The coefficient of x^{2n-1} in $h(x)$ is the coefficient of x^{2n} in $g/(1+g)$. A slightly more general version of the Lagrange Inversion Formula yields

$$[x^{2n-1}]h(x) = \frac{1}{n}[\lambda^{n-1}]\frac{d}{d\lambda}\left(\frac{\lambda}{1+\lambda}\right)(1+\lambda)^{3n} = \frac{1}{n}\binom{3n-2}{n-1}.$$

Thus $|S_{2n}| = \frac{1}{n}\binom{3n}{n-1}$ and $|S_{2n-1}| = \frac{1}{n}\binom{3n-2}{n-1}$.

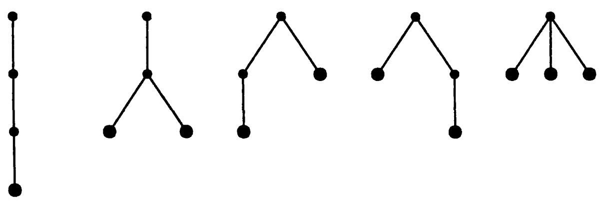
Solution III by Southwest Missouri Problems Group. Let $t_{n,k}$ denote the number of strings in S_n that end in k; this is nonzero only when $n \equiv k \pmod 2$. Since appending a 1 to a string in S_{2m} yields a string in S_{2m+1} and appending a 2 to a string in S_{2m+1} yields an element of S_{2m+2}, we have $|S_{2m}| = \sum_{k\geq 0} t_{2m,2k} = t_{2m+1,1}$ and $|S_{2m+1}| = t_{2m+2,2}$. Similarly, $t_{n,k} = t_{n-1,k-1} + t_{n,k+2}$, with $t_{1,1} = 1$ and $t_{n,k} = 0$ for $k > n$. By induction on $2n-k$, we verify that the solution to this recurrence is $t_{n,k} = \frac{2k}{3n-k}\binom{(3n-k)/2}{n}$. This specializes to the formulas for S_{2m} and S_{2m+1} obtained in the other two solutions.

Editorial comment. Bijective proofs were supplied also by O. P. Lossers and by Robin Chapman (based respectively on ballot sequences and on H. Snevily and D. B. West, The bricklayer problem and the strong cycle lemma, this MONTHLY **105** (1998) 131–143). David Beckwith gave a variant of Solution II avoiding the more general form of Lagrange Inversion via an extra application of the elementary form.

Solved also by D. Beckwith, R. J. Chapman (U. K.), O. Krafft (Germany), O. P. Lossers (The Netherlands), J. Murray (Ireland), K. Schilling, Q. Zheng, Centre College Problems Group, GCHQ Problems Group (U. K.), NSA Problems Group, and the proposer.

Leaves of Ordered Trees

10753 [1999, 777]. *Proposed by Louis Shapiro, Howard University, Washington, DC.* An *ordered tree* is a rooted tree in which the children of each node form a sequence as opposed to a set. The 5 ordered trees with 3 edges are



The number of ordered trees with n edges is the nth Catalan number $\binom{2n}{n}/(n+1)$. Therefore, if one draws each of the ordered trees with n edges, one draws a total of $\binom{2n}{n}$ nodes. Prove that exactly half of these nodes are end-nodes (i.e., leaves with no children).

Composite solution I by Richard Ehrenborg, Royal Institute of Technology, Stockholm, Sweden, and John W. Moon, University of Alberta, Edmonton, Alberta, Canada. Let C_n denote the nth Catalan number. Counting the leaves on all ordered trees with n edges is equivalent to counting the pairs consisting of an ordered tree and one marked leaf. Such an ordered tree is obtained in a unique way from an unmarked ordered tree with $n-1$ edges by attaching a marked leaf to some node of the unmarked tree. A new leaf can be attached at a node p with $d(p)$ children in $d(p)+1$ ways. For each tree with $n-1$ edges, there are thus $\sum_p (d(p)+1)$ ways of attaching a marked leaf. Since $\sum_p d(p)$ counts each edge exactly once, the number of ways to grow each tree is $2n-1$. The desired count of leaves in ordered trees with n edges is thus

$$(2n-1)C_{n-1} = \frac{2n-1}{n}\binom{2n-2}{n-1} = \frac{1}{2}\binom{2n}{n}.$$

Solution II by Michael Reid, Brown University, Providence, RI. We prove the more general statement that for $n > 0$ and $r \geq 0$, the number of nodes with exactly r children among all ordered trees with n edges is $\binom{2n-r-1}{n-1}$.

We use induction on n, with trivial basis $n = 1$. Now consider $n > 1$. Let $F(n, r)$ be the set of paris (T, v), where T is an ordered tree with n edges and v is a vertex of T with exactly r children. Let $f(n, r) = |F(n, r)|$.

For $r > 0$, we establish a bijection from $F(n, r)$ to $\bigcup_{s \geq r-1} F(n-1, s)$. Given $(T, v) \in F(n, r)$, contract the edge in T from v to its last child w. The result is (T', v'), in which the children of v' are the first $r - 1$ children of v followed by all children of w, in order. Hence $(T', v') \in \bigcup_{s \geq r-1} F(n-1, s)$. To construct the inverse for $(T', v') \in \bigcup_{s \geq r-1} F(n-1, s)$, detach all but the first $r - 1$ children of v', and introduce a new rth child w as the parent of the detached children. The result is the only tree with n edges that maps to (T', v').

Since the sets $F(n - 1, s)$ are disjoint for distinct s, the induction hypothesis and a standard binomial coefficient identity yield

$$f(n, r) = \sum_{s=r-1}^{n-1} f(n-1, s) = \sum_{s=r-1}^{n-1} \binom{2n-s-3}{n-2} = \binom{2n-r-1}{n-1}.$$

By the same identity, $\sum_{r=1}^{n} \binom{2n-r-1}{n-1} = \binom{2n-1}{n}$. Since the total number of vertices in these trees is $\binom{2n}{n}$, we have

$$f(n, 0) = \binom{2n}{n} - \sum_{r=1}^{n} f(n, r) = \binom{2n}{n} - \binom{2n-1}{n} = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}.$$

Editorial comment. Other solvers used bijections involving binary trees, bijections involving Dyck paths, generating functions, Narayana numbers, convolutions, and induction. John W. Moon notes that the problem appeared in probabilistic guise in N. Dershowitz and S. Zaks, Enumerations of ordered trees, *Disc. Math.* **31** (1980) 9–28. An analogue for weighted ordered trees is proved in L. H. Clark, A. Meir, and J. W. Moon, On the Steiner distance of trees from certain families, *Australasian J. Comb.* **20** (1999) 47–68 (see p. 64).

Solved also by C. Anderson, C. Baltus, D. Beckwith, J. C. Binz (Switzerland), D. M. Bloom, D. Callan, S. Cautis (Canada), R. J. Chapman (U. K.), C. Chauve (France), A. Del Lungo & S. Rinaldi (Italy), E. Deutsch, S. B. Ekhad, N. Hungerbuhler (Switzerland), G. Isaak, R. Johnsonbaugh, D. E. Knuth, S. C. Locke, O. P. Lossers (The Netherlands), J. H. Nieto (Venezuela), A. Nijenhuis, C. R. Pranesachar (India), H. Prodinger (Austria), D. G. Rogers, P. Simeonov, N. K. Vishnoi, GCHQ Problems Group (U. K.), NCCU Problems Group, SJSU Problems Ring, Southwest Missouri Problems Group, and the proposer.

10757 [1999, 778]. *Proposed by Mark Kidwell, United States Naval Academy, Annapolis, MD.* Given integers $a_0, a_1, a_2, \ldots, a_n$ with $a_i \neq 0$ for $i \geq 1$, write $[a_0; a_1, a_2, \ldots, a_n]$ for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Every positive rational number has a unique representation as $[a_0; a_1, a_2, \ldots, a_n]$ if we require that $a_0 \geq 0$, $a_i > 0$ for $1 \leq i \leq n-1$, and $a_n > 1$ (we call this the *standard representation*), but it can have other representations $[b_0; b_1, b_2, \ldots, b_m]$ if we permit negative values for some of the b_i or if we permit $b_m = 1$. For example, $11/3 = [3; 1, 2] = [3; 1, 1, 1] = [4; -3]$. Prove or disprove: If r is a positive rational number, $r = [a_0; a_1, a_2, \ldots, a_n]$ is the standard representation, and $r = [b_0; b_1, b_2, \ldots, b_m]$ is another representation, then $a_0 + a_1 + \cdots + a_n \leq |b_0| + |b_1| + \cdots + |b_m|$, with strict inequality if any of the b_i are negative.

A simple proof of Shapiro's Theorem

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Abstract

We prove Shapiro's Theorem by applying the well known bijection between Catalan trees and trivalent plane rooted trees, and using a simple symmetry argument.

Key words: Catalan trees, trivalent plane rooted trees, terminal vertices

1 Shapiro's Theorem

For $n \in \mathbb{N}_0$, let C_n denote the set of planted planar trees[1] with $n + 1$ edges, sometimes called *Catalan trees*. Figure 1 shows C_3. Terminal edges which are not incident with the root are called *leaves*. Shapiro observed the following:

Theorem *For $n > 0$ exactly half of the edges of the planted planar trees in C_n are leaves.*

Shapiro presented a proof of this result using generating functions in [4], but finding it so attractive, and believing that there must be other, neater, more insightful proofs, offered it also as a problem in *The American Mathematical Monthly* [3]. A detailed history of Shapiro's Theorem, and additional bibliographic remarks on Catalan Problems can be found in [2].

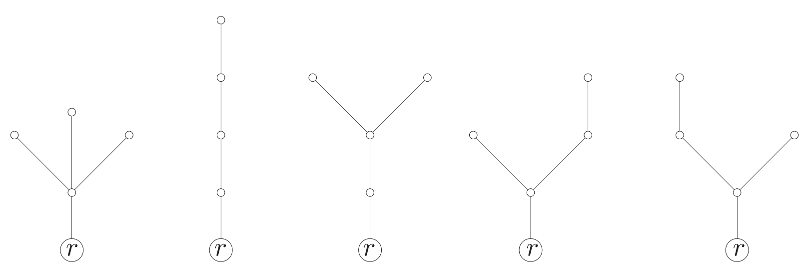

Figure 1: $C_3 = \{$Catalan trees with 4 edges$\}$. 10 among the altogether 20 edges are leaves.

2 A simple proof of Shapiro's Theorem

We recall that there is a bijection between the sets C_n of Catalan trees and T_n, the sets of planar rooted trivalent trees with $n + 1$ leaves (Figure 2 shows T_3). This bijection between C_n and T_n can be described as follows: First we bring a trivalent rooted planar tree in a special position—starting from the root, all edges run from bottom to top or from left to right. Then we contract the

[1] i.e. planar trees with a root r of degree 1

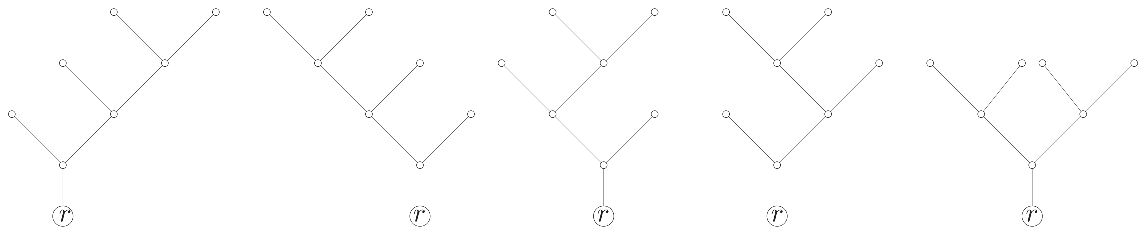

Figure 2: $T_3 = \{$trivalent planar rooted trees with 4 leaves$\}$

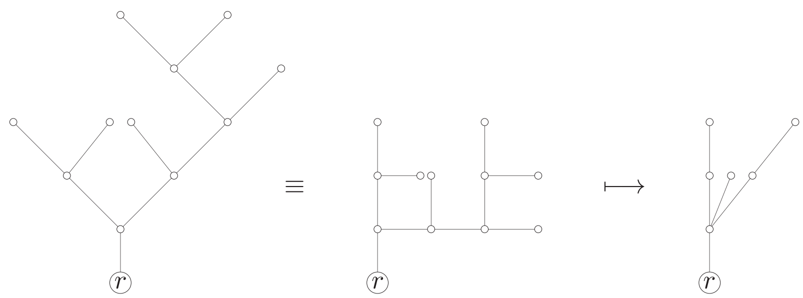

Figure 3: Bijection between T_n and C_n

horizontal edges and obtain the corresponding Catalan tree. See Figure 3 for convenience, and [1] for how to *find* this bijection. So, in particular, $|T_n| = |C_n|$.[2]

The argument of the proof is now simply the following: For $n > 0$ we observe that, by symmetry, just as many leaves in T_n are oriented to the right as to the left. Now, since exactly the edges that go to the right are contracted, there are $\frac{|T_n|(n+1)}{2}$ leaves in C_n. This is, indeed, half of the $|C_n|(n+1)$ edges in C_n!

3 Acknowledgment

The author wishes to thank Douglas G. Rogers for bringing Shapiro's problem to his attention.

References

[1] N. Hungerbühler: The isomorphism problem for Catalan families. J. Combin. Inform. System Sci. **20** (1995), 129–139.

[2] S.-H. Seo: A pairing of the vertices of rooted planar trees. Selected papers in honor of Helge Tverberg. Discrete Math. **241** (2001), no. 1–3, 471–477

[3] L. W. Shapiro: Problem 10753, *Amer. Math. Monthly*, **106** (1999), 777.

[4] L. W. Shapiro: The higher you go, the odder it gets, *Congressus Numerantium*, **138** (1999), 93–96.

[2] And of course $|C_n|$ is the n-th Catalan number.