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On Periodic Billiard Trajectories in Obtuse Triangles*

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Abstract. In 1775, J. F. de Tuschis a Fagnano observed that in every acute triangle, the orthoptic triangle represents a periodic billiard trajectory, but to the present day it is not known whether or not in every *obtuse* triangle a periodic billiard trajectory exists. The limiting case of right triangles was settled in 1993 by F. Holt, who proved that all right triangles possess periodic trajectories. The same result had appeared independently in the Russian literature in 1991, namely in the work of G. A. Gal'perin, A. M. Stepin, and Y. B. Vorobets. The latter authors discovered in 1992 a class of obtuse triangles which contain particular periodic billiard paths. In this article, we review the above-mentioned results and some of the techniques used in the proofs and at the same time show for an extended class of obtuse triangles that they contain periodic billiard trajectories.

Key words. periodic trajectories, billiard trajectories, obtuse triangles

AMS subject classifications. 37D50, 28D10

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1. Introduction.

The investigation of billiard trajectories in a plane domain Q with boundary ∂Q consisting of finitely many smooth curves has a long history and touches upon questions of geometry, partial differential equations, physics, and ergodic theory (see, e.g., [1], [3], [5], [6], [8], [10], [14], [15]). One of the first results concerning the existence of periodic trajectories was obtained by Birkhoff in 1927 for the case when Q is smooth and convex: by applying a theorem of Poincaré he showed that in that case, for any $k > 0$ and $w < \frac{k}{2}$ such that k and w are relatively prime, there exist at least two periodic reflecting paths in Q with k reflections and winding number w (see [2] and Figure 1). For nonsmooth boundary curves the situation is drastically more difficult. Using Teichmüller theory, Masur recently proved for the case of polygons that in every rational polygon there exist infinitely many periodic billiard trajectories (see [11]). Here, a polygon is called rational if all its angles α_i are rational with respect to π, i.e., $\frac{\alpha_i}{\pi} \in \mathbb{Q}$.

In this paper, we investigate periodic trajectories in triangles. Let us recall the case when the triangle is acute: the simplest periodic trajectory in it is given by the orthoptic triangle, where the trajectory has three links and its reflection points are the bases of the altitudes in the triangle (see Figure 2). A nice proof of this fact is

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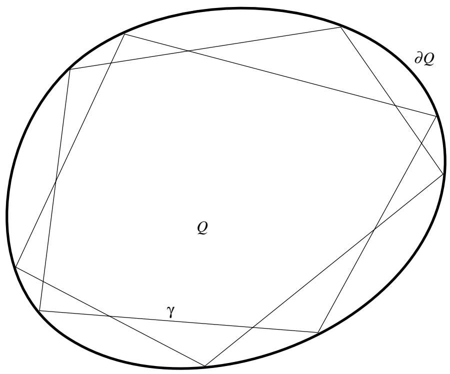

Fig. 1 *A periodic billiard trajectory γ in Q with nine reflections and winding number 2.*

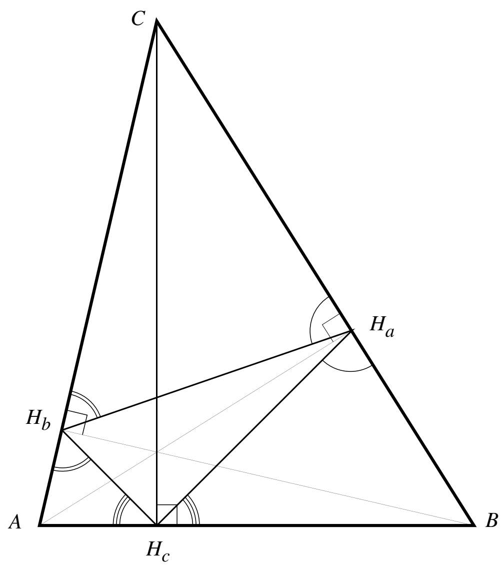

Fig. 2 *Orthoptic triangle.*

based upon the variational property of this trajectory, namely, that it is the shortest closed path through the triangle that touches all three sides. This idea goes back to K. H. A. Schwarz (1843–1921); see [12, Vol. 2, p. 344]. In [13, Vol. 2, p. 728] this proof was falsely attributed to Jacob Steiner (1796–1863). The first proof for the orthoptic triangle, however, goes back to the year 1775 and the Italian ordained priest and mathematician J. F. de Tuschis a Fagnano (see [4]). For right triangles the existence of periodic trajectories was proved by Holt in [9]. But the same result appeared earlier and independently in the Russian literature (see [7], [8]). Gal'perin, Stepin, and Vorobets discovered in [8] a class of obtuse triangles that contain stable periodic trajectories (see section 2). However, it is unknown whether periodic paths exist in *every* obtuse triangle. The aim of this article is to improve the results of Gal'perin, Stepin, and Vorobets using some new techniques for analyzing the structure of periodic orbits.

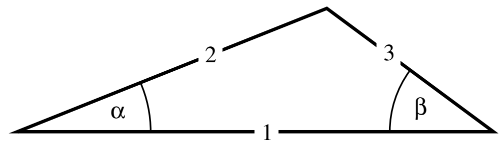

Fig. 3 *Angles and sides in an obtuse triangle.*

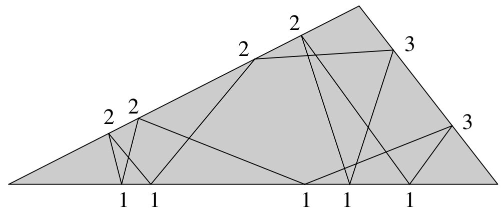

Fig. 4 *Periodic billiard path corresponding to the code* $(1, 3, 2, 1, 2, 1, 2, 1, 3, 1, 2)$.

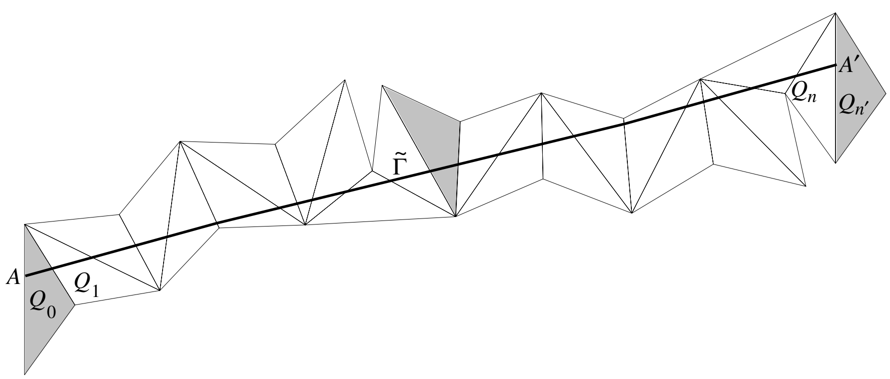

Fig. 5 *Straightened billiard trajectory.*

In order to describe our results, we parametrize the set of obtuse triangles as follows: starting with the long side of such a triangle we number the sides as 1, 2, and 3 proceeding clockwise. The angle between the sides 1 and 2 will be denoted by α, the other acute angle by β (see Figure 3). In this way the set of obtuse triangles modulo similarity is in one-to-one correspondence with the parameter set $\{(\alpha, \beta) : 0 < \alpha, 0 < \beta, \alpha + \beta < \frac{\pi}{2}\}$. We equip the set of triangles with the topology and measure inherited from the Euclidean topology and the Lebesgue measure on the parameter plane.

If we denote by $\Omega_n := \{1, 2, 3\}^{\{1,2,\ldots,n\}}$ the set of 123-sequences of length n and by $\omega_n := \Omega_n / D_n$ the set Ω_n modulo the dihedral group D_n, then we may associate with each periodic trajectory of length n a unique element in ω_n (called the *code* of the trajectory) by considering the numbers of the sides in the order in which they are visited by the reflecting path (see Figure 4).

2. Stable Trajectories. First we need to describe the method of straightening a billiard trajectory (see also [8]). We fix a periodic trajectory Γ with n reflections in a polygon Q and consider the polygons $Q_0 = Q, Q_1, Q_2, \dots, Q_{n'}$ (where $n' = n$ if n is even and $n' = 2n$ if n is odd) obtained by successive reflections of Q in those sides $i_1, i_2, \dots, i_{n'}$ in which the billiard particle moving along Γ is reflected. Since n' is even, $Q_{n'}$ has the same orientation as Q_0 and hence is obtained from Q_0 by a translation (see Figure 5). The straightened trajectory $\tilde{\Gamma}$ joins "identical" points A and A' in Q_0 and $Q_{n'}$, respectively. Hence, every line parallel to $\tilde{\Gamma}$ with a distance smaller than some positive ε from $\tilde{\Gamma}$ gives raise to a periodic trajectory in Q of length n if n is even, and of length $2n$ if n is odd. The set of those trajectories is called the *pencil of parallel periodic trajectories*.

A natural way to classify periodic trajectories in a polygon is to consider the stability of the path with respect to small deformations of the polygon (see also [8]).

Definition 1. *A periodic trajectory in a polygon Q is called stable if it is not destroyed by any small deformation of Q, and it is called odd (even) if it has an odd (even) number of links.*

Now we want to describe how one can determine whether for a given triangle (α, β) a periodic trajectory with code $(a_1, \dots, a_n)/D_n \in \omega_n$ exists. The reflection with respect to the side $j \in \{1, 2, 3\}$ is given by an affine mapping $F_j \colon \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto A_j x + c_j$, where $A_j \in O(2, \mathbb{R})$ has determinant $\det(A_j) = -1$. For $i = 1, \dots, n$ consider the affine transformation

$$G_i(x) := F_{a_i} \circ \cdots \circ F_{a_n} \circ F_{a_1} \circ \cdots \circ F_{a_{i-1}}(x) = B_i x + d_i$$

with $B_i \in O(2, \mathbb{R})$. G_i induces a mapping $\tilde{G}_i$ on the set of straight lines of the plane. Let us calculate the fixed point set of $\tilde{G}_i$: each line g may be described by $g = \{x \in \mathbb{R}^2 \,:\, n \cdot x + d = 0\}$ for a unit vector $n \in \mathbb{R}^2$. Then g is fixed under $\tilde{G}_i$ if in addition there holds $n \cdot (B_i x + d_i) + d = 0$, i.e., if $B_i^t n \cdot x + n \cdot d_i + d = 0$. Both equations $n \cdot x + d = 0$ and $B_i^t n \cdot x + n \cdot d_i + d = 0$ hold simultaneously if $(B_i^t n, n \cdot d_i + d) \| (n, d)$, i.e., if n is an eigenvector of B_i^t (and thus of B_i). If $\lambda \neq 1$ is an eigenvalue of B_i^t for the eigenvector n of B_i^t, we obtain that $d = \frac{1}{\lambda - 1} n \cdot d_i$ and we get a unique fixed line g. It follows that real fixed lines only exist if the spectrum of B_i is real and therefore a subset of $\{-1, 1\}$. Hence, we have the following three cases.

Case 1. $\operatorname{spec} B_i = \{1\}$. It follows that $B_i = \mathrm{id}$ and the (infinite) set of fixed lines is $\{n \cdot x + d = 0 \,:\, d \in \mathbb{R} \text{ arbitrary}, n \neq 0 \text{ with } n \cdot d_i = 0\}$.

Case 2. $\operatorname{spec} B_i = \{-1\}$. It follows that $B_i = -\mathrm{id}$ and the (infinite) set of fixed lines is $\{n \cdot x + d = 0 \,:\, n \text{ an arbitrary unit vector}, d = -\frac{1}{2} n \cdot d_i\}$.

Case 3. $\operatorname{spec} B_i = \{-1, 1\}$. It follows that B_i describes a reflection with respect to a line. Then $\tilde{G}_i$ has at least the fixed line.

Case 3a. $\{n \cdot x + d = 0 \,:\, n \text{ an eigenvector of } B_i \text{ to eigenvalue } -1, d = -\frac{1}{2} n \cdot d_i\}$.

Case 3b. If $n' \cdot d_i = 0$ for the eigenvector n' to eigenvalue 1, then the (infinite) family $\{n' \cdot x - d = 0, d \in \mathbb{R}\}$ consists of fixed lines.

Note, that, simultaneously for each $i = 1, \dots, n$, we have one of the cases 1, 2, 3a, or 3b.

The following theorem gives a sufficient condition for a trajectory to be stable (for a necessary and sufficient condition, see [8]).

Theorem 1. *Every odd trajectory is stable.*

Proof. We carry out the proof only for an obtuse triangle (for arbitrary polygons it is analogous). Let Γ be a periodic path with an odd number n of reflections and with code word $(a_1, \dots, a_n)$. Let us assume that the Euclidean coordinates are chosen such that side 1 of the obtuse triangle Q is the line segment with end points $(0, 0)$, $(1, 0)$ and such that the triangle lies in the upper half-plane. We consider the situation

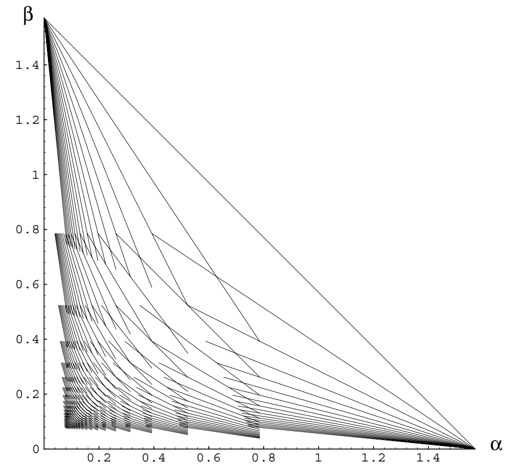

Fig. 6 *Perpendicular trajectories.*

by straightening out Γ to $\tilde{\Gamma}_i$ as described above by reflecting the triangle first with respect to side a_i. Observe that $\tilde{\Gamma}_i$ is a fixed line of the associated affine mapping $B_i x + d_i$. Since $\det B_i = -1$, we are in Case 3 above. $\tilde{\Gamma}_i$ joins corresponding points A and A'' of Q and Q_n such that the angles between $\tilde{\Gamma}_i$ and the corresponding sides are equal. Q and Q_n have different orientation, and hence every line parallel to $\tilde{\Gamma}_i$ violates the condition to join corresponding points. Hence we are in Case 3a (3b does not occur). Now, observe that the associated affine mapping $B_i x + d_i$ depends continuously on the angles α and β. The same is true for the eigenvectors of B_i and hence for the fixed line $n \cdot x - \frac{1}{2} n \cdot d_i = 0$. Now, every reflection point lies in the interior of the sides 1, 2, or 3 of the triangle Q. Since the coordinates of the top of Q also depend continuously on α and β, this translates to a set of *strict* inequalities with both sides depending continuously on α and β, and the assertion follows. □

3. Perpendicular Trajectories. In [8] it was proved that every rational n-gon contains at least $\frac{n}{2}$ pencils of periodic perpendicular trajectories, i.e., trajectories having the property that two of the reflection angles are right angles. Here, we consider one-dimensional families of obtuse triangles which contain perpendicular trajectories.

THEOREM 2. *Every triangle whose angles α and β satisfy $k\alpha + n\beta = \pi$, for some $k, n \in \mathbb{N}$, contains a pencil of perpendicular periodic trajectories with $2(n + k - 1)$ reflections, provided that the following holds:*

$$\frac{\pi}{2k}\left(1 - \frac{1}{n-1}\right) < \alpha < \frac{\pi}{2k}\left(1 + \frac{1}{k-1}\right),$$

with the convention $\frac{1}{0} = \infty$.

Remark 1. For $n = k = 2$, we recover Holt's result [9], and for $n = 2$ and k even, we recover the results in [8, section 4B]. If $n = 1$ or $k = 1$, then the triangles are acute; otherwise they are obtuse. The set of obtuse triangles that satisfy the above condition is indicated in Figure 6. For a discussion of perpendicular trajectories in the triangles with $k\alpha = n\beta < \frac{\pi}{2}$, see [8].

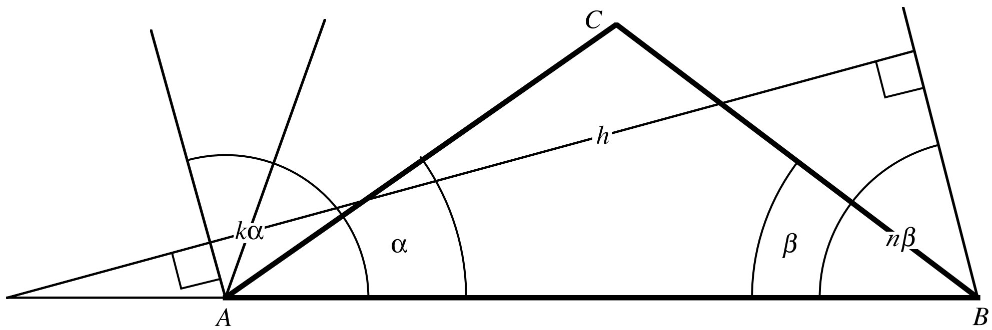

Fig. 7 *Perpendicular trajectories.*

Proof. From Figure 7 it is clear that in a triangle ABC whose angles α and β satisfy the equation

$$k\alpha + n\beta = \pi, \tag{3.1}$$

the line h generates a perpendicular periodic trajectory if either

(i) $k\alpha \le \frac{\pi}{2}$ and $\frac{\pi}{2} - k\alpha < \beta$ or

(ii) $k\alpha > \frac{\pi}{2}$ and $\frac{\pi}{2} - n\beta < \alpha$.

Using (3.1) we find that $\frac{\pi}{2} - k\alpha < \beta$ is equivalent to $(n-2)\pi < 2k\alpha(n-1)$ and that $\frac{\pi}{2} - n\beta < \alpha$ is equivalent to $(k-2)\pi < 2n\beta(k-1)$, which completes the proof. ∎

4. Generators of Stable Trajectories. First we define the notion of a generator.

DEFINITION 2. *An infinite subset of $\bigcup_{n\in\mathbb{N}} \omega_n$ is called a generator if every word of this subset is the code of a stable periodic orbit in an obtuse triangle.*

The first and only generator known up to now was discovered in [8].

THEOREM 3. $\left\{132(12)^{k-1}(13)^{l-1} \,:\, k,l \in \mathbb{N}, k+l>2\right\}$ *is a generator.*

Here, $(a_1a_2\dots a_i)^n$ denotes the concatenation of n copies of the word $a_1a_2\dots a_i$. Since the proof of this theorem is instructive, we give the main idea for the reader's convenience.

Proof. Since every word of the specified form has odd length, according to Theorem 1 it suffices to show for each pair k, l the existence of a triangle that contains a reflection path with the corresponding code. Such a triangle ABC is shown in Figure 8. The angles α and β are chosen such that $(k+\frac{1}{2})\alpha = (l+\frac{1}{2})\beta = \frac{\pi}{2}$. It is easy to see that each side in the orthoptic triangle $H_aH_bH_c$ of the "supertriangle" ABN intersects the sides AC and BC. Hence, by reflecting back the orthoptic triangle, we get a reflection path in ABC with the required code. ∎

Remark 2. The crucial condition that the line H_aH_b lies below the point C (and hence that the triangle ABC contains a periodic reflection path with code $132(12)^{k-1}(13)^{l-1}$) can be expressed algebraically by the following inequality (see [8]):

$$\big(1-\cot(k\alpha)\tan\alpha\big)\big(1-\cot(l\beta)\tan\beta\big) > 1-\tan\alpha\tan\beta. \tag{4.1}$$

The set of triangles satisfying this condition is indicated in the parameter plane in Figure 9.

4.1. Generating Generators. The idea of the method to construct new generators is to start with a *degenerate situation* and to show that a suitable and suitably

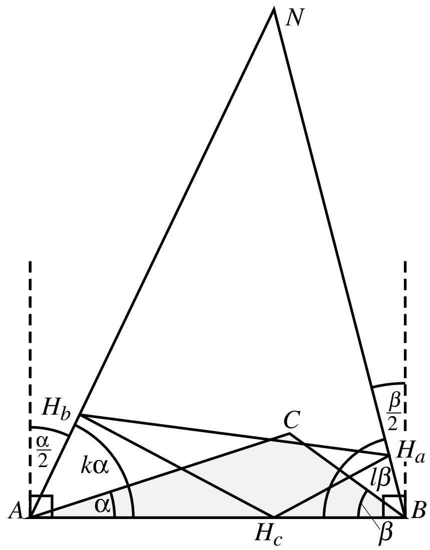

Fig. 8 *Generator* $132(12)^{k-1}(13)^{l-1}$.

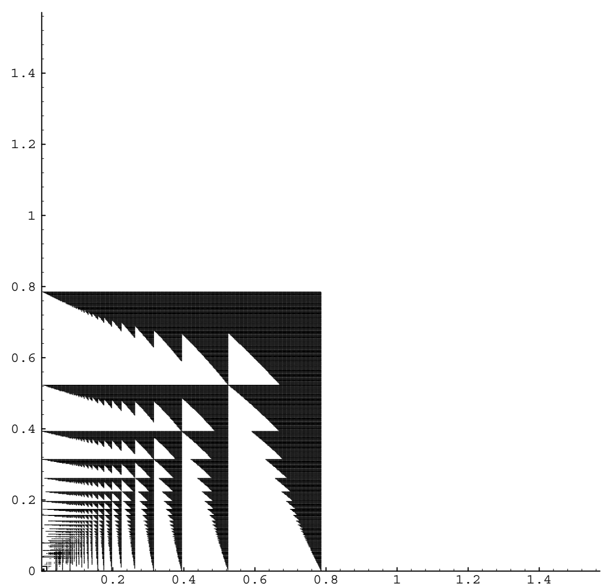

Fig. 9 *Generator* $132(12)^{k-1}(13)^{l-1}$.

small perturbation of the angles α and β leads to a *stable* trajectory. We consider two cases.

First we send a billiard particle along the basis AB (see Figure 10) of the obtuse triangle (gray in Figure 10) to the right. Instead of reflecting the particle at the sides of the triangle, we reflect the triangle and in this way straighten the trajectory. This leads to the situation shown in Figure 10. We assume (for the moment) that the

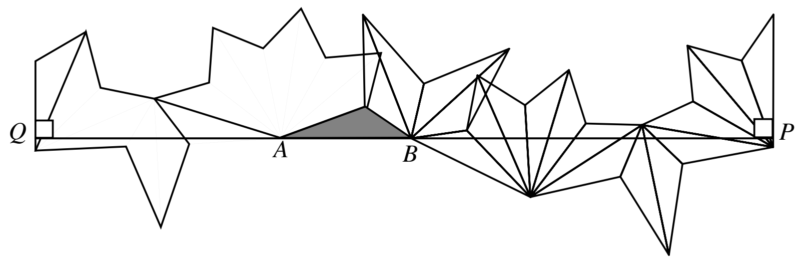

Fig. 10 *Degenerate path along the basis.*

particle does not pass through edges of reflected triangles (except for the point B) and that after a certain number of reflections, the following reflection would occur perpendicular to the path. At this point P we reflect the *particle* (no longer the *triangle*) such that it now moves back on its path in the opposite direction. Eventually the particle reaches the basis of the original triangle, now moving to the left, and we straighten the path now to the left, again assuming that the particle does not hit any of the edges (except for the point A) and that after a certain number of reflections the following reflection would be perpendicular to the path. At this point Q we reflect the particle such that we finally obtain a closed degenerate reflection path. In the example in Figure 10, the sequence of reflections to the right is

$$\underbrace{3,1,3,1,3}_{N_1=5},\underbrace{2,1,2,1,2,1}_{N_2=6},\underbrace{3,1,3,1}_{N_3=4},\underbrace{2,1,2}_{N_4=3}$$

and the sequence of reflections to the left is

$$\underbrace{2,1,2,1,2,1,2,1}_{n_1=8},\underbrace{3,1,3,1}_{n_2=4},\underbrace{2,1}_{n_3=2}.$$

The situation is fully described by the numbers N_i and n_j. In general, we formally write the code word $\langle n_k, n_{k-1}, \ldots, n_2, n_1 | \mathcal{B} | N_1, N_2, \ldots, N_l \rangle$ to describe the configuration; "$\mathcal{B}$" means that we started moving along the "basis."

Second, we start with a particle moving along side 2 in the "northeast" direction (see Figure 11). Again we assume that the particle does not hit edges of the reflected triangle except for the point C and that after a certain number of reflections the particle is reflected perpendicular to its path (in point P in Figure 11) such that it moves now in the opposite direction and eventually passes along the side 2 of the original triangle in the "southwest" direction. Now we again reflect the triangle by straightening the path until we meet another perpendicular reflection (in point Q in Figure 11) such that the path is finally closed. In the example in Figure 11 the sequence of reflections on the path from A to P is

$$\underbrace{3,1}_{N_1=2},\underbrace{2,1}_{N_2=2},\underbrace{3,1}_{N_3=2},\underbrace{2}_{N_4=1}$$

and the sequence of reflections from C to Q is

$$\underbrace{1,2,1,2,1,2,1}_{n_1=7},\underbrace{3}_{n_2=1}.$$

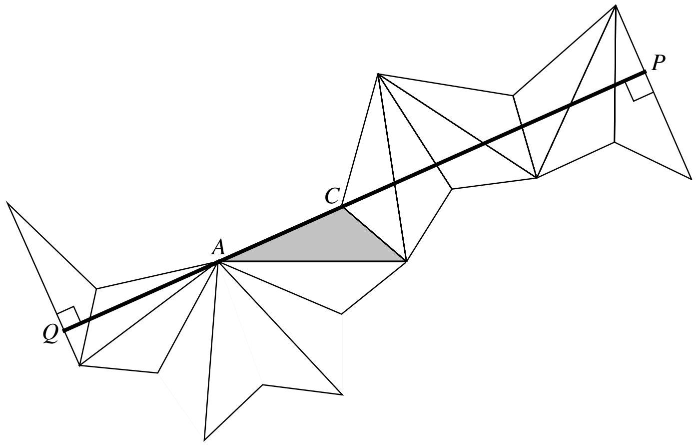

Fig. 11 *Degenerate path along side* 2.

The code for this situation is $\langle n_k, n_{k-1}, \ldots, n_2, n_1|\mathcal{L}|N_1, N_2, \ldots, N_l\rangle$; "$\mathcal{L}$" means that we have a path which is degenerate with respect to a "leg." Because of the symmetry there is no need to consider degenerate paths with respect to side 3, and notice also that $\langle n_k, n_{k-1}, \ldots, n_2, n_1|\mathcal{B}|N_1, N_2, \ldots, N_l\rangle$ and $\langle N_l, , \ldots, N_2, N_1|\mathcal{B}|n_1, n_2, \ldots, n_k\rangle$ are symmetric.

By a systematic analysis of some of these degenerate situations we obtain the following theorem.

THEOREM 4. *The following sets are generators.*

(a) *Corresponding to* $\langle n_1|\mathcal{B}|2N_1, N_2\rangle$*:*

$$\Big\{132(12)^{k-1}(13)^m(12)^n(13)^{m-1} \;:\; m,n,k\in\mathbb{N},\quad n<k\le n\,\tfrac{2m+1}{2m-1},\quad \tfrac{2m+n}{2m-1}\le k\Big\}.$$

(b) *Corresponding to* $\langle n_1|\mathcal{B}|2N_1+1, N_2\rangle$*:*

$$\Big\{132(12)^{k-1}(13)^m2(12)^{n-1}3(13)^{m-2} \;:\; m,n,k\in\mathbb{N},\quad 2\le n\le k,\quad 2\le m<\tfrac{3k-n}{2(k-n)}\Big\}.$$

(c) *Corresponding to* $\langle n_2, 2n_1|\mathcal{B}|2N_1, N_2\rangle$*:*

$$\Big\{132(12)^{l-1}(13)^k(12)^l(13)^m(12)^n(13)^{m-1} \;:\; k,l,m,n\in\mathbb{N}, n<2l,\quad \max\big(\tfrac{k+n+2l+kn}{4l-2},\tfrac{k(l+n)}{2l},\tfrac{nk}{2l-n}\big)<m<\min\big(\tfrac{k(l+n+\frac12)}{2l-1},\tfrac{n(k+\frac12)+l}{2l-n}\big)\Big\}.$$

(c) *Corresponding to* $\langle n_2, 2n_1|\mathcal{B}|2N_1+1, N_2\rangle$*:*

$$\Big\{132(12)^{l-1}(13)^k(12)^l(13)^m2(12)^{n-1}3(13)^{m-2} \;:\; k,l,m,n\in\mathbb{N}, n<2l,\quad \max\big(\tfrac{k(3l-n)}{2l},\tfrac{kn}{2l-n},\tfrac{3k+6l-n-kn}{4l-2}\big)<m<\min\big(\tfrac{k(3l+\frac32-n)}{2l-1},\tfrac{kn+3l-\frac{n}{2}}{2l-n}\big)\Big\}.$$

(e) *Corresponding to* $\langle n_2, 2n_1+1|\mathcal{B}|2N_1+1, N_2\rangle$*:*

$$\Big\{132(12)^{l-1}3(13)^{k-1}2(12)^{l-1}(13)^m2(12)^{n-1}3(13)^{m-2} \,:\, k,l,m,n\in\mathbb{N},\quad 3\big(l-\tfrac{1}{2}\big) < \tfrac{4lm-kn}{2m-k} < 3l,\;\; 3\big(m-\tfrac{1}{2}\big) < \tfrac{4lm-kn}{2n-l} < 3m\Big\}.$$

(f) *Corresponding to* $\langle n_1|\mathcal{L}|2N_1, N_2\rangle$*:*

$$\Big\{132(12)^k(13)^m(12)^{n+1}(13)^{m-1} : k,m,n\in\mathbb{N},\quad 2+n < k(2k-n),\;\; 2km(2k+2) < (k+1)(m+1)(2k+2+n)\Big\}.$$

(g) *Corresponding to* $\langle n_2, 2n_1+1|\mathcal{L}|N_1\rangle$*:*

$$\Big\{132(12)^k(13)^l(12)^k(13)^{m-1} : k,l,m\in\mathbb{N}, \max\big(2m, 2k(l-m)\big) < l+m\Big\}.$$

(h) *Corresponding to* $\langle n_2, 2n_1|\mathcal{L}|N_1\rangle$*:*

$$\Big\{132(12)^k3(13)^{l-1}2(12)^{k-1}(13)^{m-1} : k,l,m\in\mathbb{N},\quad m < l < 3m,\;\; 2k(l-m) < l+m\Big\}.$$

(i) *Corresponding to* $\langle n_2, 2n_1+1|\mathcal{L}|2N_1, N_2\rangle$*:*

$$\Big\{132(12)^k(13)^l(12)^k(13)^m(12)^{n+1}(13)^{m-1} : k,l,m,n\in\mathbb{N},\quad (m+l)(n+1) < 2km < l(k+1+n),\;\; 2k < \tfrac{(n+1)(l+m+1)}{m-1},\quad \max\big(l(2k+1+2n), ln\big) < 2m(2k+1)\Big\}.$$

(j) *Corresponding to* $\langle n_2, 2n_1|\mathcal{L}|2N_1, N_2\rangle$*:*

$$\Big\{132(12)^k3(13)^{l-1}2(12)^{k-1}(13)^m(12)^{n+1}(13)^{m-1} : k,l,m,n\in\mathbb{N},\quad l < 6m,\;\; 2m(k-1) < l(k+n),\;\; l(2k+1+2n) < 2m(2k+1),\quad m(3n+5-2k) < l(n+1) < m(3n+5-2k)+2k+1+3n\Big\}.$$

Remark 3. Notice that $\langle n_1|\mathcal{B}|N_1\rangle$ as well as $\langle n_1|\mathcal{L}|N_1\rangle$ leads to the generator described in Theorem 3.

Proof. As a prototype we prove (a). The remaining cases are proved similarly.

In order to simplify the calculation, we define $k := n_1+1 \geq 1$, $n := N_2+1 \geq 1$, $m := N_1 \geq 1$. For the angles $\varepsilon_1, \varepsilon_2$ in Figure 12, we have

$$\varepsilon_1 = k\alpha,$$
$$\varepsilon_2 = 2m\beta - n\alpha.$$

If we choose $\alpha_0 = \frac{\pi}{2k}$ and $\beta_0 = \frac{\pi}{4km}(k+n)$, we get $\varepsilon_1 = \varepsilon_2 = \frac{\pi}{2}$. Since

$$\det\begin{pmatrix} k & 0 \\ -n & 2m \end{pmatrix} = 2km > 0,$$

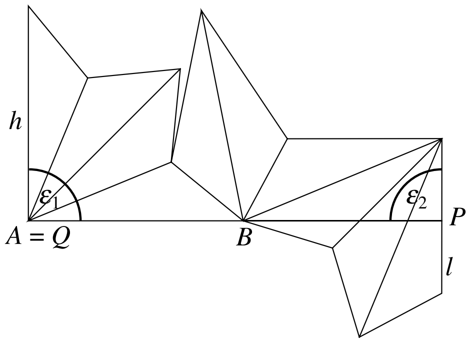

Fig. 12 $\langle n_1|\mathcal{B}|2N_1, N_2\rangle$.

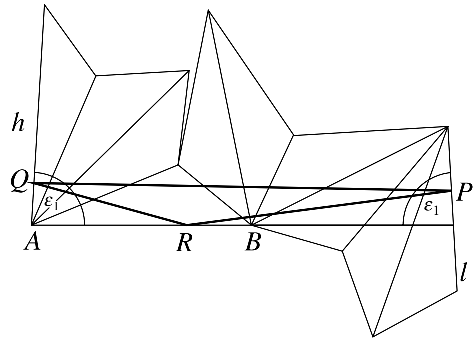

Fig. 13 *Orthoptic triangle PQR.*

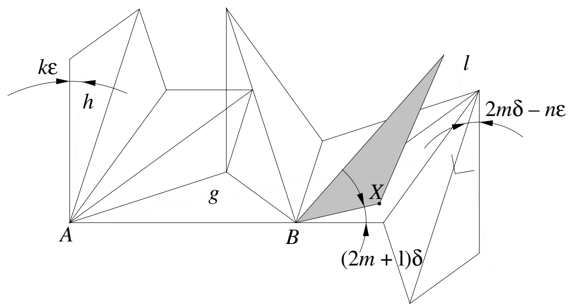

Fig. 14 *Limiting situation.*

the Jacobian of the mapping $(\alpha, \beta) \mapsto (\varepsilon_1, \varepsilon_2)$ is regular. Therefore, a suitable perturbation of the values α_0 and β_0 allows one to obtain any prescribed value of ε_1 and ε_2 near $\frac{\pi}{2}$. Observe that

(1) $n < k \implies 2m\beta_0 < \pi$,

(2) $n(2m+1) \geq k(2m-1) \implies (2m+1)\beta_0 \geq \pi$,

(3) $k(2m-1) \geq 2m + n \implies \alpha_0 + \beta_0 \leq \frac{\pi}{2}$.

If we assume for the moment that we have ">" in (2) and (3) above, then we can choose (α, β) near (α_0, β_0) such that ε_1 and ε_2 are simultaneously slightly smaller than $\frac{\pi}{2}$, $\alpha + \beta < \frac{\pi}{2}$, and the orthoptic triangle PQR in the supertriangle with sides AB, h, and l is contained in the reflected triangles in such a way that the point R is between A and B (see Figure 13). Hence, the orthoptic triangle generates a periodic reflection path in the original triangle.

Now, we allow an "=" in (2). Figure 14 shows the reflected triangles with the critical values (α_0, β_0). By choosing $\beta = \beta_0 - \delta$ and $\alpha = \alpha_0 - \varepsilon$ (with $\varepsilon, \delta > 0$ sufficiently small), we also obtain a situation in which the orthoptic triangle generates a periodic orbit and $\alpha + \beta < \frac{\pi}{2}$. In fact, if we choose ε and δ such that $(m - \frac{1}{2})\delta > n\varepsilon$, we have $2m\delta - n\varepsilon > \frac{1}{2}(2m+1)\delta$, and hence the critical point X (in the perturbed situation, gray in Figure 14) lies below the perpendicular line g from A to l. (The case "=" in (3) is handled similarly.)

Thus, the conditions (1)–(3) imply that in the neighborhood of the corresponding degenerate situation, there is a nondegenerate periodic orbit in an obtuse triangle. This completes the proof of (a). ∎

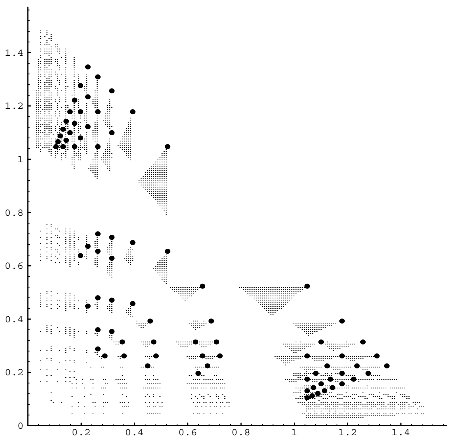

Fig. 15 *Parameter set of generator* (a).

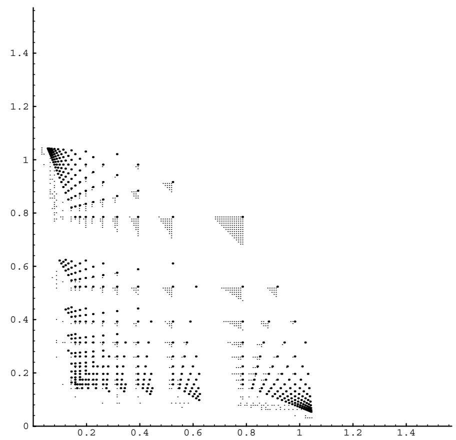

Fig. 16 *Parameter set of generator* (b).

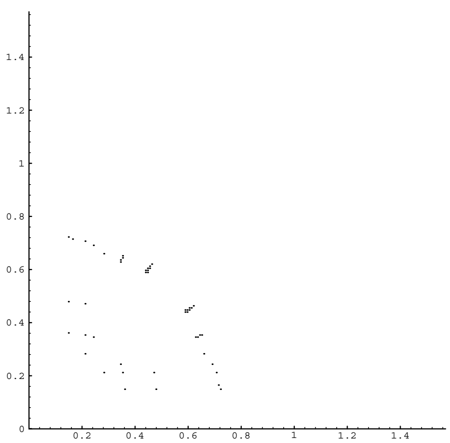

Fig. 17 *Parameter set of generator* (c).

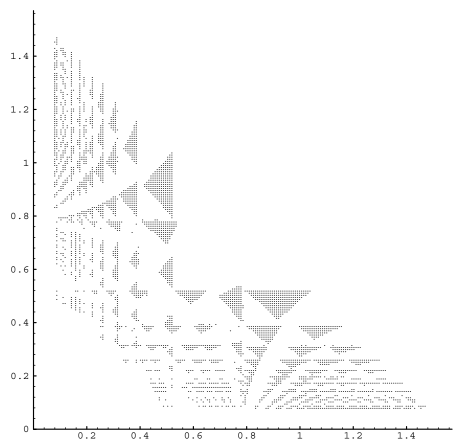

Fig. 18 *Parameter set of generator* (f).

The parameter sets of triangles which contain a periodic reflection path corresponding to some of the generators are indicated in Figures 15–20. The fat dots represent some of the points (α_0, β_0) where the paths degenerate.

4.2. Obtuse Triangles with Periodic Trajectories. In addition to the generators presented in the previous section, we checked systematically for all code words in ω_n of length n less than or equal to 25, whether or not they generate a reflection path in some obtuse triangle. Figure 21 shows the open parameter set of all triangles that we found by this method and by using the generators of the previous section. Numerically, this adds up to about 50% of all obtuse triangles. Sharp estimates are, of course, very hard to derive. However, the hope would be to detect enough generators to fill the whole parameter set.

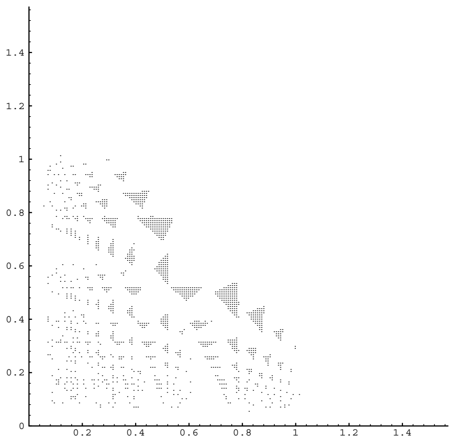

Fig. 19 *Parameter set of generator* (g).

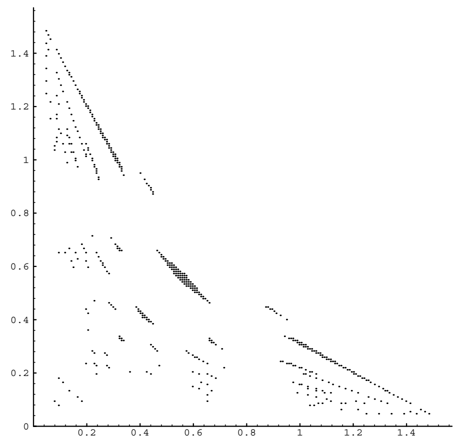

Fig. 20 *Parameter set of generator* (h).

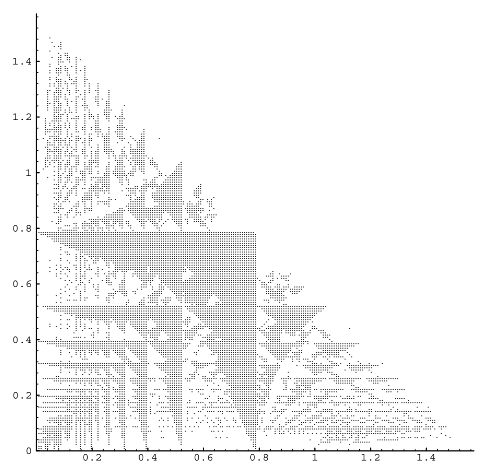

Fig. 21 *Parameter set.*

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