Linear Algebra and its Applications 609 (2021) 413–441

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Magic sets for polynomials of degree n

Lorenz Halbeisen, Norbert Hungerbühler *, Salome Schumacher [1]

ARTICLE INFO

Article history:
Received 20 August 2020
Accepted 16 September 2020
Available online 23 September 2020
Submitted by R. Brualdi

MSC:
26C05
11C20

Keywords:
Sets of range uniqueness
Polynomials
Magic sets
Unique range

ABSTRACT

Let $\mathcal{P}_n$ be the family of all real, non-constant polynomials with degree at most n and let $\mathcal{Q}_n$ be the family of all complex, non-constant polynomials with degree at most n. A set $S \subseteq \mathbb{R}$ is called a set of range uniqueness (SRU) for a family $\mathcal{F} \in \{\mathcal{P}_n, \mathcal{Q}_n\}$ if for all $f, g \in \mathcal{F}$, $f[S] = g[S] \Rightarrow f = g$. And S is called a magic set if for all $f, g \in \mathcal{F}$, $f[S] \subseteq g[S] \Rightarrow f = g$. In this paper we will show that there are magic sets for $\mathcal{P}_n$ and $\mathcal{Q}_n$ of size s for every $s \geq 2n+1$. However, there are no SRUs of size at most $2n$ for $\mathcal{P}_n$ and $\mathcal{Q}_n$. Moreover we will show that SRUs and magic sets are not the same by giving examples of SRUs for $\mathcal{P}_2$ and $\mathcal{P}_3$ that are not magic.

1. Introduction

Let $\mathcal{F}$ be a set of functions with a common domain X and a common range Y. A set $S \subseteq X$ is called a set of range uniqueness (SRU) for $\mathcal{F}$ if the following holds: For all $f, g \in \mathcal{F}$

$$f[S] = g[S] \Rightarrow f = g.$$

Furthermore, S is called a magic set for $\mathcal{F}$ if for all $f, g \in \mathcal{F}$

* Corresponding author.
E-mail address: norbert.hungerbuehler@math.ethz.ch (N. Hungerbühler).

1 Partially supported by Swiss National Science Foundation (SNF) grant 200021_178851.

https://doi.org/10.1016/j.laa.2020.09.026

$$f[S] \subseteq g[S] \Rightarrow f = g.$$

Note that every magic set is also an SRU. The existence of magic sets and SRUs has already been studied for several families of functions:

- Berarducci and Dikranjan proved in [1] that under the continuum hypothesis (CH) there exists a magic set for the family $C^n(\mathbb{R})$ of all nowhere constant, continuous functions. Halbeisen, Lischka and Schumacher showed in [6] that we can weaken the requirement by replacing CH by the assumption that the union of less than continuum many meager sets is meager, i.e. $\text{add}(\mathcal{M}) = \mathfrak{c}$. However, the existence of a magic set for $C^n(\mathbb{R})$ is not provable in ZFC as Ciesielski and Shelah proved in [3].
- In [2], Burke and Ciesielski proved that SRUs always exist for the family of all Lebesgue-measurable functions on $\mathbb{R}$.
- In [4], Diamond, Pomerance and Rubel constructed SRUs for the family $C^\omega(\mathbb{C})$ of entire functions.
- In [5] the authors of this paper proved that there exist SRUs for the family $\mathcal{P}_n$ of all real, non-constant polynomials of degree at most n of size $2n + 1$ but none of size $2n$.

In this paper we consider magic sets for the family $\mathcal{P}_n$ of all real, non-constant polynomials of degree at most n and for the family $\mathcal{Q}_n$ of all complex, non-constant polynomials of degree at most n. We will show that there exist no SRUs, and therefore also no magic sets, of size at most $2n$ for $\mathcal{P}_n$ and $\mathcal{Q}_n$. Then we will give examples of SRUs for $\mathcal{P}_2$ and $\mathcal{P}_3$ that are not magic. And finally we will answer one of the open questions in [5] and show that for every $s \geq 2n + 1$ there is a magic set of size s for the families $\mathcal{P}_n$ and $\mathcal{Q}_n$.

2. There are no SRUs of size at most $2n$ for $\mathcal{P}_n$

In [5] we have already shown that there are no SRUs of size $2n$: For points $x_0 < x_1 < \cdots < x_{2n}$ we constructed two functions $f, g \in \mathcal{P}_n$ such that $f = 1 - g$ and

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

for all $1 \leq i < n$. In a similar way we can prove that there are no SRUs of size $2n - 1$:

Lemma 1. *There are no SRUs of size* $2n - 1$.

Proof. Let $0 < x_1 < x_2 = x_3 < x_4 < \cdots < x_{2n}$. As in [5] define

$$Y^n := \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid y_i \in \{x_{2i-1}, x_{2i}\} \text{ for all } 1 \leq i \leq n\}$$

and

$$A_n = A_n(x_1, x_2, \ldots, x_{2n}) = \begin{pmatrix} x_1 + x_2 & x_1^2 + x_2^2 & \ldots & x_1^n + x_2^n \\ x_3 + x_4 & x_3^2 + x_4^2 & \ldots & x_3^n + x_4^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{2n-1} + x_{2n} & x_{2n-1}^2 + x_{2n}^2 & \ldots & x_{2n-1}^n + x_{2n}^n \end{pmatrix}$$

For all $y_1, y_2, \ldots, y_n \in \mathbb{R}$ let

$$V_n(y_1, y_2, \ldots, y_n) = \begin{pmatrix} y_1 & y_1^2 & \ldots & y_1^n \\ y_2 & y_2^2 & \ldots & y_2^n \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_n^2 & \ldots & y_n^n \end{pmatrix}.$$

By [5, Lemma 23] we have that

$$\begin{aligned} \det(A_n(x_1, x_2, x_3, \ldots, x_{2n})) &= \sum_{(y_1, y_2, \ldots, y_n) \in Y^n} \det(V_n(y_1, y_2, \ldots, y_n)) \\ &= \sum_{\substack{(y_1, y_2, \ldots, y_n) \in Y^n \\ y_1 \neq y_2}} \det(V_n(y_1, y_2, \ldots, y_n)) > 0, \end{aligned}$$

because $\det(V_n(y_1, y_2, \ldots, y_n)) > 0$ whenever $|\{y_1, y_2, \ldots, y_n\}| = n$. So, as in [5] we can conclude that there are functions $f, g \in \mathcal{P}_n$ with

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

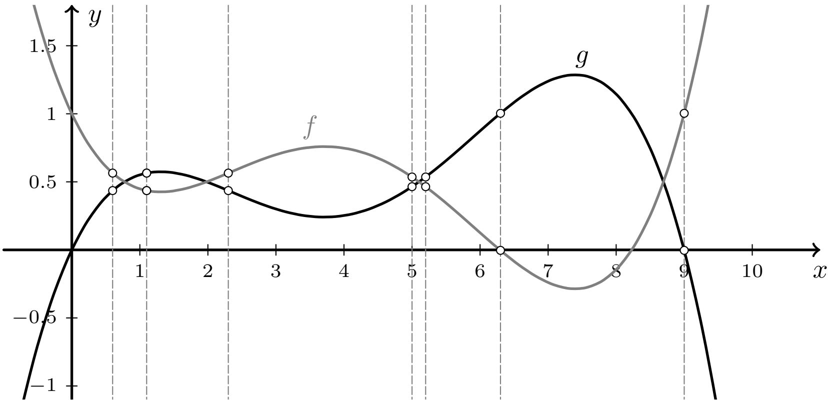
and therefore, there does not exist an SRU of size $2n - 1$. $\square$

Remark 2. The polynomials f and g we constructed in [5] and in Lemma 1 have degree n. To see this, note that for all $1 \leq i \leq n$ we have that

$$(f - g)(x_{2i-1}) = -(f - g)(x_{2i}).$$

By the intermediate value theorem, $(f - g)(x)$ has at least n pairwise different zeros. Since $f - g \not\equiv 0$ and since by construction $f - g$ has degree at most n, it follows that $\deg(f - g) = n$. By construction $f - g = 1 - 2g$. Therefore, $\deg(f) = \deg(g) = n$.

Example 3. Let $S := \left\{\frac{3}{5}, \frac{11}{10}, \frac{23}{10}, 5, \frac{26}{5}, \frac{63}{10}, 9\right\}$. In the following picture we can see two polynomials f and g of degree 4 with $f[S] = g[S]$ but $f \neq g$. These polynomials indicate that S is not an SRU for $\mathcal{P}_4$.



Proposition 4. *There does not exist an SRU of size less than* $2n - 1$.

Proof. Let $1 \leq s < 2n - 1$. Let $x_1 < x_2 < \cdots < x_s$. We want to show that $S := \{x_1, x_2, \ldots, x_s\}$ is not an SRU for $\mathcal{P}_n$.

Case 1: s is an even number.

Choose $\{x_{s+1}, x_{s+2}, \ldots, x_{2n}\} \subseteq \mathbb{R}$ with $x_s < x_{s+1} < x_{s+2} < \cdots < x_{2n}$. By [5, Lemma 23] we can find two functions $f, g \in \mathcal{P}_n$ with

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

for all $1 \leq i \leq n$. Therefore we have that

$$f[S] = g[S] \text{ and } f[\{x_{s+1}, x_{s+2}, \ldots, x_{2n}\}] = g[\{x_{s+1}, x_{s+2}, \ldots, x_{2n}\}].$$

So S is not an SRU for $\mathcal{P}_n$.

Case 2: s is an odd number.

Choose $\{x_{s+1}, x_{s+2}, \ldots, x_{2n-1}\} \subseteq \mathbb{R}$ with $x_s < x_{s+1} < x_{s+2} < \ldots x_{2n-1}$. By [5, Lemma 23] we can find two functions $f, g \in \mathcal{P}_n$ with

$$f[S] = g[S] \text{ and } f[\{x_{s+1}, x_{s+2}, \ldots, x_{2n-1}\}] = g[\{x_{s+1}, x_{s+2}, \ldots, x_{2n-1}\}].$$

So S is not an SRU for $\mathcal{P}_n$. □

3. There are no SRUs of size at most $2n$ for $\mathcal{Q}_n$

We define $\mathcal{Q}_n$ to be the set of all non-constant polynomials of degree at most n with complex coefficients. Let $S := \{x_1, x_2, \ldots, x_{2n}\} \subseteq \mathbb{C}$ be a set of cardinality $2n$. Our goal is to find two polynomials $f, g \in \mathcal{Q}_n$ with $f[S] = g[S]$ but $f \neq g$. By rotating the set S around the origin of the complex plane we can assume without loss of generality that all real parts of the points in S are pairwise different. By renaming the elements in the set, we can assume that

$$\mathrm{Re}(x_1) < \mathrm{Re}(x_2) < \cdots < \mathrm{Re}(x_{2n}).$$

Define

$$Y^n := \{(y_1, y_2, \ldots, y_n) \in \mathbb{C}^n \mid y_i \in \{x_{2i-1}, x_{2i}\} \text{ for all } 1 \leq i \leq n\}$$

and let π_n be the set of all permutations of $\{1, 2, \ldots, n\}$. By translating the set S to the right in the complex plane we can also assume that for all $(y_1, y_2, \ldots, y_n) \in Y^n$, all $M_0 \subseteq \{1, 2, \ldots, n\}$ and all $M_1 \subseteq [\{1, 2, \ldots, n\}]^2$ (where $[\{1, 2, \ldots, n\}]^2$ is the family of all 2-element subsets of $\{1, 2, \ldots, n\}$)

$$\begin{aligned} \left| \prod_{k \in M_0} \mathrm{Im}(y_k) \prod_{\substack{1 \leq i < j \leq n \\ \{i,j\} \in M_1}} \big(\mathrm{Im}(y_j) - \mathrm{Im}(y_i)\big) \right| &\leq \\ &\leq \frac{1}{2^n 2^{\binom{n}{2}}} \prod_{k \in M_0} \mathrm{Re}(y_k) \prod_{\substack{1 \leq i < j \leq n \\ \{i,j\} \in M_1}} \big(\mathrm{Re}(y_j) - \mathrm{Re}(y_i)\big). \end{aligned} \tag{1}$$

We will show that there are $f, g \in \mathcal{Q}_n$ with

$$f(x_{2i}) = g(x_{2i-1}) \text{ and } f(x_{2i-1}) = g(x_{2i})$$

for all $1 \leq i \leq n$. The two polynomials will have the form

$$g(x) = \sum_{j=1}^{n} b_j x^j \text{ with } b_j \in \mathbb{C} \text{ for } j = 1, 2, \ldots, n$$

and

$$f(x) = 1 - g(x).$$

In order to prove that such polynomials f and g exist we have to show that the following linear equation is solvable:

$$\underbrace{\begin{pmatrix} x_1 + x_2 & x_1^2 + x_2^2 & \ldots & x_1^n + x_2^n \\ x_3 + x_4 & x_3^2 + x_4^2 & \ldots & x_3^n + x_4^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{2n-1} + x_{2n} & x_{2n-1}^2 + x_{2n}^2 & \ldots & x_{2n-1}^n + x_{2n}^n \end{pmatrix}}_{=:A_n} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

To do this we have to show that $\det(A_n) \neq 0$ for every $n \in \mathbb{N}^*$. By [5, Lemma 23] we have that

$$\det(A_n) = \sum_{(y_1,\dots,y_n)\in Y^n} \det(V_n(y_1, y_2, \dots, y_n)),$$

where

$$V_n(y_1, y_2, \dots, y_n) = \begin{pmatrix} y_1 & y_1^2 & \cdots & y_1^n \\ y_2 & y_2^2 & \cdots & y_2^n \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_n^2 & \cdots & y_n^n \end{pmatrix}.$$

Note that

$$\det(V_n(y_1, y_2, \dots, y_n)) = \Big(\prod_{k=1}^{n} y_k\Big)\Big(\prod_{1\le i<j\le n} (y_j - y_i)\Big).$$

In particular we have that

$$\mathrm{Re}(\det(V_n(y_1, \dots, y_n))) = \Big(\prod_{k=1}^{n} \mathrm{Re}(y_k)\Big)\Big(\prod_{1\le i<j\le n} (\mathrm{Re}(y_j) - \mathrm{Re}(y_i))\Big) + R$$

where each summand in R has the form

$$\pm \prod_{k\in M_0} \mathrm{Im}(y_k) \prod_{\substack{1\le i<j\le n \\ \{i,j\}\in M_1}} \big(\mathrm{Im}(y_j) - \mathrm{Im}(y_i)\big) \prod_{k\notin M_0} \mathrm{Re}(y_k) \prod_{\substack{1\le i<j\le n \\ \{i,j\}\notin M_1}} \big(\mathrm{Re}(y_j) - \mathrm{Re}(y_i)\big)$$

where $M_0 \subseteq \{1, 2, \dots, n\}$ and $M_1 \subseteq [\{1, 2, \dots, n\}]^2$ are not both empty and $M_0 \cup M_1$ has even cardinality. Since R contains less than $2^n 2^{\binom{n}{2}}$ summands and by (1) we have that

$$\mathrm{Re}(\det(V_n(y_1, y_2, \dots, y_n))) > 0$$

for all $(y_1, \dots, y_n) \in Y^n$. Therefore

$$\det(A_n(y_1, y_2, \dots, y_n)) \neq 0.$$

This implies that there are $f, g \in \mathcal{Q}_n$ with $f[S] = g[S]$ but $f \neq g$.
Note that as in Section 2 we can show that there are no SRUs for $\mathcal{Q}_n$ of size less than $2n$.

4. SRUs that are not magic for $\mathcal{P}_2$ and $\mathcal{P}_3$

Let $\mathcal{P}_n$ be the family of all real, non-constant polynomials of degree at most n. For the family $\mathcal{P}_1$ magic sets and SRUs are the same: Let $S \subseteq \mathbb{R}$ and assume that S is an SRU. If S were not magic, there were two functions $f, g \in \mathcal{P}_1$ with $f[S] \subseteq g[S]$ but $f \neq g$.

But since f and g are both bijective, it follows that $f[S] = g[S]$ which then implies that $f = g$ because S is an SRU. But we assumed that $f \neq g$, which is a contradiction.
However, the following Lemmas show that magic sets and SRUs for $\mathcal{P}_2$ and $\mathcal{P}_3$ are not the same:

Lemma 5. *The set* $S := \left\{-2, -1, 2, \sqrt{8}, \sqrt{14 - \sqrt{8}}\right\}$ *is an SRU for* $\mathcal{P}_2$ *but not a magic set.*

Proof. The set S is not a magic set because for $f(x) := x^2$ and $g(x) := 2x^2 - x - 2$ we have that

$$f[S] = \left\{1, 4, 8, 14 - \sqrt{8}\right\} \subseteq \left\{1, 4, 8, 14 - \sqrt{8}, 26 - 4\sqrt{2} - \sqrt{14 - \sqrt{8}}\right\} = g[S].$$

On the other hand, we now show that $S = \{x_1, x_2, x_3, x_4, x_5\}$ is an SRU for $\mathcal{P}_2$. First of all note that $f[S] = g[S]$ with $|f[S]| \leq 2$ immediately implies $f = g = \text{const}$. Observe also that there is no polynomial $f \in \mathcal{P}_2$ with $|f[S]| = 3$. So we only have to deal with the case that $|f[S]| \geq 4$. Assume towards a contradiction that there are

$$f(x) = a_0 + a_1 x + a_2 x^2 \quad \text{and} \quad g(x) = b_0 + b_1 x + b_2 x^2$$

with $f[S] = g[S]$, $|f[S]| = |g[S]| \geq 4$ and $f \neq g$. In other words, f and g satisfy a linear equation of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & -1 & -x_{i_1} & -x_{i_1}^2 \\ 1 & x_2 & x_2^2 & -1 & -x_{i_2} & -x_{i_2}^2 \\ 1 & x_3 & x_3^2 & -1 & -x_{i_3} & -x_{i_3}^2 \\ 1 & x_4 & x_4^2 & -1 & -x_{i_4} & -x_{i_4}^2 \\ 1 & x_5 & x_5^2 & -1 & -x_{i_5} & -x_{i_5}^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\{i_1, i_2, \ldots, i_5\} \subseteq \{1, 2, 3, 4, 5\}$ and $|\{i_1, \ldots, i_5\}| \geq 4$. By checking all cases, one finds that the only solution of such a linear equation with $f \neq g$ is

$$f(x) = 1 + \frac{1}{2}x^2 \quad \text{and} \quad g(x) = -\frac{1}{2}x + x^2.$$

But $f[S] \neq g[S]$. So S is indeed an SRU. □

Lemma 6. *The set*

$$S := \left\{1, 2, 4, 10, 31, \frac{1}{2}\left(3 + \sqrt{68581}\right), \frac{1}{2}\left(3 - \sqrt{550558 + 13347\sqrt{68581}}\right)\right\}$$

is an SRU for $\mathcal{P}_3$ *but not a magic set.*

Proof. The set S is not a magic set for $\mathcal{P}_3$ because for

$$f(x) = 18(x-1)(x-2) \text{ and } g(x) := (x-1)(7x^2 + 120x - 160)$$

we have that $f[S] \subsetneq g[S]$. Observe also that there is no polynomial $f \in \mathcal{P}_3$ with $|f[S]| = 3$. So we only have to deal with the case that $|f[S]| \geq 4$.

Assume towards a contradiction that there are

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad \text{and} \quad g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

with $f[S] = g[S]$, $|f[S]| = |g[S]| \geq 4$ and $f \neq g$. In other words, f and g satisfy a linear equation of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & -1 & -x_{i_1} & -x_{i_1}^2 & -x_{i_1}^3 \\ 1 & x_2 & x_2^2 & x_2^3 & -1 & -x_{i_2} & -x_{i_2}^2 & -x_{i_2}^3 \\ 1 & x_3 & x_3^2 & x_3^3 & -1 & -x_{i_3} & -x_{i_3}^2 & -x_{i_3}^3 \\ 1 & x_4 & x_4^2 & x_4^3 & -1 & -x_{i_4} & -x_{i_4}^2 & -x_{i_4}^3 \\ 1 & x_5 & x_5^2 & x_5^3 & -1 & -x_{i_5} & -x_{i_5}^2 & -x_{i_5}^3 \\ 1 & x_6 & x_6^2 & x_6^3 & -1 & -x_{i_6} & -x_{i_6}^2 & -x_{i_6}^3 \\ 1 & x_7 & x_7^2 & x_7^3 & -1 & -x_{i_7} & -x_{i_7}^2 & -x_{i_7}^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\{i_1, i_2, \ldots, i_7\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$ and $|\{i_1, \ldots, i_7\}| \geq 4$. By checking all cases, one finds that the only solution of such a linear equation with $f \neq g$ is

$$f(x) = \frac{18}{7}x^2 - \frac{54}{7}x - \frac{124}{7} \quad \text{and} \quad g(x) = x^3 + \frac{113}{7}x^2 - 40x.$$

But $f[S] \neq g[S]$. So S is indeed an SRU. □

In the above Lemma, the two polynomials showing that the set S is not magic for $\mathcal{P}_3$, are of degree 2 and 3. In the next Lemma we show that there is an SRU S and two polynomials of degree 3 showing that S is not magic.

Lemma 7. *The set*

$$S := \{1, 2, 5, 12, 23, 27, \alpha\}$$

with

$$\alpha = \frac{8}{3} - \frac{13}{3\sqrt[3]{3197764 - 9\sqrt{126243143179}}} - \frac{1}{3}\sqrt[3]{3197764 - 9\sqrt{126243143179}}$$

is an SRU for $\mathcal{P}_3$ but not a magic set.

Proof. The set S is not a magic set for $\mathcal{P}_3$ because for

$$f(x) = 21(x-1)(x-2)(x-5) \text{ and } g(x) := (x-1)(-1150x^2 + 17213x - 13656)$$

we have that $f[S] \subsetneq g[S]$. Observe also that there is no polynomial $f \in \mathcal{P}_3$ with $|f[S]| = 3$. So we only have to deal with the case that $|f[S]| \geq 4$.

Assume towards a contradiction that there are

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{and} \quad g(x) = b_0 + b_1x + b_2x^2 + b_3x^3$$

with $f[S] = g[S]$, $|f[S]| = |g[S]| \geq 4$ and $f \neq g$. In other words, f and g satisfy a linear equation of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & -1 & -x_{i_1} & -x_{i_1}^2 & -x_{i_1}^3 \\ 1 & x_2 & x_2^2 & x_2^3 & -1 & -x_{i_2} & -x_{i_2}^2 & -x_{i_2}^3 \\ 1 & x_3 & x_3^2 & x_3^3 & -1 & -x_{i_3} & -x_{i_3}^2 & -x_{i_3}^3 \\ 1 & x_4 & x_4^2 & x_4^3 & -1 & -x_{i_4} & -x_{i_4}^2 & -x_{i_4}^3 \\ 1 & x_5 & x_5^2 & x_5^3 & -1 & -x_{i_5} & -x_{i_5}^2 & -x_{i_5}^3 \\ 1 & x_6 & x_6^2 & x_6^3 & -1 & -x_{i_6} & -x_{i_6}^2 & -x_{i_6}^3 \\ 1 & x_7 & x_7^2 & x_7^3 & -1 & -x_{i_7} & -x_{i_7}^2 & -x_{i_7}^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\{i_1, i_2, \ldots, i_7\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$ and $|\{i_1, \ldots, i_7\}| \geq 4$. By checking all cases, one finds that the only solution of such a linear equation with $f \neq g$ is

$$f(x) = \frac{6933}{575} - \frac{357}{1150}x + \frac{84}{575}x^2 - \frac{21}{1150}x^3 \quad \text{and} \quad g(x) = \frac{30869}{1150}x - \frac{18363}{1150}x^2 + x^3.$$

But $f[S] \neq g[S]$. So S is indeed an SRU. $\square$

5. Magic sets for $\mathcal{P}_n$

In this section we will show that for every $s \geq 2n+1$ there is a magic set of size s for the set $\mathcal{P}_n$ of all real, non-constant polynomials of degree at most n.

Remark 8. For $n \geq 1$ the condition that $\mathcal{P}_n$ does not contain any constant polynomials is necessary for the existence of a magic set. Otherwise let $M \subseteq \mathbb{R}$ be a non-empty set, $f(x) \equiv c$ for a $c \in \mathbb{R}$ and let g be a non-constant polynomial with $g(m) = c$ for an $m \in M$. Then we have that

$$\{c\} = f[M] \subseteq g[M]$$

but $f \neq g$.

First of all we want to give some general definitions:

Definition 9. A *directed graph* H is a pair (V, E), where V is a set (the vertices of H) and $E \subseteq V \times V$ (the edges of H). For every $v \in V$ we define

$$\begin{aligned} \text{indegree}_H(v) &:= |\{v' \in V \mid (v', v) \in E\}|, \\ \text{outdegree}_H(v) &:= |\{v' \in V \mid (v, v') \in E\}| \text{ and} \\ \deg_H(v) &:= \text{indegree}_H(v) + \text{outdegree}_H(v). \end{aligned}$$

Definition 10. Let $H = (V, E)$ be a directed graph.

- A *cycle* is a subgraph $C = (V_C, E_C)$ of H with $V_C = \{c_0, c_1, \dots, c_{m-1}\}$ and $E_C = \{(c_i, c_{(i+1) \bmod m}) \mid i \in \mathbb{N}\}$ for an $m \geq 2$.
- A *loop* is a subgraph $L = (V_L, E_L)$ of H with $V_L = \{w\}$ and $E_L = \{(w, w)\}$.
- A *solitary path* is a directed path $P = (\{v_0, v_1, \dots, v_m\}, \{(v_i, v_{i+1}) \mid i = 0, 1, \dots, m-1\})$ with $\text{indegree}_H(v_0) = 0$, $\deg_H(v_m) > 2$ and $\deg_H(v_i) = 2$ for all $1 \leq i \leq m - 1$.

Definition 11. Let $l \in \mathbb{N}$. Cycles and loops $C_0 = (V_{C_0}, E_{C_0}), \dots, C_l = (V_{C_l}, E_{C_l})$ are called *obviously different* if for every $0 \leq i \leq l$ there is a

$$y_i \in V_{C_i} \setminus \left(\bigcup_{j=0, j \neq i}^{l} V_{C_j} \right).$$

Definition 12. Let H be a directed graph and let H_1 and H_2 be two subgraphs of H. Then H_1 and H_2 are called *undirected edge disjoint* iff H_1 and H_2 do not share any edges even if we replace all edges in H_1 and H_2 by undirected edges.

Let $k, n \in \mathbb{N}^*$ with $k \geq 2n$ and let $\{x_0, x_1, \dots, x_k\} \subseteq \mathbb{R}$. For all $0 \leq i \leq k$ let $v_i := (x_i, x_i^2, \dots, x_i^n)$. The following family $\mathcal{H}$ will play a crucial role in the construction of magic sets of size $k + 1$ for the set $\mathcal{P}_n$.

Definition 13. Let $\mathcal{H}$ be the family of all directed graphs $H = (V, E)$ with vertex set $V = \{v_0, v_1, \dots, v_k\}$ and a set E of directed edges such that for each $v \in V$ we have that

$$\text{outdegree}_H(v) \geq 1.$$

We now partition the family $\mathcal{H}$ into three parts, namely the graphs of type α_n, β_n and γ_n.

Definition 14. A graph $H \in \mathcal{H}$ is of type

- γ_n iff there are more than $n - 1$ solitary paths in H.
- β_n iff there are more than n obviously different loops and cycles in H and H is not of type γ_n.

- α_n iff H is neither of type γ_n nor of type β_n.

In Section 5.1, we will consider graphs of type α_n and we will show in Corollary 23, that for every graph $H = (V, E)$ of type α_n, there is a $(2n+1) \times (2n+1)$-matrix

$$M_H(x_0, x_1, \ldots, x_k) = \begin{pmatrix} 1 & v_{i_0} & -v_{j_0} \\ 1 & v_{i_1} & -v_{j_1} \\ \vdots & \vdots & \vdots \\ 1 & v_{i_{2n}} & -v_{j_{2n}} \end{pmatrix}$$

with $i_l, j_l \in \{0, 1, \ldots, k\}$ (for all $0 \leq l \leq 2n$) and $(v_{i_l}, v_{j_l}) \in E$ (for all $0 \leq l \leq 2n$), such that for all open sets $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ with

$$\det\big(M_H(x_0, x_1, \ldots, x_k)\big) \neq 0 \tag{2}$$

for all $(x_0, x_1, \ldots, x_k) \in U_H$.

Concerning graphs $H = (V, E)$ of type β_n, let $C_0 = (V_{C_0}, E_{C_0}), \ldots, C_n = (V_{C_n}, E_{C_n})$ be $n+1$ obviously different loops and cycles. Let $x_{i_0}, x_{i_1}, \ldots, x_{i_n}$ be $n+1$ vertices of H such that for each $0 \leq l \leq n$,

$$x_{i_l} \in V_{C_l} \setminus \left(\bigcup_{m=0, m\neq l}^{n} V_{C_m} \right).$$

We will show in Section 5.2 that for every open set $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ such that for all $(x_0, x_1, \ldots, x_k) \in U_H$ we have

$$\det\big(N_H(x_0, x_1, \ldots, x_k)\big) \neq 0\,, \tag{3}$$

where

$$N_H(x_0, x_1, \ldots x_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x \in V_{C_0}} x & \sum_{x \in V_{C_0}} x^2 & \cdots & \sum_{x \in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \cdots & \sum_{x \in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_n}} x^2 & \cdots & \sum_{x \in V_{C_n}} x^n \end{pmatrix}.$$

In Section 5.3 we will show that for every graph H of type γ_n there is an $n \times n$-matrix

$$L_H(x_0, x_1, \ldots, x_k) = \begin{pmatrix} v_{j_0} - v_{i_0} \\ v_{j_1} - v_{i_1} \\ \vdots \\ v_{j_{n-1}} - v_{i_{n-1}} \end{pmatrix}$$

such that

- $j_l, i_l \in \{0, 1, \dots, k\}$ for all $0 \le l \le n-1$;
- v_{i_l} and v_{j_l} are different but have the same successor in H and
- for all open sets $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ such that for all $(x_0, x_1, \dots, x_k) \in U_H$ we have that

$$\det\bigl(L_H(x_0, x_1, \dots, x_k)\bigr) \neq 0\,. \tag{4}$$

As a consequence of (2), (3) and (4) and since $|\mathcal{H}| < \infty$, we can find a point $(m_0, m_1, \dots, m_k) \in \mathbb{R}^{k+1}$ such that for all $H \in \mathcal{H}$ of type α_n

$$\det(M_H(m_0, \dots, m_k)) \neq 0,$$

for all $H \in \mathcal{H}$ of type β_n

$$\det(N_H(m_0, \dots, m_k)) \neq 0,$$

and for all $H \in \mathcal{H}$ of type γ_n

$$\det(L_H(m_0, \dots, m_k)) \neq 0.$$

This leads to the following

Theorem 15. *The set $M := \{m_0, m_1, \dots, m_k\}$ is a magic set for $\mathcal{P}_n$.*

Proof. Assume towards a contradiction that M is not a magic set for $\mathcal{P}_n$. So, there are two non-constant polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

such that $f[M] \subseteq g[M]$ but $f \neq g$. Let $H = (V, E)$ with

$$V := M \text{ and } E := \{(m_i, m_j) \mid f(m_i) = g(m_j)\}.$$

Note that $H \in \mathcal{H}$. There are three cases:

<u>*Case 1:*</u> H is of type α_n.

In this case

$$M_H(m_0, m_1, \dots, m_k) = \begin{pmatrix} 1 & v_{i_0} & -v_{j_0} \\ 1 & v_{i_1} & -v_{j_1} \\ \vdots & \vdots & \vdots \\ 1 & v_{i_{2n}} & -v_{j_{2n}} \end{pmatrix}$$

has non-zero determinant. Note that for all $0 \leq l \leq n$ we have that

$$f(m_{i_l}) = g(m_{j_l}) \iff$$
$$(a_0 - b_0) + (a_1 m_{i_l} + \cdots + a_n m_{i_l}^n) - (b_1 m_{j_l} + \cdots + b_n m_{j_l}^n) = 0.$$

So, f and g satisfy the following system of linear equations:

$$M_H(m_0, \ldots, m_k) \cdot \begin{pmatrix} a_0 - b_0 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det\left(M_H(m_0, \ldots, m_k)\right) \neq 0$, this equation has only the trivial solution. Therefore, $f = g$, which is a contradiction to our assumption that M is not a magic set.

<u>*Case 2:*</u> H is of type β_n.
In this case

$$N_H(m_0, \ldots, m_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x \in V_{C_0}} x & \sum_{x \in V_{C_0}} x^2 & \cdots & \sum_{x \in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \cdots & \sum_{x \in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_n}} x^2 & \cdots & \sum_{x \in V_{C_n}} x^n \end{pmatrix}$$

with $n+1$ obviously different cycles $C_0 = (V_{C_0}, E_{C_0}), C_1 = (V_{C_1}, E_{C_1}), \ldots, C_n = (V_{C_n}, E_{C_n})$. For all $0 \leq i \leq n$ we have that

$$\sum_{m \in V_{C_i}} (f - g)(m) = 0.$$

In other words, we have to solve the following system of linear equations:

$$N_H(m_0, \ldots, m_k) \cdot \begin{pmatrix} a_0 - b_0 \\ a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(N_H(m_0, \ldots, m_k)) \neq 0$ this equation has only the trivial solution. Therefore, $f = g$, which is again a contradiction.

<u>*Case 3:*</u> H is of type γ_n.

In this case

$$L_H(m_0, m_1, \ldots, m_k) = \begin{pmatrix} v_{j_0} - v_{i_0} \\ v_{j_1} - v_{i_1} \\ \vdots \\ v_{j_{n-1}} - v_{i_{n-1}} \end{pmatrix}$$

has non-zero determinant. For all $0 \leq l \leq n-1$ the points m_{i_l} and m_{j_l} have the same successors in H. Therefore,

$$f(m_{j_l}) = f(m_{i_l}) \iff a_1(m_{j_l} - m_{i_l}) + a_2(m_{j_l}^2 - m_{i_l}^2) + \cdots + a_n(m_{j_l}^n - m_{i_l}^n) = 0$$

for all $0 \leq l \leq n-1$. In other words, f satisfies the following system of linear equations:

$$L_H(m_0, m_1, \ldots, m_k) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(L_H(m_0, m_1, \ldots, m_k)) \neq 0$ this equation has only the trivial solution. Therefore, f is a constant polynomial. This is a contradiction. □

5.1. Graphs and matrices of type α_n

Remark 16. From now on we assume that there is at least one solitary path in every graph of type α_n. If a graph H of type α_n has no solitary path, it is of type 1_n (*i.e.*, it has at most n obviously different cycles and loops) and we can find a suitable matrix as in [5].

Definition 17. Let $G = (V, E)$ be a graph. Assume, that for each edge in E either the foot or the head is marked. The marked vertices are called *relevant*. Then $v \in V$ is called a *unique vertex* iff

$$\text{indegree}_G(v) = 0, \;\; \text{outdegree}_G(v) = 1$$

and v is the relevant vertex of the edge incident with v.

Definition 18. Let $n \in \mathbb{N}^*$ and let $H = (V, E)$ be a graph of type α_n with $|V| \geq 2n + 1$. A *good sequence* of length $m \in \mathbb{N}$ of H is a sequence of graphs

$$(\emptyset, \emptyset) = H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \cdots \subseteq H_m = (V_m, E_m) \subseteq H = (V, E)$$

such that for all $0 \leq l < m$ the set $E_{l+1} \setminus E_l$ has one of the following forms:

(a) $E_{l+1} \setminus E_l = \{(v_i, v_j), (v_j, v_t)\}$ with $0 \leq i, j, t \leq k$, $i \neq j$ and $j \neq t$. Moreover, if v_j is contained in an edge in E_l together with a v_s, then v_s is a unique vertex of H_l. The relevant vertex of both edges (v_i, v_j) and (v_j, v_t) is v_j.

(b) $E_{l+1} \setminus E_l = \{(v_i, v_j), (v_s, v_t)\}$ with $0 \leq i, j, s, t \leq k$, $i \neq j$, $i \neq t$ and $s \neq t$. Moreover, if v_t or v_i is contained in an edge in E_l together with a v_p then v_p is a unique vertex of H_l. The relevant vertex of (v_i, v_j) is v_i and the relevant vertex of (v_s, v_t) is v_t.

(c) $E_{l+1} \setminus E_l = \{(v_i, v_i), (v_j, v_t)\}$ with $0 \leq i, j, t \leq k$ and $j \neq t$. Moreover, if v_i and v_j are contained in an edge in E_l together with a v_s, then v_s is a unique vertex of H_l. The relevant vertex of (v_i, v_i) is v_i and the relevant vertex of (v_j, v_t) is v_j.

(d) $E_{l+1} \setminus E_l = \{(v_i, v_i), (v_t, v_j)\}$ with $0 \leq i, j, t \leq k$ and $j \neq t$. Moreover, if v_i and v_j are contained in an edge in E_l together with a v_s, then v_s is a unique vertex of H_l. The relevant vertex of (v_i, v_i) is v_i and the relevant vertex of (v_t, v_j) is v_j.

(e) $E_{l+1} \setminus E_l = \{(v_i, v_j), (v_s, v_t)\}$ with $i \neq j$ and $s \neq t$. We have that $\text{indegree}_H(v_i) = 0$ and for all $0 \leq q \leq l$ we have that $E_q \setminus E_{q-1}$ contains an edge with a unique vertex of H_l. Moreover we assume that if there is an edge in E_l containing v_t and a v_p we have that either v_p is a unique vertex of H_l or $(v_t, v_p) \in E_l$. The relevant vertex of (v_i, v_j) is v_i and the relevant vertex of (v_s, v_t) is v_t.

Lemma 19. *Let $n \in \mathbb{N}^*$. Every graph $H = (V_H, E_H)$ of type α_n with $|V_H| \geq 2n + 1$ has a good sequence*

$$(\emptyset, \emptyset) = H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \cdots \subseteq H_m = (V_m, E_m) \subseteq H$$

of length m with $|E_m| \geq 2n$ and an edge $z = (z_0, z_1) \notin E_m$ such that neither z_0 nor z_1 is a relevant vertex of any edge in E_m.

Proof. Let $H = (V_H, E_H)$ be a graph of type α_n. If there is a vertex $v \in V_H$ with $\text{outdegree}_H(v) \geq 2$ and $\text{indegree}_H(v) = 0$ remove all but one edge containing v. The resulting graph is still of type α_n. Let $\mathcal{L}$ be the set of all isolated loops of H. To be more precise

$$\mathcal{L} := \{(\{v\}, \{(v, v)\}) \subseteq H \mid \deg_H(v) = 2\}.$$

Let $\mathcal{T} = \{S_0, S_1, \ldots, S_l\}$ (for an $l \in \mathbb{N}$) be the set of all solitary paths in H. Let $0 \leq i \leq l$. If S_i ends in a vertex v in which only solitary paths end we have that $(v, v) \in E_H$. Add this edge to S_i iff this loop has not already been added to a S_j with $j < i$. Define $Z := S_0$. Note that $|\mathcal{T}| \geq 1$ by Remark 16. Remove Z from $\mathcal{T}$. Let $\mathcal{S}$ be the set of all first edges of the remaining solitary paths in $\mathcal{T}$ that contain an odd number of edges.

Step 1: Removing isolated loops with solitary paths.
Assume that $\mathcal{S} \neq \emptyset$ and $\mathcal{L} \neq \emptyset$. Let $s = (s_0, s_1) \in \mathcal{S}$ and let $t = (t_0, t_0) \in \mathcal{L}$. Add s, t and the corresponding edges to H_0. Call the resulting graph H_1. Note that $E_1 \setminus E_0$ has the form (c) and that s contains a unique vertex. Remove t from $\mathcal{L}$ and remove s from $\mathcal{S}$.

The relevant vertex of s is s_0 and the relevant vertex of t is t_0. Redo this construction until either $\mathcal{S} = \emptyset$ or $\mathcal{L} = \emptyset$.
From now on we assume that $\mathcal{L} = \emptyset$. The construction in the other case is similar. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_0}$$

with $m_0 \in \mathbb{N}$ be the good sequence we constructed so far.

Step 2: Adding cycles.
Let $C_0 = (V_{C_0}, E_{C_0}), C_1 = (V_{C_1}, E_{C_1}), \dots, C_{l_1} = (V_{C_{l_1}}, E_{C_{l_1}})$ be a maximal family of pairwise disjoint cycles in H. If there is a cycle $C = C_j$ for a $0 \le j \le l_1$ that contains a vertex to which Z points, assume that $C = C_{l_1}$. This is important because we might have to add edges of the form (e). Assume that we have already added $C_0, C_1, \dots, C_{i-1}$ for a $0 \le i \le l_1$ to H_{m_1} and defined a good sequence

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m'}$$

for an $m' \ge m_0$. Now we want to add C_i. If the solitary path Z points to a vertex in V_{C_i} mark this vertex v_0 with a cross.

Case 1: There is a vertex $v_0 \in V_{C_i}$ that is marked with a cross.
If $\mathcal{S} \ne \emptyset$ let $\mathcal{M}_i \subseteq \mathcal{S}$ be maximal with $0 \le |\mathcal{M}_i| + 1 \le |E_{C_i}|$ and such that $|\mathcal{M}_i| + |E_{C_i}|$ is even. If $\mathcal{S} = \emptyset$ let $\mathcal{M}_i = \emptyset$. Remove $\mathcal{M}_i$ from $\mathcal{S}$.

Case 1.1: $|\mathcal{M}_i| + |E_{C_i}|$ is even.

There are two subcases:

- $\mathcal{M}_i \ne \emptyset$.
  Let $e = (e_0, e_1)$ be the first edge in E_{C_i} coming after v_0 and let $s = (s_0, s_1) \in \mathcal{M}_i$. Add e, s and the corresponding vertices to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). Remove e and s from E_{C_i} and $\mathcal{M}_i$. The relevant vertex of e is e_1 and the relevant vertex of s is s_0. Note that $e_1 \ne v_0$ because $|\mathcal{M}_i| + 1 \le |E_{C_i}|$. In particular v_0 is not a relevant vertex of any edge in $H_{m'+1}$.
- $\mathcal{M}_i = \emptyset$.
  There is a vertex $w \in V_{C_i} \setminus \{v_0\}$ such that both edges $e = (e_0, e_1)$ and $f = (f_0, f_1)$ containing w are still in E_{C_i}. We assume that w is the first vertex with this property coming after v_0 in C_i. Add e, f and the corresponding vertices to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (a). Remove e and f from E_{C_i}. The relevant vertex of e and of f is w. Note that v_0 is not a relevant vertex of any edge in $H_{m'+1}$.

Case 1.2: $|\mathcal{M}_i| + |E_{C_i}|$ is odd.

Note that we are only in this case when $\mathcal{M}_i = \emptyset$ and C_i is still the original cycle. Let $y = (y_0, y_1)$ be the first edge in E_{C_i} coming after v_0. By the assumption in Case 1 we have in particular that $i = l_1$. So there is no cycle C_{i+1}. If $|E_Z|$ is even, add y, the third last (or if this is not possible the first) edge $f = (f_0, f_1)$ of Z and the corresponding vertices to $E_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (b). Remove f from Z and y from E_{C_i}. The relevant vertex of y is y_1 and the relevant vertex of f is f_0. If there is no cycle C_{i+1} and $|E_Z|$ is odd, remove y from E_{C_i}.

Case 2: There is no vertex in V_{C_i} that is marked with a cross.

Let $\mathcal{M}_i \subseteq \mathcal{S}$ be maximal with $|\mathcal{M}_i| \leq |E_{C_i}|$. Remove $\mathcal{M}_i$ from $\mathcal{S}$.

Case 2.1: $|\mathcal{M}_i| + |E_{C_i}|$ is odd.

Note that in this case $|\mathcal{M}_i| < |E_{C_i}|$ and therefore, $\mathcal{S} = \emptyset$ (we removed $\mathcal{M}_i$ from $\mathcal{S}$). So for all $j > i$ we will have that $\mathcal{M}_j = \emptyset$.

- There is a $j > i$ such that $|E_{C_j}| = |E_{C_j}| + |\mathcal{M}_j|$ is odd.
  Let $e = (e_0, e_1) \in E_{C_i}$ be an arbitrary edge. Note that C_i is still equal to the original cycle. Otherwise we would not be in this subcase. Let $f = (f_0, f_1) \in E_{C_j}$ be an arbitrary edge. That is, if possible, ending in a vertex that is marked with a cross. Add e, f and the corresponding edges to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). Remove e from E_{C_i} and remove f from E_{C_j}. The relevant vertex of e is e_1 and the relevant vertex of f is f_0.
- There is no $j > i$ such that $|E_{C_j}| = |E_{C_j}| + |\mathcal{M}_j|$ is odd.
  If $|E_Z|$ is even, let $e = (e_0, e_1) \in E_{C_i}$ be an arbitrary edge and let $f = (f_0, f_1)$ be the third last (or if this is not possible the first) edge in Z. Add e, f and the corresponding vertices to $H_{m'}$. Call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (b). Remove f from Z and e from E_{C_j}.
  If $|E_Z|$ is odd let $e = (e_0, e_1) \in E_{C_i}$ be an arbitrary edge. Remove e from E_{C_i}.

Case 2.2: $|\mathcal{M}_i| + |E_{C_i}|$ is even.

There are two subcases:

- $\mathcal{M}_i \neq \emptyset$.
  If E_{C_i} does not contain all edges of the original cycle C_i let $e = (e_0, e_1)$ be the first edge in E_{C_i}. Otherwise let e be an arbitrary edge in E_{C_i}. Let $s = (s_0, s_1) \in \mathcal{M}_i$. Add e, s and the corresponding edges to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (b) or (e). Remove e and s from $\mathcal{M}_i$ and E_{C_i}. The relevant variable of e is e_1 and the relevant variable of s is s_0.
- $\mathcal{M}_i = \emptyset$.
  In this case let w be the first vertex in C_i with $\deg_{C_i}(w) = 2$ (or if C_i is still the original cycle choose a $w \in V_{C_i}$ with $\deg_{C_i}(w) = 2$). Add the edges $e, f \in E_{C_i}$ that contain w to $H_{m'}$. We call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ has the form (a). Remove e and f from E_{C_i}. The relevant vertex of e and of f is w.

Assume that we have done this construction for all cycles $C_0, C_1, \ldots, C_{l_1}$. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_1}$$

with $m_1 \geq m_0$ be the good sequence we constructed so far.

Step 3: Adding paths.
Let $P_0 = (V_{P_0}, E_{P_0})$ be a maximal path in H which is undirected edge disjoint from H_{m_1}. In addition we require that all vertices (except possibly the first or the last one) are disjoint from the vertices in H_{m_1}. If possible let P_0 be a path such that Z points to a vertex v_0 in $V_{P_0} \setminus V_{m_1}$. Let $p_0 \in \mathbb{N}$ be the number of vertices in V_{P_0} that are not in V_{m_1}.

<u>*Case 1:*</u> The solitary path Z points to a vertex $v_0 \in V_{P_0} \setminus V_{m_1}$.
If $\mathcal{S} \neq \emptyset$ let $\mathcal{N}_0 \subseteq \mathcal{S}$ be maximal with $|\mathcal{N}_0| + 1 \leq p_0$ such that $|\mathcal{N}_0| + p_0$ is even. If $\mathcal{S} = \emptyset$ let $\mathcal{N}_0 := \emptyset$. Remove $\mathcal{N}_0$ from $\mathcal{S}$.

Case 1.1: $|\mathcal{N}_0| + p_0$ is even.
There are two subcases:

- $\mathcal{N}_0 \neq \emptyset$.
  Let $e = (e_0, e_1)$ be the first edge in P_0. If it points to v_0 remove it from P_0 and from H. Otherwise let $s = (s_0, s_1) \in \mathcal{S}$. Add e, s and the corresponding vertices to H_{m_2}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b), (c) or (d). The relevant vertex of e is e_1 and the relevant vertex of s is s_0. Remove s and e from $\mathcal{N}_0$ and from E_{P_0}.
- $\mathcal{N}_0 = \emptyset$.
  Let $w \neq v_0$ be the first vertex in the path that is contained in exactly two edges of P_0. Let e and f be the two edges containing w. Add them and the corresponding vertices to H_{m_1} and call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ has the form (a), (c) or (d). Remove e and f from P_0. The relevant vertex of e and of f is w.

Repeat the procedure described in Case 1.1 until $|E_{P_0}| \leq 1$. Remove the remaining edge from E_{P_0}.

Case 1.2: $|\mathcal{N}_0| + p_0$ is odd.
Note that we are only in this case when $\mathcal{N}_0 = \emptyset$.

- On the right or on the left of v_0 there is an even number of edges.
  Let $w \neq v_0$ be the first vertex in the path that is contained in exactly two edges of P_0 and $w \notin \{z_0, z_1\}$ if we have already defined an edge $z = (z_0, z_1)$. Let e and f be the two edges containing w. Add e, f and the corresponding vertices to H_{m_1} and call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ has the form (a), (c) or (d). Remove e and f from E_{P_0} or from E_Z. The relevant vertex of e and of f is w.
- We are not in the first subcase and v_0 is the first vertex in $V_{P_0} \setminus V_{m_1}$.
  Let e be the first edge in P_0. Remove e from H and add back the original Z to H. This graph H is of type α_n. Redo the whole construction. Note that at one point we will never be in this case anymore.

- We are not in the first two subcases.
  If P_0 ends in a vertex of a cycle C_i that is relevant for an edge in E_{m_1} mark that last vertex of P_0 with a cross and redo the whole construction with the same cycles and paths. If necessary remove one edge s from $\mathcal{M}_i$ and add it to $\mathcal{N}_0$. So we can now assume that the last vertex in P_0 is not relevant for any edge in E_{m_1}. There are two cases we have to look at:
  - If $|\mathcal{N}_0| = 1$, let $e = (e_0, e_1)$ be the first edge in P_0 (note that $e_1 \neq v_0$) and let $s = (s_0, s_1) \in \mathcal{N}_0$. Add e, s and the corresponding vertices to H_{m_1} and remove them from P_0 and from $\mathcal{N}_0$. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \backslash E_{m_1}$ is of the form (b). The relevant vertex of e is e_1 and the relevant vertex of s is s_0.
  - If $\mathcal{N}_0 = \emptyset$, let $e = (e_0, e_1)$ be the first edge in P_0. Note that by assumption $e_1 \neq v_0$. Let $f = (f_0, f_1)$ be the third last (or if this is not possible the first) edge in Z. Add e, f and the corresponding vertices to H_{m_1}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b), (c) or (d). Remove e from P_0 and f from Z.
    If now $|E_Z| = 0$ let $z = (z_0, z_1)$ be the first edge coming after v_0 in P_0. In particular we have that $z_0 = v_0$. Note that neither z_0 nor z_1 is a relevant vertex of an edge we added to H_0 so far. Moreover, it will never be a relevant vertex of any edge we will add in the future.

Repeat the procedure described in Case 1.2 until $|E_{P_0}| \leq 1$. Remove the remaining edge from P_0.

Case 2: The solitary path Z does not point to a vertex in P_0.
Let $\mathcal{N}_0 \subseteq \mathcal{S}$ be maximal with $|\mathcal{N}_0| \leq p_0$. Remove $\mathcal{N}_0$ from $\mathcal{S}$.

Case 2.1: $\mathcal{N}_0 \neq \emptyset$.

Let $e = (e_0, e_1)$ be the first edge in P_0 and let $f = (f_0, f_1) \in \mathcal{N}_0$. Add e, f and the corresponding vertices to H_{m_1}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (b). Remove e and f from $\mathcal{N}_0$ and from E_{P_0}. The relevant vertex of e is e_1 and the relevant vertex of f is f_0

Case 2.2: $\mathcal{N}_0 = \emptyset$.

Let w be the first vertex in P_0 that is contained in exactly two edges $e, f \in E_{P_0}$. Add e, f and the corresponding vertices to H_{m_1}. Call the resulting graph H_{m_1+1}. Note that $E_{m_1+1} \setminus E_{m_1}$ is of the form (a). Remove e and f from E_{P_0}. The relevant vertex of e and of f is w.
Repeat this procedure until $|E_{P_0}| \leq 1$. Remove the remaining edges from P_0.
Do the same procedure for all paths in H. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_2}$$

with $m_2 \geq m_1$ be the good sequence we constructed so far.

Step 4: Adding the rest of the solitary paths.

Add Z to $\mathcal{T}$. And if $|E_Z| \geq 2$ is odd, add the first edge of Z to $\mathcal{S}$. Define

$$\mathcal{T}_2 := \{S \in \mathcal{T} \mid |E_S| \geq 2\} = \{T_0, T_1, \ldots, T_{l_3}\}$$

for an $l_3 \in \mathbb{N}$. Assume that $Z = T_{l_3}$ if $|E_Z| \geq 2$. Note that if Z ends in a vertex v in which only solitary paths end, Z contains the loop (v, v).
Let $F = \emptyset$. Assume that we have already added $T_0, T_1, \ldots, T_{i-1}$ for a $0 \leq i \leq l_2$ to H_{m_2} and we defined a good sequence

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m'}$$

with a $m' \geq m_2$. Now we want to add $T_i = (V_{T_i}, E_{T_i})$.

Case 1: $|E_{T_i}| > 2$ is even and $\mathcal{S} \neq \emptyset$.

Let $s = (s_0, s_1)$ be the third last edge in E_{T_i} and let $t = (t_0, t_1) \in \mathcal{S}$. Add s, t and the corresponding vertices to $H_{m'}$. Call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). Remove t from $\mathcal{S}$ and s from E_{T_i}. If t is contained in a T_j, $j > i$, remove t from E_{T_j}. The relevant vertex of s is s_1 and the relevant vertex of t is t_0.

Case 2: $|E_{T_i}| > 2$ is even and $\mathcal{S} = \emptyset$.

Let w be the first vertex in T_i with $\deg_{T_i}(w) = 2$. Let e and f be two edges containing w. Add e, f and the corresponding vertices to $H_{m'}$. Call the resulting graph $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (a) or (d). Remove e and f from E_{T_i}. The relevant vertex of e and of f is w.

Case 3: $|E_{T_i}| > 2$ is odd and $\mathcal{S} \setminus E_{T_i} \neq \emptyset$.

Let $e = (e_0, e_1)$ be the third last edge in E_{T_i} and let $f = (f_0, f_1) \in \mathcal{S} \setminus \{e\}$. Add e, f and the corresponding vertices to $H_{m'}$. The resulting graph is called $H_{m'+1}$. Note that $E_{m'+1} \setminus E_{m'}$ is of the form (b). The relevant vertex of e is e_1 and the relevant vertex of f is f_1. Remove e from E_{T_i} and f from $\mathcal{S}$. Remove the first edge of E_{T_i} from $\mathcal{S}$.

Case 4: $|E_{T_i}| > 2$ is odd and $\mathcal{S} \setminus E_{T_i} = \emptyset$.

Let $z = (z_0, z_1)$ be the first edge in E_{T_i}. Remove z from E_{T_i} and from $\mathcal{S}$. Note that neither z_0 nor z_1 will ever be a relevant vertex of an edge we add to H_0.

Case 5: $|E_{T_i}| = 2$.

There are two subcases:

- $T_i = Z$ and we haven't defined an edge z yet.
  Let $z = (z_0, z_1)$ be the last edge in E_{T_i}. Remove both edges from E_{T_i}. Note that neither z_0 nor z_1 are relevant vertices of any edge in $E_{m'}$.
- We are not in the first subcase and E_{T_i} does not contain a loop.
  Add the two edges in E_{T_i} to the set F and remove them from E_{T_i}.
- We are not in the first subcase and E_{T_i} does contain a loop.
  Do the same as in Case 2.

Repeat the procedure with all solitary paths. Let

$$(\emptyset, \emptyset) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{m_3}$$

with $m_3 \geq m_2$ be the good sequence we constructed so far.

Step 5: Adding the set F.

Let $F = \{\{e_0, f_0\}, \{e_1, f_1\}, \ldots, \{e_{l_4}, f_{l_4}\}\}$ with a $l_4 \in \mathbb{N}$. The pairs of edges are enumerated in the order we added them to F. Now add e_0, f_0 and the corresponding vertices to H_{m_3}. Call the resulting graph H_{m_3+1}. Note that $E_{m_3+1} \setminus E_{m_3}$ has the form (a). The relevant vertex of e_0 and of f_0 is the vertex they share. Repeat the procedure with $\{f_1, e_1\}$, $\{f_2, e_2\}$ and so on. □

Example 20. In this example we will construct a good sequence for the following graph H of type α_n (Figs. 1–8).

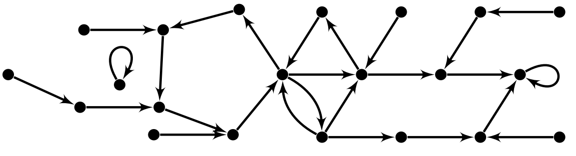


Fig. 1. Graph $H = (V, E)$.

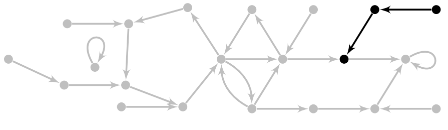


Fig. 2. Solitary path Z.

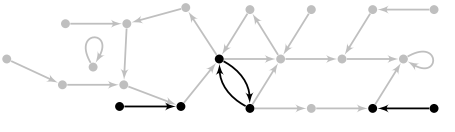


Fig. 3. $\mathcal{M}_0$ and cycle C_0.

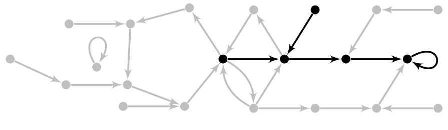


Fig. 4. $\mathcal{N}_0$ and path P_0.

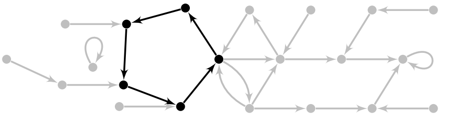


Fig. 5. Path P_1.

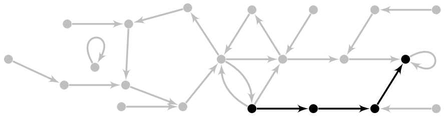


Fig. 6. Path P_2.

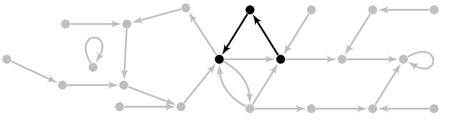


Fig. 7. Path P_3.

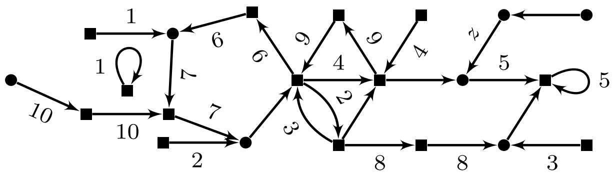


Fig. 8. Graph $H = (V, E)$. The squared vertices are relevant vertices of an edge. The numbers show the order in which the edges are added.

(end example)

Let $k \geq n$, and for all $0 \leq i, j \leq k$ and all $0 \leq s \leq n$ define

$$v_i\_\_ - v_j := (x_i, x_i^2, \dots, x_i^s, -x_j, -x_j^2, \dots, -x_j^s)$$

and

$$1\_v_i\_\_ - v_j := (1, x_i, x_i^2, \dots, x_i^s, -x_j, -x_j^2, \dots, -x_j^s).$$

For every graph $H = (V, E)$ of type α_n choose a good sequence

$$(\emptyset, \emptyset) = H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \dots \subseteq H_n = (V_n, E_n)$$

with $|E_n| = 2n$ and an additional edge $z = (z_0, z_1)$ such that neither z_0 nor z_1 is a relevant vertex for any edge in E_n. For every graph H of type α_n and all $0 \leq l \leq n$ let $M_{H_l}(x_0, \dots, x_k)$ be a square matrix with pairwise different rows $v_i\_\_ - v_j$ where $(v_i, v_j) \in E_{H_l}$. For all $0 \leq l \leq n$ we define

$$\mathcal{C}_l := \left\{ M_{H_l}(x_0, \dots, x_k) \mid H \text{ is a graph of type } \alpha_n \right\}.$$

Furthermore, we define M_H to be the square matrix with $2n + 1$ pairwise different rows $1\_v_i\_\_ - v_j$ where $(v_i, v_j) \in E_n$ or $(v_i, v_j) = z$.

Definition 21. Let $R_0 := \emptyset$ and $p_0(x_0, \dots, x_k) := 1$. For every $1 \leq l \leq n$ let R_l be the set of all relevant vertices of the edges in $E_l \setminus E_{l-1}$. We define

$$p_l(x_0, x_1, \dots, x_k) = \Big(\prod_{v_i \in R_l} x_i^l \Big) p_{l-1}(x_0, x_1, \dots, x_k).$$

The polynomial p_l is called the *relevant polynomial* of $M_{H_l}(x_0, x_1, \dots, x_k)$.

Lemma 22. *Let H be a graph of type α_n, let $1 \leq l \leq n$ and let $M_{H_l} \in \mathcal{C}_l$. Then we have that*

$$\det(M_{H_l}) = \overline{p_l} + q_l,$$

where $\overline{p_l}$ is plus or minus the relevant polynomial of H_l and q_l is a polynomial that contains no term of the form $\pm p_l$.

Proof. We prove the Lemma by induction on l. For $l = 1$ it is clear. So assume that $2 \leq l \leq n$. By the induction hypothesis we have that

$$\det(M_{H_{l-1}}) = \overline{p_{l-1}} + q_{l-1}$$

with the properties described in the Lemma. There are five cases:

Case 1: $E_l \setminus E_{l-1}$ has the form (a).

There are two rows

$$Z_0 = v_{i\_\_} - v_j$$
$$Z_1 = v_{j\_\_} - v_t$$

in M_{H_l} such that v_j and $-v_j$ are only contained in these two rows and in rows that also contain a unique vertex of H_l. We first do a Laplace expansion of M_{H_l} along Z_0. So we have that

$$\det(M_{H_l}) = \epsilon_0 x_j^l \det(\overline{M_{H_l}}) + \gamma,$$

where γ is a polynomial, $\epsilon_0 \in \{-1, 1\}$ and $\overline{M_{H_l}}$ is the matrix we obtain from M_{H_l} when we delete the row Z_0 and the $2l$-th column. Now we do a Laplace expansion along the remainders of the row Z_1. We get

$$\det(\overline{M_{H_l}}) = \epsilon_1 x_j^l \det(M_{H_{l-1}}) + \delta = \epsilon_1 x_j^l (\overline{p_{l-1}} + q_{l-1}) + \delta,$$

where δ is a polynomial and $\epsilon_1 \in \{-1, 1\}$. So we have that

$$\det(M_{H_l}) = \epsilon_0 \epsilon_1 x_j^{2l} (\overline{p_{l-1}} + q_{l-1}) + \epsilon_0 x_j^l \delta + \gamma.$$

Define

$$\overline{p_l} := \epsilon_0 \epsilon_1 x_j^{2l} \overline{p_{l-1}} \text{ and } q_l := \epsilon_0 \epsilon_1 x_j^{2l} q_{l-1} + \epsilon_0 x_j^l \delta + \gamma.$$

It remains to prove that q_l does not contain a term of the form $\pm p_l$. First we show that γ does not contain a term of the form $\pm p_l$. If γ does not contain a term containing x_j^{2l} we are done. So there are terms in γ containing x_j^{2l}. But then not the whole x_j^{2l} comes from the rows Z_0, Z_1. Since outside of Z_0 and Z_1 the vertex v_j is only contained in rows together with unique vertices of H_{l-1}, there is a unique variable (i.e. the variable belonging to a unique vertex) which is not contained in the term with x_j^{2l} in it. So there are no terms in γ of the form $\pm p_l$.

Similarly we can show that there are no terms in $\epsilon_0 x_j \delta$ of the form $\pm p_l$. By the properties of q_{l-1} also $\epsilon_0\epsilon_1 x_j^{2l} q_{l-1}$ does not contain a term of the form $\pm p_l$. So q_l has the desired properties.

Case 2: $E_l \setminus E_{l-1}$ has the form (b).

There are two rows

$$Z_0 = v_{i\_\_} - v_j$$
$$Z_1 = v_{s\_\_} - v_t$$

in M_{H_l} such that $v_i, -v_i, v_t$ and $-v_t$ are only contained in these two rows and in rows together with a unique vertex of H_{l-1}. After doing two Laplace expansions we see that

$$\det(M_{H_l}) = \epsilon_0\epsilon_1 x_i^l x_t^l (\overline{p_{l-1}} + q_{l-1}) + \epsilon_0 x_i^l \delta + \gamma.$$

Define

$$\overline{p_l} := \epsilon_0\epsilon_1 x_i^l x_t^l \overline{p_{l-1}} \text{ and } q_l := \epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1} + \epsilon_0 x_i^l \delta + \gamma.$$

If γ does not contain a term containing $x_i^l x_t^l$ we are done. Otherwise not the whole $x_i^l x_t^l$ comes from the rows Z_0 and Z_1. Since outside of Z_0 and Z_1 the vertices v_i and v_j are only contained in rows together with unique vertices of H_{l-1}, there is a unique variable (i.e. the variable belonging to a unique vertex) which is not contained in the term with $x_i^l x_t^l$ in it. So there are no terms in γ of the form $\pm p_l$. Similarly we can show that $\epsilon_0 x_i^l \delta$ does not contain terms of the form $\pm p_l$. By the properties of q_{l-1} the polynomial $\epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1}$ does not contain a term of the form $\pm p_l$.

Case 3: $E_l \setminus E_{l-1}$ has the form (c).

This case is similar to Case 2.

Case 4: $E_l \setminus E_{l-1}$ has the form (d).

This case is similar to Case 2.

Case 5: $E_l \setminus E_{l-1}$ has the form (e).

There are two rows

$$Z_0 = v_{i\_\_} - v_j$$
$$Z_1 = v_{s\_\_} - v_t$$

in M_{H_l} such that $\text{indegree}_H(v_i) = 0$ and such that v_t is only contained in rows together with a unique variable or on the left side. Moreover, for all $0 \leq l' < l$ we have that one of the edges in $E_{l'} \setminus E_{l'-1}$ contains a unique vertex of H_{l-1}. After doing two Laplace expansions we see that

$$\det(M_{H_l}) = \epsilon_0\epsilon_1 x_i^l x_t^l (\overline{p_{l-1}} + q_{l-1}) + \epsilon_0 x_i^l \delta + \gamma.$$

Define

$$\overline{p_l} := \epsilon_0\epsilon_1 x_i^l x_t^l \overline{p_{l-1}} \text{ and } q_l := \epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1} + \epsilon_0 x_i^l \delta + \gamma.$$

Note that there is no term in γ that contains x_i^l because Z_0 is the only row in M_{H_l} containing x_i. So γ does not contain a term of the form $\pm p_l$.
Assume towards a contradiction that there is a term in δ containing x_t^l. But then x_t^l contains an $x_t^{l'}$ with $0 < l' < l$ maximal from an other row than Z_1. If this $x_t^{l'}$ comes from a row that also contains a unique variable, then the term containing x_t^l does not contain this unique variable. So this is not possible. Therefore, the $x_t^{l'}$ comes from a row of the form

$$v_{t\_\_} - v_p$$

for a $p \in \{0, 1, \ldots, k\} \setminus \{t\}$. But then the term does not contain the unique variable in p_{l-1} that has power l'. This is a contradiction. So $\epsilon_0 x_i^l \delta$ does not contain a term of the form $\pm p_l$. By the properties of q_{l-1} the polynomial $\epsilon_0\epsilon_1 x_i^l x_t^l q_{l-1}$ does not contain a term of the form $\pm p_l$. □

Corollary 23. *Let H be a graph of type α_n. For every open set $U \subseteq \mathbb{R}^{k+1}$ there is an open subset $U_H \subseteq U$ such that for all $(x_0, x_1, \ldots, x_k) \in U_H$*

$$\det(M_H(x_0, x_1, \ldots, x_k)) \neq 0.$$

Proof. It suffices to prove that

$$\det(M_H(x_0, x_1, \ldots, x_k)) \not\equiv 0.$$

By Lemma 22 we have that

$$\det(M_{H_n}) = \overline{p_n} + q_n,$$

where $\overline{p_n}$ is plus or minus the relevant polynomial of H_n and q_n is a polynomial that contains no term of the form $\pm p_n$. Let $z = (v_i, v_j)$ be the edge in E_H that does not contain a relevant vertex of any edge in E_n. Do a Laplace expansion of M_H along the row

$$1\_v_{i\_\_} - v_j.$$

We have that

$$\det(M_H(x_0, \ldots, x_k)) = \det(M_n) + \gamma = \overline{p_n} + q_n + \gamma,$$

where γ is a polynomial in which each term either contains x_i or x_j. Since $\overline{p_n}$ does not contain terms with x_i or x_j in it, we have that

$$\det(M_H(x_0,\dots,x_k)) \not\equiv 0.$$

This finishes the proof. □

5.2. Graphs of type β_n

Let $H = (V,E) \in \mathcal{H}$ be a graph of type β_n. So H contains at least $n+1$ obviously different loops and cycles $C_0 = (V_{C_0}, E_{C_0}), C_1 = (V_{C_1}, E_{C_1}), \dots, C_n = (V_{C_n}, E_{C_n})$. Without loss of generality we can assume that for all $0 \le i \le n$ we have that

$$x_i \in V_{C_i} \setminus \left(\bigcup_{j=0, j\neq i}^{n} V_{C_j} \right).$$

Let

$$N_H(x_0,x_1,\dots,x_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x\in V_{C_0}} x & \sum_{x\in V_{C_0}} x^2 & \dots & \sum_{x\in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x\in V_{C_1}} x & \sum_{x\in V_{C_1}} x^2 & \dots & \sum_{x\in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x\in V_{C_n}} x & \sum_{x\in V_{C_n}} x^2 & \dots & \sum_{x\in V_{C_n}} x^n \end{pmatrix}$$

Then we have that

$$\det(N_H(x_0,x_1,\dots,x_n,0,\dots,0)) = \det \begin{pmatrix} |V_{C_0}| & x_0 & x_0^2 & \dots & x_0^n \\ |V_{C_1}| & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}$$

$$= \sum_{l=0}^{n} (-1)^{l+2} |V_{C_l}| \det \begin{pmatrix} x_0 & x_0^2 & \dots & x_0^n \\ x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{l-1} & x_{l-1}^2 & \dots & x_{l-1}^n \\ x_{l+1} & x_{l+1}^2 & \dots & x_{l+1}^n \\ x_{l+2} & x_{l+2}^2 & \dots & x_{l+2}^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \dots & x_n^n \end{pmatrix}$$

$$= \sum_{l=0}^{n} (-1)^{l} |V_{C_l}| \prod_{\substack{0\le i<j\le n \\ i,j\neq l}} (x_j - x_i) \not\equiv 0.$$

Therefore, $\det(N_H(x_0,\dots,x_k)) \not\equiv 0$. So, for every open set $U \subseteq \mathbb{R}^{k+1}$ there is an open set $U_H \subseteq U$ such that for all $(x_0,\dots,x_k) \in U_H$

$$\det(N_H(x_0,\dots,x_k)) \neq 0.$$

5.3. Graphs of type γ_n

Let $H = (V,E) \in \mathcal{H}$ be a graph of type γ_n. Let $V_0 \subseteq V$ be a maximal subset such that the direct successors of the vertices in V_0 are pairwise different. Since H contains at least n solitary paths there is a set $W_0 \subseteq V \setminus V_0$ which contains at least n points. We define the matrix $L_H(x_0,x_1,\dots,x_k)$ belonging to H as follows:

$$L_H(x_0,\dots,x_k) = \begin{pmatrix} v_{j_0} - v_{i_0} \\ v_{j_1} - v_{i_1} \\ \vdots \\ v_{j_{n-1}} - v_{i_{n-1}} \end{pmatrix},$$

where for all $0 \leq l \leq n-1$ the vertices $v_{i_l} \in V_0$ and $v_{j_l} \in W_0$ have the same successor in H and the vertices v_{j_l}, $0 \leq l \leq n-1$, are pairwise different.

Lemma 24. *Let $H = (V,E) \in \mathcal{H}$ be a graph of type γ_n and let*

$$L_H(x_0,\dots,x_k)$$

be a matrix belonging to H. Then we have that $\det(L_H(x_0,x_1,\dots,x_k)) \not\equiv 0$.

Proof. Let V_0 and W_0 be as above. Without loss of generality we assume that $x_{j_l} = x_l$ for all $0 \leq l \leq n-1$. Since $V_0 \cap W_0 \neq \emptyset$ we have that $V_0 \subseteq \{x_{n+1},\dots,x_k\}$. Let $x_{n+1} = x_{n+2} = \cdots = x_k = 0$. Then we have that

$$L_H(x_0,x_1,\dots,x_n,0,\dots,0) = \begin{pmatrix} x_0 & x_0^2 & \dots & x_0^n \\ x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^n \end{pmatrix}.$$

This is a Vandermonde matrix. Its determinant is not constantly equal to zero. Therefore, $\det(L_H(x_0,x_1,\dots,x_k)) \not\equiv 0$. □

6. Magic sets for $\mathcal{Q}_n$

To construct a magic set for $\mathcal{Q}_n$ we could redo the construction from Section 5. However, this is not necessary:

Fact 25. Let $M \subseteq \mathbb{R}$ be a magic set for $\mathcal{P}_n$ and let $f \in \mathcal{P}_n$. Then we have that $|f[M]| \geq n+1$.

Proof. Let $M = \{m_1, m_2, \dots, m_k\} \subseteq \mathbb{R}$ be a magic set for $\mathcal{P}_n$ and assume towards a contradiction that there is an $f \in \mathcal{P}_n$ with $|f[M]| \leq n$. Note that $k \geq 2n+1$ by Section 2. So, there is a non-constant polynomial $g \in \mathcal{P}_n$ with $g \neq f$ and $g[\{m_1, \dots, m_n\}] = f[M]$. Therefore, $f[M] \subseteq g[M]$ but $f \neq g$ which contradicts the assumption that M is a magic set for $\mathcal{P}_n$. □

Lemma 26. *Every magic set for $\mathcal{P}_n$ is also a magic set for $\mathcal{Q}_n$.*

Proof. Let $M \subseteq \mathbb{R}$ be a magic set for $\mathcal{P}_n$ and let $f, g \in \mathcal{Q}_n$ with $f[M] \subseteq g[M]$. Let

$$f(x) = f_0(x) + if_1(x) \quad \text{and} \quad g(x) = g_0(x) + ig_1(x)$$

where f_0, f_1, g_0 and g_1 are polynomials of degree at most n with real coefficients. By our assumption we have that

$$f_0[M] \subseteq g_0[M] \quad \text{and} \quad f_1[M] \subseteq g_1[M],$$

because $f[M] \subseteq g[M]$ and M contains only real numbers. Note that f_0 or f_1 is not constant. Without loss of generality we assume that f_1 is not constant. Since $f_1[M] \subseteq g_1[M]$, g_1 is also not constant. So, we have that $f_1 = g_1$ because M is a magic set for $\mathcal{P}_n$. If f_0 is also not constant, it follows that $f_0 = g_0$ and therefore $f = g$. So, assume that f_0 is constantly equal to $c \in \mathbb{R}$. By Fact 25 there are $m_1, m_2, \dots, m_{n+1} \in M$ such that $f_1(m_1), f_1(m_2), \dots, f_1(m_{n+1})$ are pairwise different. Since $f[M] \subseteq g[M]$ there are pairwise different $m_{i_1}, m_{i_2}, \dots, m_{i_{n+1}} \in M$ such that for $1 \leq k \leq n+1$ we have

$$c + if_1(m_k) = g_0(m_{i_k}) + ig_1(m_{i_k}) \;\Rightarrow\; f_1(m_k) = g_1(m_{i_k}) \wedge c = g_0(m_{i_k}).$$

So, $g_0(x) - c$ is a polynomial of degree at most n that has at least $n+1$ zeros. This shows that g_0 is constantly equal to c. Therefore we have $f_0 = g_0$ which implies $f = g$. □

Declaration of competing interest

None declared.

Acknowledgement

We would like to thank the referee for his or her careful reading of our manuscript and the comment on the proof of Lemma 26.

References

[1] Alessandro Berarducci, Dikran Dikranjan, Uniformly approachable functions and spaces, in: Proceedings of the Eleventh International Conference of Topology, Trieste, 1993, vol. 25, 1994, pp. 23–55, 1993.

[2] Maxim R. Burke, Krzysztof Ciesielski, Sets on which measurable functions are determined by their range, Can. J. Math. 49 (6) (1997) 1089–1116.

[3] Krzysztof Ciesielski, Saharon Shelah, A model with no magic sets, J. Symb. Log. 64 (4) (1999) 1467–1490.

[4] Harold G. Diamond, Carl Pomerance, Lee Rubel, Sets on which an entire function is determined by its range, Math. Z. 176 (3) (1981) 383–398.

[5] Lorenz Halbeisen, Norbert Hungerbühler, Salome Schumacher, Sets and multisets of range uniqueness for polynomials, Linear Algebra Appl. 589 (2020) 39–61.

[6] Lorenz Halbeisen, Marc Lischka, Salome Schumacher, Magic sets, Real Anal. Exch. 43 (1) (2018) 187–204.