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Heat flow of p-harmonic maps with values into spheres

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0 Introduction

Let M be a compact m-dimensional smooth Riemannian manifold without boundary and N is the unit sphere S^n of $\mathbb{R}^{n+1}$. In local coordinates $(x_1, x_2, \ldots, x_m)$ the metric g on M is represented by the matrix $(g_{\alpha\beta})_{m\times m}$. For a C^1-map $u\colon M \to N \subset \mathbb{R}^{+1}$ in the local coordinates on M we denote

$$\nabla u := \left(\frac{\partial u^i}{\partial x_\alpha}\right)_{m\times(n+1)}$$

and

$$|\nabla u|^2 = \sum_{\alpha\beta}\sum_i g^{\alpha\beta}\frac{\partial u^i}{\partial x_\alpha}\frac{\partial u^i}{\partial x_\beta}$$

where $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$.

For $1 < p < \infty$, the p-energy of the map is given by

$$E(u) = \int_M |\nabla u|^p \sqrt{|g|}\, dx \tag{0.1}$$

where $g = \det(g_{\alpha\beta})$.

A critical point u on $C^2(M, N)$ of the p-energy E is called a p-harmonic map. The Euler–Lagrange equation of E satisfied by p-harmonic maps from M to $N = S^n$ is

$$-\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_\alpha}\left(|\nabla u|^{p-2} g^{\alpha\beta}\sqrt{|g|}\frac{\partial u}{\partial x_\beta}\right) = |\nabla u|^p u\,. \tag{0.2}$$

If a map $u \in W^{1,p}(M, N)$ satisfies (0.2) in the sense of distributions, u is called a weakly p-harmonic map. Here the Sobolev space $W^{1,p}(M, N)$ is

$$W^{1,p}(M, N) = \{u \in W^{1,p}(M, \mathbb{R}^{n+1}) \,|\, u \in N \text{ for a.e. } x \in M\}\,.$$

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The regularity of minimizing p- harmonic maps between two compact smooth Riemannian manifold has been widely discussed, see [HL, GM, FH, L, CGu]. Motivated by these results we study the existence of p-harmonic maps. The following question arises: Given a map $u_0 \in W^{1,p}(M, N)$, can u_0 be deformed into a p-harmonic map $u: M \to N$? As well known, the "heat flow" method has been studied by Eells and Sampson [ES] to derive the existence of harmonic maps in the case where $N \subset \mathbb{R}^k$ has nonpositive sectional curvatures. In such a case, it has been shown that there exists a global smooth solution $u: \mathbb{R}_+ \times M \to N$ to the following Cauchy problem

$$u_t - \Delta_M u = A(u)(du, du), \quad \forall t > 0, x \in M \,, \tag{0.3}$$

$$u(0, x) = u_0(x), \quad \forall x \in M \,. \tag{0.4}$$

where $A(u): T_u N \times T_u N \in (T_u N)^{\perp}$ is the second fundamental formof N in $\mathbb{R}^k$ and $u_0 \in C^{2,\alpha}(M, N)$. Without the assumptions on the curvature, the solution to (0.3)–(0.4) may blow up in a finite time, see [CGa, CD]. Recently Struwe [S1, S2], Chen [C], Chen and Struwe [CS] proved the global existence and partial regularity of weak solution to (0.3) and (0.4).

The main purpose of this paper is to investigate the global existence of weak solutions to the heat flow of p-harmonic maps in the case where $N = S^n \subset \mathbb{R}^{n+1}$ and $2 \leqq p < \infty$. That is to study the global existence of weak solutions to the following evolution problem

$$\partial_t u - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_\alpha} \left(|\nabla u|^{p-2} g^{\alpha\beta} \sqrt{|g|} \frac{\partial u}{\partial x_\beta} \right) = |\nabla u|^p u \,, \tag{0.5}$$

$$u(0, x) = u_0 \,, \tag{0.6}$$

$$|u|^2 = 1 \quad u \in \mathbb{R}^{n+1} \,. \tag{0.7}$$

We approximate a solution to (0.5)–(0.7) by the solutions to the following penalized equation

$$\begin{aligned} \partial_t u^k - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_\alpha} \left(|\nabla u|^{p-2} g^{\alpha\beta} \sqrt{|g|} \frac{\partial u^k}{\partial x_\beta} \right) & \\ + k|1 - |u^k|^2|^{2\alpha-2} (|u^k|^2 - 1) u^k = 0, \quad \forall t > 0, x \in M, \ k = 1, \dots, & \end{aligned} \tag{0.8}$$

where α is a constant such that $\frac{1}{2} < \alpha < \frac{p}{4(m-p)} + \frac{1}{2}$ if $p \leqq m$ and $\alpha = 1$ if $p > m$. For fixed k, we can use Galerkin methods to prove the existence of weak solutions of the penalized Eq. (0.8). The "monotonicity trick" is used in this proof. However, due to the higher nonlinearity of $\nabla \cdot (|\nabla u^k|^{p-2} \nabla u^k)$ we have difficulties to prove that when $k \to \infty$, u^k converges to a map u which is a weak solution of (0.5)–(0.7). From the energy inequality, we know that $u^k(t)$ is uniformly bounded in $L^\infty(0, T; W^{1,p}(M))$ and $\partial_t u^k$ is uniformly bounded in $L^2(0, T; L^2(M))$. By a modification of Kondrachov's compactness theorem, we can prove that u^k strongly converges to u in $L^p(0, T; L^p(M))$. The main difficulty is to prove that $|\nabla u^k|^{p-2} \nabla u^k$ converges to $|\nabla u|^{p-2} \nabla u$ weakly in $L^{p'}(0, T; L^{p'}(M))$. Fortunately, the term $k||u^k|^2 - 1|^{2\alpha-2}(|u^k|^2 - 1)u^k$ is not "too bad" and uniformly bounded in $L^1(0, T; L^1(M))$. Therefore, we can modify a compactness assertion present by Evans in [Ev] to obtain the strong convergence of ∇u_k in $L^q(0, T; L^q(M))$ for each $1 \leqq q < p$ and to overcome this difficulty.

1 Preliminaries and the penalized approximation equation

In this section we shall give some lemmas and prove the global existence of weak solutions to the penalized approximation equation.

First, we state a lemma so called the "decisive monotonicity trick" (see Zeidler's book [Z]).

Lemma 1.1 *Suppose that X is a reflexive Banach space and X^* is the dual space of X, suppose also that an operator $A: X \to X^*$ satifies*

(i) *A is monotone on X, i.e. $\langle Au - Av, u - v\rangle \geqq 0$ for all, $u, v \in X$.*

(ii) *A is hemicontinuous, i.e. the map $t \to \langle A(u + tv), w\rangle$, is continuous on $[0, 1]$ for all $u, v, w \in X$.*

Then, it follow from

(iii) *$u_n \rightharpoonup u$ weakly in X as $n \to \infty$, $Au_n \to b$ weakly in X^* as $n \to \infty$ and*

$$\overline{\lim_{n\to\infty}} \langle Au_n, u_n\rangle \leqq \langle b, u\rangle$$

that

$$Au = b\,.$$

For $2 \leqq p < \infty$, let $X = L^p_{\text{loc}}(0, \infty; W^{1,p}(M))$ and $X^* = L^{p'}_{\text{loc}}(0, \infty; W^{-1,p'}(M))$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Let also

$$(1.1) \qquad A\varphi = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_\alpha}\left(|\nabla\varphi|^{p-2} g^{\alpha\beta}\sqrt{|g|}\,\frac{\partial\varphi}{\partial x_\beta}\right), \quad \text{for all } \varphi \in X\,.$$

For two maps $u, v: M \to N \subset \mathbb{R}^{n+1}$, we define

$$\nabla u\cdot\nabla v = \sum_{\alpha,\beta}\sum_i g^{\alpha\beta}\nabla u^i{}_\alpha \nabla v^i{}_\beta\,.$$

Lemma 1.2 *Let $p \geqq 2$. Then there holds for all $a, b \in \mathbb{R}^{nk}$*

$$(1.2) \qquad (|a|^{p-2}a - |b|^{p-2}b)\cdot(a - b) \geqq 2^{p-2}|a - b|^p\,.$$

Proof. By a suitable rotation and dilatation, the problem reduces to two dimensions where the verification is elementary. □

By this lemma, it is easy to verify that A defined in (1.1) satisfies the assumptions (i) and (ii) of Lemma 1.1. Of course, the monotonicity of A also follows from the convexity of the p-energy functional (see [Ev]).

Therefore we have

Corollary 1.3. *Let A be a operator defined in* (1.1). *Suppose that $\{u_n\}$ converges to u weakly in $L^p_{\text{loc}}(0, \infty; W^{1,p}(M))$ and for any $T > 0$,*

$$(1.3) \qquad \overline{\lim_{n\to\infty}} \int_0^T \langle Au_n, u_n\rangle_{W^{-1,p'},W^{1,p}}\,dt \leqq -\int_0^T \langle w, \nabla u\rangle_{L^{p'},L^p}\,dt$$

where w is the weak limit of the sequence $\{|\nabla u_n|^{p-2}\nabla u_n\}$ *in* $L^{p'}_{\mathrm{loc}}(0, \infty; L^{p'}(M))$. *Then* $\{Au_n\}$ *converges to Au weakly in* $L^{p'}_{\mathrm{loc}}(0, \infty; W^{-1,p'}(M))$.

Proof. The weak convergence of $\{u_n\}$ in $L^p_{\mathrm{loc}}(0, \infty; W^{1,p}(M))$ implies the weak convergence of $\{Au_n\}$ in $L^{p'}_{\mathrm{loc}}(0, \infty; W^{-1,p'})$. Au_n converges to $\nabla \cdot w$ weakly in $L^{p'}_{\mathrm{loc}}(0, \infty; W^{-1,p'}(M))$. Then, by Lemma 1.1 and (1.3) we have

$$\nabla \cdot w = Au .$$

Thus, the proof of this corollary is completed. □

Lemma 1.4 *Suppose that* $\{u_l\}$ *is bounded in* $L^\infty(0, T; W^{1,p}(M))$, $1 \leqq p < \infty$ *and* $\{\partial_t u_l\}$ *is bounded in* $L^2(0, T; L^2(M))$. *Then* $\{u_l\}$ *subconverges to u strongly in* $L^r(0, T; L^r(M))$ *for each* r, $p \leqq r < \dfrac{mp}{m-p}$.

Proof. By the Kondrachov compactness theorem (see [GT]), we know that there exists a subsequence of $\{u_l\}$, without changing notation, such that as $l \to \infty$

$$(1.4) \qquad u_l \to u \quad \text{strongly in} \quad L^1(0, T; L^1(M)), \quad \text{and} \quad u \in L^\infty(0, T; W^{1,p}(M)) .$$

On the other hand, by using Theorem 2.2 of Chap. II of [LSU] (see pp. 62–68), we have

$$(1.5) \qquad \|u_l - u\|_r \leqq \left\| u_l - u - \frac{1}{|M|}\int_M (u_l - u)\, dx \right\|_r + C \left| \int_M (u_l - u)\, dx \right|^r$$
$$\leqq C \|\nabla_x(u_l - u)\|_p^a \left\| u_l - u - \frac{1}{|M|}\int_M (u_l - u)\, dx \right\|_1^{1-a} + C\|u_l - u\|_1^r$$
$$\leqq C \|\nabla_x(u_l - u)\|_p^a (2\|u_l - u\|_1)^{1-a} + C\|u_l - u\|_1^r$$

for all $0 \leqq t \leqq T$, where

$$a = \left(1 - \frac{1}{r}\right)\left(\frac{1}{m} - \frac{1}{p} + 1\right)^{-1} .$$

By the assumption of $p \leqq r < \dfrac{mp}{m-p}$, we know

$$(1.6) \qquad 1 - a = \frac{\dfrac{1}{m} - \dfrac{1}{p} + \dfrac{1}{r}}{\dfrac{1}{m} - \dfrac{1}{p} + 1} > 0 .$$

Since $\{u_l(t)\}$ is bounded in $L^\infty(0, T; W^{1,p}(M))$, for $t \leqq T$, there exists a constant $C > 0$ such that

$$(1.7) \qquad \|u_l - u\|_1 \leqq C .$$

Integrating both sides of (1.5) from 0 to T, using (1.4) and (1.7) we have

$$(1.8) \qquad \left(\int_0^T \|u_l - u\|_r^r\, dt\right)^{\frac{1}{r}} \leqq C \sup_t \|\nabla(u_l(x,t) - u(x,t))\|_p^a \left(\int_0^T \|u_l - u\|_1^{(1-a)r}\, dt\right)^{\frac{1}{r}}$$
$$+ C\|u_l - u\|_1^r \to 0$$

as $l \to \infty$. □

We approximate a solution to (0.5)–(0.7) by solving the penalized Eq. (0.8) with the initial data

$$u^k(0, x) = u_0(x), \quad x \in M, \quad k = 1, \ldots, . \tag{1.9}$$

We may use Galerkin's method and the decisive monotonicity trick to solve the problem (0.8) and (1.9) for every fixed $k \geqq 1$.

Theorem 1.5 *Let $u_0 : M \to S^n$ satisfying $u_0(x) \in W^{1,p}(M, S^n)$, $2 \leqq p < \infty$. Then for every fixed $k \geqq 1$, there exists a weak solution for the penalized approximation eqs.* (0.8) *and* (1.9).

Proof. Assume that $\{w_j\}_{j=1,2,\ldots,}$ is a base in the space $W^{1,p}(M)$. For any fixed positive integer l, we try to find an approximate solution $u_l = u_l^k$,

$$u_l(t) = \sum_{i=1}^{l} g_{il}(t) w_i$$

such that

$$\langle u_l', w_j \rangle + \langle |\nabla u_l|^{p-2} \nabla u_l, \nabla w_j \rangle_{p,p'} + k \langle \, | |u_l|^2 - 1|^{2\alpha - 2} (|u_l|^2 - 1) u_l, w_j \rangle_{p',p} = 0 \tag{1.10}$$

for $1 \leqq j \leqq l$, with

$$u_l(0) = u_{0l} \tag{1.11}$$

where

$$u_{0l} = \sum_{i=1}^{l} \xi_i w_i \to \varphi \text{ strongly in } W^{1,p}(M) \text{ as } l \to \infty . \tag{1.12}$$

$\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle_{p',p}$ denote the inner product in $L^2(M)$ and the dual product in $L^p(M)$, $(p \geqq 2)$ respectively.

By the well-known results on the systems of ordinary differential equations, the Cauchy problem (1.10)–(1.11) has a unique short time solution u_l. Then, by the uniform estimate of the local solution and the extensive methods, we can get a global solution u_l such that $u_l \in L^\infty(0, \infty; W^{1,p}(M))$, $u_l' \in L^2(0, \infty; L^2(M))$ and

$$\int_0^t \|u_l'(\tau)\|^2_{L^2(M)} d\tau + \frac{1}{p} \|\nabla u_l\|^p_{L^p(M)}(t) + \frac{k}{4} \int_M | |u_l|^2 - 1|^{2\alpha} dM = \frac{1}{p} \|\nabla u_{0l}\|^p_{L^p(M)} + \frac{k}{4} \int_M | |u_{0l}|^2 - 1|^2 dM, \quad \forall t > 0 . \tag{1.13}$$

Then, by Pioncaré's inequality we have

$$\{u_l'\} \text{ is a bounded set in } L^2(0, \infty; L^2(M)) , \tag{1.14}$$

$$\{u_l\} \text{ is a bounded set in } L^\infty(0, \infty; W^{1,p}(M)) \tag{1.15}$$

and

$$\{u_l\} \text{ is a bounded set in } L^\infty(0, \infty, L^{4\alpha}(M)) . \tag{1.16}$$

By the compactness of these spaces mentioned in (1.14)–(1.15) and Lemma 1.4, one can pass to a subsequence, without changing notation, to get that as $l \to \infty$,

$$u_l \rightharpoonup u \quad \text{weakly* in } L^\infty(0, \infty; W^{1,p}(M)), \tag{1.17}$$

$$u_l' \rightharpoonup u' \quad \text{weakly in } L^2(0, \infty; L^2(M)). \tag{1.18}$$

and

$$u_l \to u \text{ strongly in } L^p_{\text{loc}}(0, \infty; L^p(M)) \text{ and a.e. on } \mathbb{R}_+ \times M, \tag{1.19}$$

Moreover, from (1.15) we have

$$|\nabla u_l|^{p-2}\nabla u_l \rightharpoonup \omega \text{ weakly* in } L^{p'}_{\text{loc}}(0, \infty; L^{p'}(M)). \tag{1.20}$$

From (1.4), (1.18) and (1.20), we get that as $l \to \infty$,

$$k\langle ||u_l|^2 - 1|^{2\alpha-2}(|u_l|^2 - 1)u_l, w_j\rangle_{p',p} \to -(u', w_j) - (w, \nabla w_j\rangle_{p',p}. \tag{1.21}$$

Since $\{w_j\}$ is a base in $W^{1,p}(M)$, (1.21) shows that

$$k||u_l|^2 - 1|^{2\alpha-2}(|u_l|^2 - 1)u_l \to -u' + \nabla\cdot w \quad \text{weakly in } L^{p'}_{\text{loc}}(0, \infty; W^{-1,p'}(M)).$$

However, from (1.15)–(1.16) and the assumption on α, we know that

$$k||u_l|^2 - 1|^{2\alpha-2}(|u_l|^2 - 1)u_l \quad \text{is bounded in } L^{p'}_{\text{loc}}(0, \infty; L^{p'}(M)). \tag{1.22}$$

Therefore, we conclude that

$$k||u_l|^2 - 1|^{2\alpha-2}(|u_l|^2 - 1)u_l \rightharpoonup -u' + \nabla\cdot w \text{ weakly in } L^{p'}_{\text{loc}}(0, \infty; L^{p'}(M)). \tag{1.23}$$

Denote $V = W^{1,p}(M)$ and $V' = W^{-1,p'}(M)$.

From (1.4) and (1.18), (1.19) and (1.23) we know that for any $T > 0$

$$\begin{aligned} \overline{\lim_{l\to\infty}} \int_0^T \langle Au_l, u_l\rangle_{V',V}\, dt &= \overline{\lim_{l\to\infty}} - \int_0^T \langle |\nabla u_l|^{p-2}\nabla u_l, \nabla u_l\rangle_{p',p} dt \\ &= \overline{\lim_{l\to\infty}} \left\{ \int_0^T \langle u_l', u_l\rangle\, dt + k\int_0^T \langle ||u_l|^2 - 1|^{2\alpha-2}(|u_l|^2 - 1)u_l, u_l\rangle_{p',p} dt \right\} \\ &= \int_0^T \langle u', u\rangle\, dt + \int_0^T \{-\langle u', u\rangle - \langle w, \nabla u\rangle_{p',p}\}\, dt \\ &= -\int_0^T \langle w, \nabla u\rangle_{p',p} dt. \end{aligned} \tag{1.24}$$

By Corollary 1.3 we get that as $l \to \infty$,

$$Au_l \rightharpoonup Au \quad \text{weakly in } L^{p'}_{\text{loc}}(0, \infty; W^{-1,p'}(M)). \tag{1.25}$$

Moreover, from (1.19) and (1.22) we know that as $l \to \infty$,

$$k||u_l|^2 - 1|^{2\alpha-2}(|u_l|^2 - 1)u_l \rightharpoonup k||u|^2 - 1|^{2\alpha-2}(|u|^2 - 1)u \quad \text{weakly in } L^{p'}_{\text{loc}}(0, \infty; L^{p'}(M)). \tag{1.26}$$

It follows from (1.18), (1.25) and (1.26) that for every $k \geqq 1$, the problem (0.8) and (1.9) has a global solution $u = u^k(t, x)$ satisfying

$$u^k \in L^\infty(0, \infty; W^{1,p}(M)), \tag{1.27}$$

$$\partial_t u^k \in L^2(0, \infty; L^2(M)), \tag{1.28}$$

and

$$(1.29) \qquad \int_0^T \| \partial_t u^k(\tau)\|^2_{L^2(M)} d\tau + \frac{1}{p}\int_M |\nabla u^k|^p(t)\,dM + \frac{k}{4}\int_M (|u^k|^2 - 1)^{2\alpha}(t)\,dM$$

$$\leqq \frac{1}{p}\int_M |\nabla u_0|^p dM, \quad \forall t > 0 .$$

This proves Theorem 1.5. □

2 Global existence of the weak solutions

In this section, we shall prove that the sequence $\{u^k\}$ of the solutions of the penalized Eqs. (0.8) and (1.9) weakly converges to a map u which is a weak soluton to (0.5)–(0.7).

Definition. A function u is said to be a *global weak solution* to (0.5)–(0.7), if u is defined a.e. on $\mathbb{R}_+ \times M$, such that

(D1) $u \in L^\infty(0, \infty; W^{1,p}(M))$, $\partial_t u \in L^2(0, \infty; L^2(M))$.

(D2) u is weakly continuous in $t > 0$ with value in $W^{1,p}(M)$, i.e. for any test function $v \in C^\infty(M)$, $f_1(t) = \int_M u \cdot v\,dM$ and $f_2(t) = \int_M \nabla u \cdot \nabla v\,dM$ are continuous in $t > 0$, with eventual modification in a set of measure zero on $(0, \infty)$.

(D3) $|u|^2 = 1$ a.e. on $\mathbb{R}_+ \times M$.

(D4) u satisfies (0.5)–(0.7) in the sense of distributions.

First, let us state a compactness theorem which was presented in [Ev] for $p = 2$.

Theorem 2.1 *For* $k = 1, 2, \ldots$, let $u_k = (u_k^1, \ldots, u_k^{n+1})$ *be vector functions of* (t, x) *on* $[0, t] \times M$ *satisfying the following equation*

$$\partial_t u_k - \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_\alpha}\left(|\nabla u_k|^{p-2} g^{\alpha\beta}\sqrt{|g|}\,\frac{\partial u_k}{\partial x_\beta}\right) = f_k, \quad \forall 0 \leqq t \leqq T, x \in M$$

in the sense of distribution. Assume further that $\{u_k\}_{k=1}^\infty$ *is bounded in* $L^\infty(0, T; W^{1,p}(M))$, $\{\partial_t u_k\}_{k=1}^\infty$ *is bounded in* $L^2(0, T{:}\,L^2(M))$, *and* $\{f_k\}_{k=1}^\infty$ *is bounded in* $L^1(0, T; L^1(M))$. *Then,* $\{u_k\}_{k=1}^\infty$ *is precompact in* $L^q(0, T; W^{1,q}(M))$ *for each* $1 \leqq q < p$.

Proof. By Lemma 1.4, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$ satisfying

$$u_{k_j} \rightharpoonup u \text{ weakly in } L^p(0, T; W^{1,p}(M)) \text{ and } u_{k_j} \to u \text{ strongly in } L^p(0, T; L^p(M)) .$$

Take a K such that $K \geqq \|\nabla u_i\|_{L^p(0,T;ML^p(M))}$ and $K \geqq \|f_i\|_{L^1(0,T;L^1(M))}$ for all i. Take a subsequence $\{u_j\} \subset \{u_i\}$ so that $u_j \rightharpoonup u$ in $H^{1,p}(M, R^m)$ for some $u \in H^{1,p}$. $K \geqq \|\nabla u\|_{H^{1,p}}$. We show that $u_j \to u$ in $H^{1,q}(M, R^m)$.

For $1 \geqq \delta > 0$, let $E_\delta^i = \{x \in M \times [0, T]\colon |u_i(x, t) - u(x, t)| \geqq \delta\}$. Then

$$(2.1) \qquad \int_{E_\delta^i} |\nabla u_i - \nabla u|^q\,dx\,dt \leqq K^q |E_\delta^i|^{(p-q)/p} .$$

On the other hand, let $\eta(y) = \min\{\delta/|y|, 1\}\, y\colon R^m \to R^m$ with $|\eta| \leqq \delta$. So for some constant C,

$$
\begin{aligned}
C & \int_{M\times[0,T]\setminus E_\delta^i} |\nabla u_i - \nabla u|^p\, dx\, dt \\
& \leqq \int_{M\times[0,T]\setminus E_\delta^i} [|\nabla u_i|^{p-2}\nabla u_i - |\nabla u|^{p-2}\nabla u](\nabla u_i - \nabla u)\, dx\, dt \\
& = \int_{M\times[0,T]} |\nabla u_i|^{p-2}\nabla u_i \nabla(\eta(u_i - u))\, dx\, dt - \delta \int_{E_\delta^i} |\nabla u_i|^{p-2}\nabla u_i \nabla \frac{u_i - u}{|u_i - u|}\, dx\, dt \\
& \quad - \int_{M\times[0,T]\setminus E_\delta^i} |\nabla u|^{p-2}\nabla u(\nabla u_i - \nabla u)\, dx\, dt \\
& = \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
\end{aligned}
$$

Now

$$
|\mathrm{I}| \leqq \left| \int_{M\times[0,T]} f_i \eta(u_i - u) \right| dx\, dt \leqq \delta K
$$

where one uses the equation and Hölder's inequality. For II,

$$
\begin{aligned}
\mathrm{II} = & - \int_{E_\delta^i} \frac{\delta}{|u_i - u|^3} [|\nabla u_i|^{p-2}\nabla u_i^\alpha][(\nabla u_i^\alpha - \nabla u^\alpha)|u_i - u|^2 \\
& - (u_i^\alpha - u^\alpha)(u_i^\beta - u^\beta)(\nabla u_i^\beta - \nabla u^\beta)]\, dx\, dt \\
= & - \int_{E_\delta^i} \frac{\delta}{|u_i - u|^3} [|\nabla u_i|^{p-2}\nabla u_i^\alpha][\nabla u_i^\alpha |u_i - u|^2 \\
& - (u_i^\alpha - u^\alpha)(u_i^\beta - u^\beta)\nabla u_i^\beta]\, dx\, dt \\
& + \int_{E_\delta^i} \frac{\delta}{|u_i - u|^3} [|\nabla u_i|^{p-2}\nabla u_i^\alpha][\nabla u^\alpha |u_i - u|^2 \\
& - (u_i^\alpha - u^\alpha)(u_i^\beta - u^\beta)\nabla u^\beta]\, dx\, dt \\
= & \; \mathrm{II}' + \mathrm{II}''\,.
\end{aligned}
$$

Note that $\mathrm{II}' \leqq 0$ and by Hölder inequality,

$$
|\mathrm{II}''| \leqq 2 \int_{E_\delta^i} |\nabla u_i|^{p-1}|\nabla u|\, dx\, dt \leqq 2K^{p-1}\left(\int_{E_\delta^i} |\nabla u|^p\, dx\, dt \right)^{1/p}.
$$

For III, we use the weak convergence of u_i and Hölder's inequality again to get

$$
\begin{aligned}
|\mathrm{III}| & \leqq \int_{M\times[0,T]} |\nabla u|^{p-2}\nabla u \nabla(u_i - u)| + \left| \int_{E_\delta^i} |\nabla u|^{p-2}\nabla u \nabla(u_i - u) \right| dx\, dt \\
& \leqq o(1) + 2K \left(\int_{E_\delta^i} |\nabla u|^p\, dx\, dt \right)^{(p-1)/p}.
\end{aligned}
$$

So

$$
(2.2) \qquad c \int_{M\times[0,T]\setminus E_\delta^i} |\nabla u_i - \nabla u|^p\, dx\, dt \leqq \mathrm{I} + \mathrm{II}'' + \mathrm{III} \leqq |\mathrm{I}| + |\mathrm{II}''| + |\mathrm{III}|\,.
$$

Choosing δ to be small, using the fact that u is absolutely continuous, $|E_\delta^i| \to 0$ as $i \to \infty$ and (2.1)–(2.2), one sees the convergence of u_i to u in $H^{1,q}$. □

Theorem 2.2 *Let $u_0: M \to S^n$ satisfying $u_0(x) \in W^{1,p}(M, S^n)$. Then, there exists a global weak solution to Eq.* (0.5)–(0.7) *in the sense of* (D1)–(D4).

Proof. By Theorem 1.5 and (1.29), we know that there exists the global weak solution u_k to (0.8) and (1.9) such that

(2.3) $$\{u^k\} \text{ is a bounded set in } L^\infty(0, \infty; W^{1,p}(M)),$$

(2.4) $$\{\partial_t u^k\} \text{ is a bounded set in } L^2(0, \infty; L^2(M)).$$

By the weakly compactness of these spaces mentioned in (2.3)–(2.4), there exists a subsequence of u^k, without changing notation, such that as $k \to \infty$,

(2.5) $$u^k \rightharpoonup u \quad \text{weakly* in } L^\infty(0, \infty; W^{1,p}(M))$$

(2.6) $$\partial_t u^k \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0, \infty; L^2(M))$$

which implies by Lemma 1.4

(2.7) $$u^k \rightharpoonup u \text{ strongly in } L^2_{\text{loc}}(0, \infty; L^2(M)) \text{ and a.e. on } \mathbb{R}_+ \times M.$$

As $k \to \infty$ (2.7) leads to

(2.8) $$|u^k|^2 - 1 \to |u|^2 - 1, \text{ a.e. on } \mathbb{R}_+ \times M.$$

Moreover, from (1.29) we find that as $k \to \infty$,

(2.9) $$|u^k|^2 - 1 \to 0, \quad \text{weakly in } L^{2\alpha}_{\text{loc}}(0, \infty; L^{2\alpha}(M)).$$

The combination of (2.8) and (2.9) shows that

(2.10) $$|u|^2 = 1 \text{ a.e. on } \mathbb{R}_+ \times M.$$

In order to verify (D2), we observe, that (2.5) and (2.7) imply for f_1 and after integration by parts for f_2

$$f_1, f_2 \in L^\infty(0, \infty; \mathbb{R}); \quad \frac{d}{dt} f_1^k(t), \frac{d}{dt} f_2^k(t) \in L^2(0, \infty; \mathbb{R}).$$

Hence $f_1, f_2 \in W^{1,2}_{\text{loc}}(0, \infty; \mathbb{R})$, and the Sobolev embedding theorem gives (D2).

To check (D4) we start with

Lemma 2.3 *The solutions u^k to the penalized Eqs.* (0.8) *and* (1.9) *satisfy* $|u^k| \leqq 1$ *for all* $t \geqq 0$.

Proof. By testing (0.8) with the function

$$u^k - \frac{u^k}{|u^k|} \min\{1, |u^k|\}$$

we find

$$\frac{1}{2} \partial_t \int_{|u^k| \geqq 1} (u^k)^2 \left(1 - \frac{1}{|u^k|}\right)^2 dM + \int_{|u^k| \geqq 1} |\nabla u^k|^p \left(1 - \frac{1}{|u^k|}\right) dM \leqq 0.$$

Since $|u^k| = 1$ for $t = 0$ we get $|u^k| \leqq 1$ for all $t \geqq 0$. □

Now we finish the proof of Theorem 2.2. By Lemma 2.3 it follows from Lebesgues theorem and (2.7) that

$$u^k \to u \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}_+ \times M) \tag{2.11}$$

for all $q \in [1, \infty)$.

On account of (2.3–4) and Lemma 2.3 it is now easy to check that

$$\{k||u^k|^2 - 1|^{2\alpha-2}(|u^k|^2 - 1)u^k\}_{K \in N}$$

is a bounded sequence in $L^1(0, T; L^1(M))$:

$$\int_0^T \int_M k||u^k|^2 - 1|^{2\alpha-1}|u^k| dM\,dt$$

$$\leqq \int_0^T \int_M k||u^k|^2 - 1|^{2\alpha} dM\,dt + \int_0^T \int_M k||u^k|^2 - 1|^{2\alpha-1}|u^k|^2 dM\,dt \leqq C .$$

Applying Theorem 2.1 we get

$$\nabla u^k \to \nabla u \quad \text{strongly in } L^q(0, T; L^q(M)) \quad \text{for each } 1 \leqq q \leqq p . \tag{2.12}$$

Therefore, we have from (2.3) that

$$|\nabla u^k|^{p-2}\nabla u^k \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{weakly in } L^{p'}_{\text{loc}}(0, \infty; L^{p'}(M)) . \tag{2.13}$$

Now, by taking the wedge product of (0.8) with u^k, we get

$$0 = \partial_t u^k \wedge u^k - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_\alpha}\left(|\nabla u^k|^{p-2} g^{\alpha\beta} \sqrt{|g|} \frac{\partial u^k}{\partial x_\beta} \wedge u^k \right) . \tag{2.14}$$

From (2.6) and (2.11), we know that as $k \to \infty$

$$\partial_t u^k \wedge u^k \to \partial_t u \wedge u . \tag{2.15}$$

The combination of (2.11) and (2.13) leads to

$$|\nabla u^k|^{p-2}\nabla u^k \wedge u^k \rightharpoonup |\nabla u|^{p-2}\nabla u \wedge u, \quad \text{as } k \to \infty . \tag{2.16}$$

Due to (2.15) and (2.16), one can pass to the limit in (2.14) to get

$$0 = \partial_t u \wedge u - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_\alpha}\left(|\nabla u|^{p-2} g^{\alpha\beta} \sqrt{|g|} \frac{\partial u}{\partial x_\beta} \wedge u \right)$$

in the sense distribution. This implies that

$$\partial_t u - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_\alpha}\left(|\nabla u|^{p-2} g^{\alpha\beta} \sqrt{|g|} \frac{\partial u}{\partial x_\beta} \right) = \lambda u \tag{2.17}$$

for some function $\lambda : M \to \mathbb{R}$. By taking the scalar product of (2.17) with u and noticing (D1) and (D3), we get

$$\lambda = |\nabla u|^2 u .$$

Therefore, u satisfies (D4). This proves Theorem 2.2. □

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References

[C] Chen, Y.: Weak solutions to the evolution problem of harmonic maps. Math. Z. **201** 69–74 (1989)

[CD] Chen, Y., Ding, W.: Blow-up and global existence for heat flows of harmonic maps. Invent. Math. **99**, 567–578 (1990)

[CS] Chen, Y., Struwe, M.: Existence and partial regularity results for the heat flow for harmonic maps. Math. Z. **201**, 83–103 (1989)

[CGa] Coron, J.M., Gadalgia, J.M.: Explosion en temps fini pour le flot des applications harmoniques. C.R. Acad. Sci. Paris **308**, 339–344 (1989)

[CGu] Coron, J.M., Gulliver, R.: Minimizing p-harmonic maps into spheres. J. Reine Angew. Math. **401**, 82–100 (1989)

[EL] Eells, J., Lemaire, L.: Another report on harmonic maps. Bull. Lond. Math. Soc. **20**, 385–524 (1988)

[ES] Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)

[Ev] Evans, L.C.: Weak convergence methods for nonlinear partial differential equations (CBMS Reg. Conf. Ser. Math. vol., 74) Providence, RI.: Am. Math. Soc. 1990

[FH] Fusco, N., Hutchinson, J.: Partial regularity for minimizers of certain functionals having non quadratic growth. Canberra CMA: Preprint (1985)

[GM] Giaquinta, M., Modica, G.: Remarks on the regularity of the minimizers of certain degenerate functionals. Manuscr. Math. **57**, 55–99 (1986)

[GT] Gilbarg, D., Trudinger, N.: Elliptic partial differential equations of second order, 2nd ed. Berlin Heidelberg New York: Springer 1983

[HL] Hardt, R., Lin, F.H.: Mappings minimizing the L^p norm of the gradient. Commun. Pure Appl. Math. **15**, 555–588 (1987)

[LSU] Ladyzenskaja, O.A., Solonnikov, V.A., Urel'ceva, N.N.: Linear and quasilinear equations of parabolic type. (Transl. Math. Monogr. vol. 23) Providence, R.I: Am. Math. Soc. 1968

[L] Luckhaus, S.: Partial Holder continuity of energy minimizing p-harmonic maps between Riemannian manifolds. Canberra CMA: Preprint (1986)

[S1] Struwe, M.: On the evolution of harmonic maps of Riemannian surfaces. Comm. Math. Helv. **60**, 558–581 (1985)

[S2] Struwe, M.: On the evolution of harmonic maps in high dimensions. J. Differ. Geom. **28**, 485–502 (1988)

[Z] Zeidler, E.: Nonlinear Functional Analysis and its Applications II/B (Nonlinear Monotonic Operators). Berlin Heidelberg New York: Springer 1990