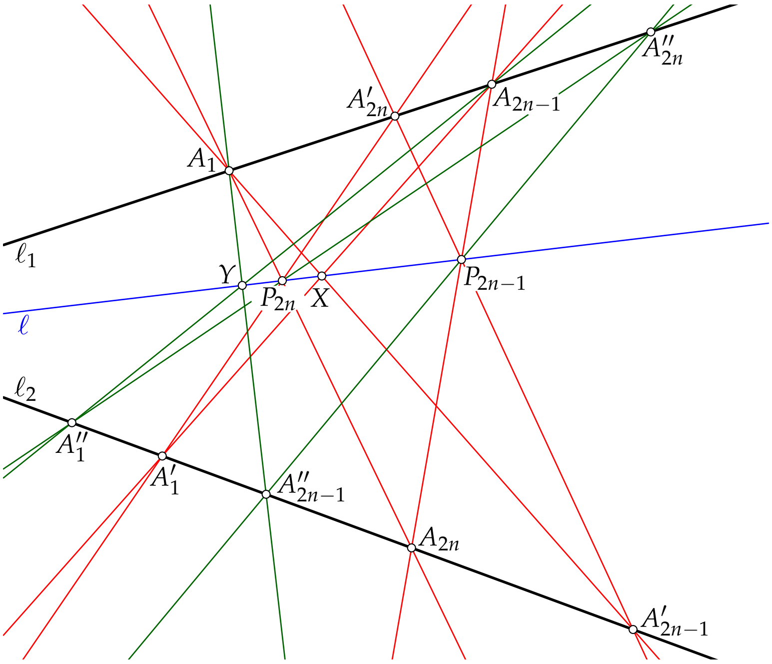

Figure 9. The blue line ℓ is determined by the points $P_1, \ldots, P_{2n-2}$.

Now the converse: Let us first choose an arbitrary point A_1 on ℓ_1 for which (3.7) does not close. Then, we choose two different points A'_1, A''_1 on ℓ_2 for which (3.7) does not close either. This defines the polygonal chains

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{2n-3}} A_{2n-2} \xrightarrow{P_{2n-2}} A_{2n-1}$$
$$A'_1 \xrightarrow{P_1} A'_2 \xrightarrow{P_2} A'_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{2n-3}} A'_{2n-2} \xrightarrow{P_{2n-2}} A'_{2n-1}$$
$$A''_1 \xrightarrow{P_1} A''_2 \xrightarrow{P_2} A''_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{2n-3}} A''_{2n-2} \xrightarrow{P_{2n-2}} A''_{2n-1}$$

Then the intersection X of the lines $A_1 A'_{2n-1}$ with the line $A'_1 A_{2n-1}$ and the intersection Y of the lines $A_1 A''_{2n-1}$ with the line $A''_1 A_{2n-1}$ are different and define a line ℓ (see Figure 9). Choose a point P_{2n-1} on ℓ such that the line $A_{2n-1} P_{2n-1}$ intersects ℓ_2 in a point A_{2n}. The line $A_{2n} A_1$ then intersects ℓ in a point P_{2n}. Hence, $A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n}$ and $A_{2n} \xrightarrow{P_{2n}} A_1$.

Now we consider the intersection A'_{2n} of the lines $A'_{2n-1}P_{2n-1}$ and A'_1P_{2n} and the intersection A''_{2n} of the lines $A''_{2n-1}P_{2n-1}$ and A''_1P_{2n}. By the Braikenridge-Maclaurin Theorem applied to the hexagon $H_1 = A_1A'_{2n-1}A'_{2n}A'_1A_{2n-1}A_{2n}$ it follows that $A'_{2n} \in \ell_1$. Similarly, by considering the hexagon $H_2 = A_1A''_{2n-1}A''_{2n}A''_1A_{2n-1}A_{2n}$ it follows that $A''_{2n} \in \ell_1$. Hence, the porism

$$A'_1 \xrightarrow{P_1} A'_2 \xrightarrow{P_2} A'_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A'_{2n-1} \xrightarrow{P_{2n-1}} A'_{2n} \xrightarrow{P_{2n}} A'_1$$

is valid for the initial points A'_1 and A''_1 on ℓ_2, and hence by Corollary 9 for all initial points on ℓ_2. On the other hand, we know from Theorem 8 that a unique point $\tilde{P}_{2n}$ exists with the property that

$$A_1 \xrightarrow{P_1} A_2 \xrightarrow{P_2} A_3 \xrightarrow{P_3} \dots \xrightarrow{P_{2n-2}} A_{2n-1} \xrightarrow{P_{2n-1}} A_{2n} \xrightarrow{\tilde{P}_{2n}} A_1$$

holds for all A_1 on ℓ_1. But P_{2n} has this property for the original initial point A_1 we started with. Hence, $P_{2n}, \tilde{P}_{2n}$ and A_1 are collinear. In view of Remark 11 this implies that $P_{2n} = \tilde{P}_{2n}$ and we are done. □

Notice that the proof gives a concrete and simple construction for the points P_{2n-1} and P_{2n}.

At the end of this section we considered the question whether porisms with more than two concurrent support lines are also possible. Indeed, the proof of Theorem 8 carries over to an arbitrary number of concurrent lines $\ell_1, \dots, \ell_k$, $k \geq 2$:

Theorem 13. *Let $\ell_1, \dots, \ell_k$ be lines (not necessarily distinct, but $\ell_k \neq \ell_1$) and $P_1, \dots, P_{k-1}$ points not incidents with the given lines. Then there exists a point P_k such that the porism*

$$A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} A_3 \xrightarrow[\ell_3 \quad \ell_4]{P_3} \dots \xrightarrow[\ell_{k-1} \quad \ell_k]{P_{k-1}} A_k \xrightarrow[\ell_k \quad \ell_1]{P_k} A_1$$

is valid for every A_1 on ℓ_1.

An example is shown in Figure 10.

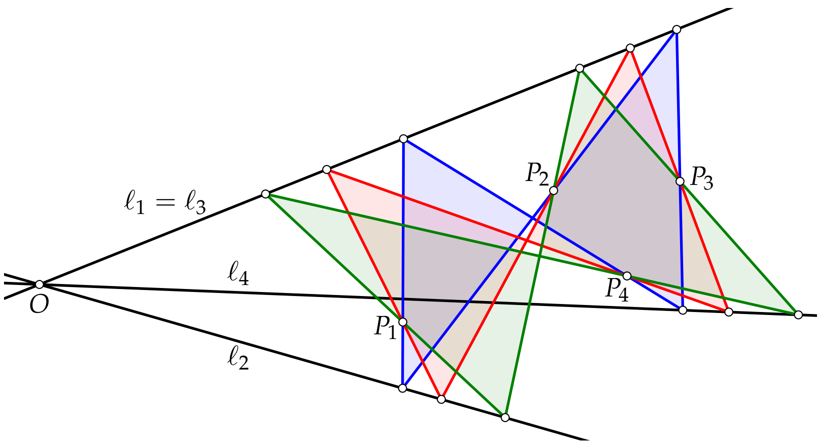

Figure 10. Illustration for Theorem 13: P_1, P_2, P_3 determine P_4.

The next theorem shows that one can construct a porism also by adding a suitable support line instead of a reversion point.

Theorem 14. *Let ℓ_1, ℓ_2 be lines which intersect in O and P_1, P_2, P_3 three collinear points not incident with ℓ_1 and ℓ_2. Then there exists a line ℓ_3 such that the porism*

$$A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} A_3 \xrightarrow[\ell_3 \quad \ell_1]{P_3} A_1$$

is valid for all A_1 on ℓ_1.

Proof. Choose A_1 on ℓ_1 and consider the point $A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2$. Denote the intersection of the lines A_1P_3 and A_2P_2 by A_3. Then the points $P_1, A_2, A_3, P_3, A_1, P_2$ form a complete quadrilateral (see Figure 11). In particular, if we consider the lines OP_2, ℓ_1 as conjugate

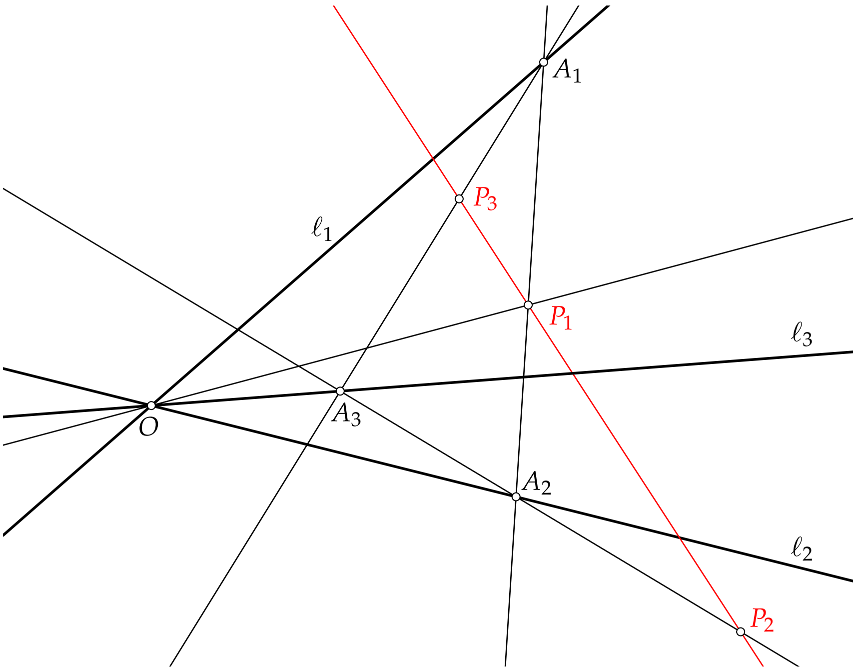

Figure 11. Proof of Theorem 14.

lines and also ℓ_2, OP_3, then these four lines define a line involution with respect to which the lines OA_3 is the conjugate line of OP_1. Hence, the line $\ell_3 = OA_3$ is determined by the lines $OP_1, OP_2, OP_3, \ell_1, \ell_2$, and the porism must close on this line. □

Notice that the previous theorem is not valid if P_1, P_2, P_3 are not collinear. However, it generalizes easily to an arbitrary number of lines and reversion points:

Theorem 15. *Let $\ell_1, \ldots, \ell_n$ be concurrent lines and $P_1, \ldots, P_n$ reversion points not incident with the given lines. Then there exists a line ℓ with the following property: For any point P_{n+1}*

on ℓ, not incident with the lines $\ell_1, \dots, \ell_n$, there is a line ℓ_{n+1} such that the porism

$$A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A_n \xrightarrow[\ell_n \quad \ell_{n+1}]{P_n} A_{n+1} \xrightarrow[\ell_{n+1} \quad \ell_1]{P_{n+1}} A_1$$

is valid for every A_1 on ℓ_1.

Proof. Consider three points A_1, A_1', A_1'' on ℓ_1 and the points

$$A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A_n$$

$$A_1' \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2' \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A_n'$$

$$A_1'' \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2'' \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A_n''$$

Then we have for the cross ratios $(O, A_1, A_1', A_1'') = (O, A_n, A_n', A_n'')$. Thus the points A_1A_n, $A_1'A_n'$ and $A_1''A_n''$ are concurrent in a point P. Now we are in the situation of Theorem 14: We can choose an arbitrary point P_{n+1} on the line ℓ through the points P and P_n and construct the line ℓ_{n+1} by a line inversion such that the porism closes. □

4. NON-COLLINEAR LINES

In this section we consider lines $\ell_1, \dots, \ell_n$ which are not collinear. Also in this case one can construct reversion porisms:

Theorem 16. *Let $\ell_1, \dots, \ell_n$ be lines, $\ell_1 \neq \ell_n, \ell_2$, and $P_1, \dots, P_{n-2}$ be points not on these lines. Then there exists a line ℓ such that one can choose an arbitrary point P_{n-1} on ℓ which then determines a point P_n such that the porism*

$$A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} A_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A_n \xrightarrow[\ell_n \quad \ell_1]{P_n} A_1$$

is valid for every A_1 on ℓ_1.

Proof. Let O_1 be the intersection of ℓ_1 and ℓ_2, and A_1, A_1', A_1'' points on ℓ_1 (see Figure 12). Consider the points

$$A_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} A_{n-1}$$

$$A_1' \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2' \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} A_{n-1}'$$

$$A_1'' \xrightarrow[\ell_1 \quad \ell_2]{P_1} A_2'' \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} A_{n-1}''$$

$$O_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} O_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} O_{n-1}$$

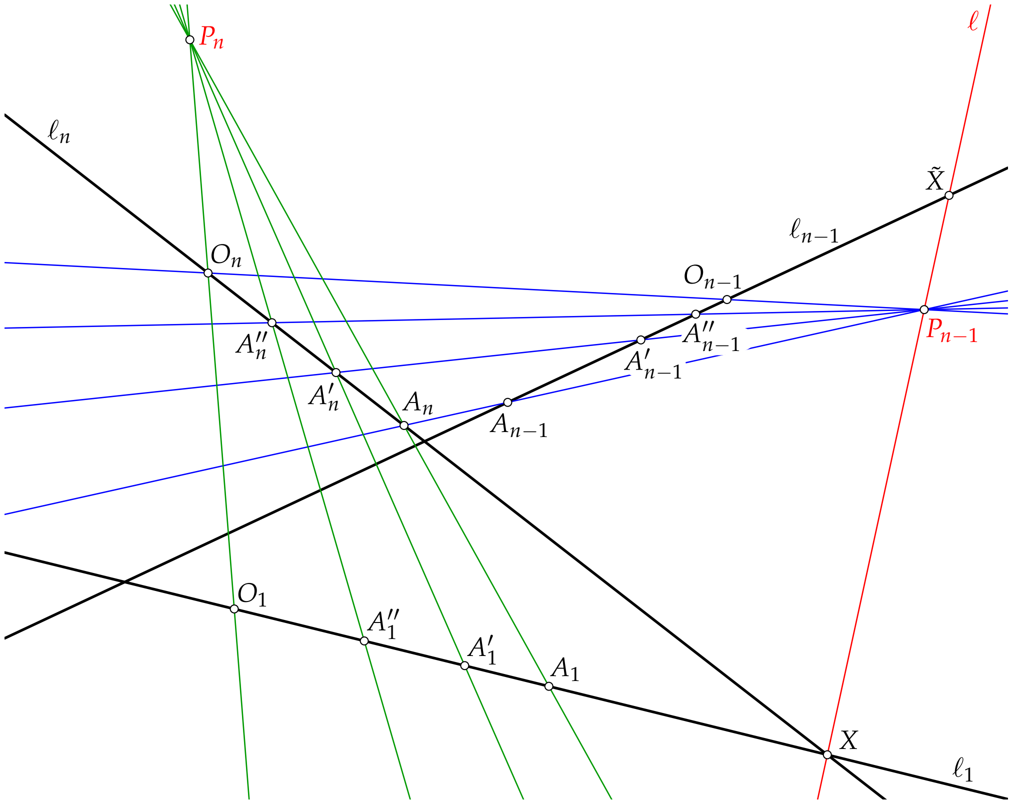

Figure 12. Proof of Theorem 16.

and observe that $(O_1, A_1, A'_1, A''_1) = (O_{n-1}, A_{n-1}, A'_{n-1}, A''_{n-1})$. Let X denote the intersection of ℓ_1 and ℓ_n. Then there is a unique point $\tilde{X}$ on ℓ_{n-1} with the property

$$(X, A_1, A'_1, A''_1) = (\tilde{X}, A_{n-1}, A'_{n-1}, A''_{n-1}).$$

Now let ℓ be the line joining X and $\tilde{X}$ and choose P_{n-1} on ℓ, not incident with ℓ_1 and ℓ_{n-1}. In particular, we have $\tilde{X} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X$. Consider the points

$$A_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A_n, \quad A'_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A'_n, \quad A''_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} A''_n,$$

$$\text{and } O_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} O_n.$$

The cross ratio of four of the points $A_{n-1}, A'_{n-1}, A''_{n-1}, O_{n-1}, \tilde{X}$ equals the cross ratio of the four corresponding image points A_n, A'_n, A''_n, O_n, X. In particular, the lines $A_1 A_n$, $A'_1 A'_n$, $A''_1 A''_n, O_1 O_n$ are concurrent in a point P_n. □

An Example is shown in Figure 13.

We remark that a particularly simple case occurs if ℓ_n passes through O_1: Then $\tilde{X}$ is the point O_{n-1}.

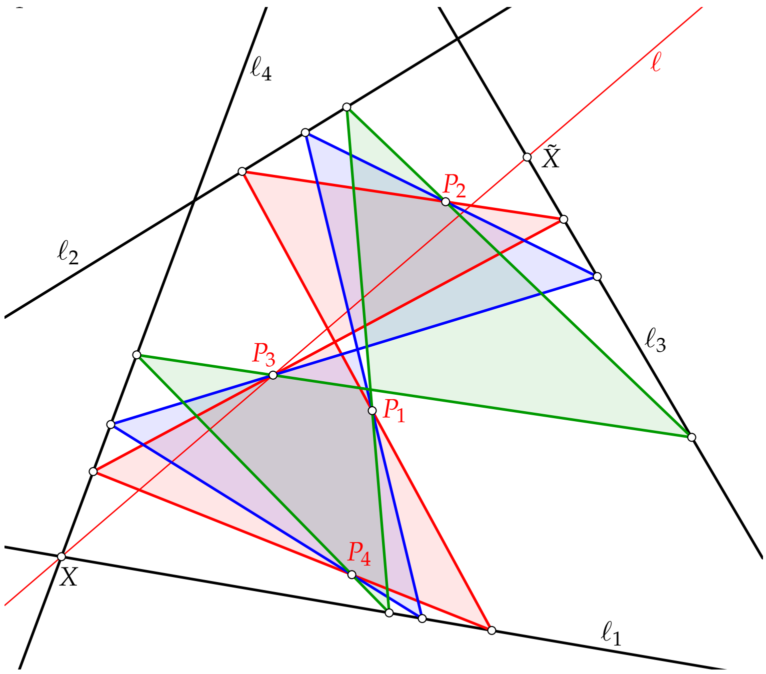

Figure 13. Theorem 16 for $n = 4$: The lines $\ell_1, \ldots, \ell_4$ and the reversion points P_1, P_2 are given and determine the line ℓ. Then the point P_3 can be freely chosen on ℓ and P_4 is uniquely determined thereafter.

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