Global Weak Solutions of the p–harmonic Flow into Homogeneous Spaces

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Abstract. For $2 < p < \dim M$ we establish existence of global weak solutions of the p-harmonic flow between Riemannian manifolds M and N for arbitrary initial data having finite p-energy in the case when the target N is a homogeneous space with a left invariant metric. In particular we construct a solution $f : M \times [0, \infty) \to N$ which satisfies the energy inequality

$$\frac{1}{2}\int_0^T \int_M |\partial_t f|^2 d\mu\, dt + \frac{1}{p}\int_M |df(T)|^p d\mu \le \frac{1}{p}\int_M |df(0)|^p d\mu$$

for all $T > 0$. In the proof we combine a (time-) discrete scheme with certain compactness properties of the p-harmonic flow into homogeneous spaces.

1. Introduction. Let M and N be compact smooth Riemannian manifolds without boundaries with metrics γ and g respectively. Let m and n denote the dimensions of M and N. For a C^1-map $f : M \to N$ the p-energy density is defined by

$$e(f)(x) := \frac{1}{p}\,|df_x|^p \tag{1.1}$$

and the p-energy by

$$E(f) := \int_M e(f)\, d\mu\,. \tag{1.2}$$

Here, p denotes a real number in $[2, \infty[$, $|df_x|$ is the Hilbert-Schmidt norm with respect to γ and g of the differential $df_x \in T_x^*(M) \otimes T_{f(x)}(N)$ and μ is the measure on M which is induced by the metric. For concrete calculations we need an expression for $E(f)$ in local coordinates:

$$E_U(f) = \frac{1}{p} \int_\Omega \left(\gamma^{\alpha\beta}(g_{ij} \circ f)\partial_\alpha f^i \partial_\beta f^j\right)^{p/2} \sqrt{\gamma}\, dx\,.$$

Here, $U \subset M$ and $\Omega \subset \mathbb{R}^m$ denote the domain and the range of the coordinates on M and it is assumed that $f(U)$ is contained in the domain of the coordinates chosen on N. Upper indices denote components, whereas ∂_α denotes the derivative with respect to the coordinate variable x^α. We use the usual summation convention.

Variation of the energy-functional yields the Euler-Lagrange equations of the p-energy which are

$$\Delta_p f = -\left(\gamma^{\alpha\beta} g_{ij}\partial_\alpha f^i \partial_\beta f^j\right)^{p/2-1} \gamma^{\alpha\beta}\Gamma^l_{ij}\partial_\alpha f^i \partial_\beta f^j \tag{1.3}$$

in local coordinates. Here, the operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}}\partial_\beta \left(\sqrt{\gamma}\left(\gamma^{\alpha\beta} g_{ij}\partial_\alpha f^i \partial_\beta f^j\right)^{p/2-1} \gamma^{\alpha\beta}\partial_\alpha f^l\right)$$

is called p-Laplace operator (for $p = 2$ this is just the Laplace-Beltrami operator and does not depend on N). On the right hand side of (1.3) the Γ^l_{ij} denote the Christoffel-symbols related to the manifold N. According to Nash's embedding theorem we can think of N as being isometrically embedded in some Euclidean space $\mathbb{R}^k$ since N is compact. Then, if we denote by F the function f regarded as a function into $N \subset \mathbb{R}^k$, equation (1.3) admits a geometric interpretation, namely

$$\Delta_p F \perp T_F N$$

with Δ_p being the p-Laplace operator with respect to the manifolds M and $\mathbb{R}^k$.

For $p > 2$ the p-Laplace operator is degenerately elliptic. (Weak) solutions of (1.3) are called (weakly) p-harmonic maps. One possibility to generate p-harmonic maps is to investigate the heat flow related to the p-energy, i.e., to look at the heat flow-equation

$$\partial_t f - \Delta_p f \quad \perp \quad T_f N \tag{1.4}$$

$$f\big|_{t=0} \quad = \quad f_0 \tag{1.5}$$

or explicitly for (1.4)

$$\partial_t f - \Delta_p f = (p\, e(f))^{1-2/p} A(f)(\nabla f, \nabla f) \tag{1.6}$$

where $A(f)(\,\cdot\,,\,\cdot\,)$ is the second fundamental form on N. For $p = 2$ Eells and Sampson showed in their famous work [5] of 1964, that there exist global solutions of (1.4)–(1.5) provided N has nonpositive sectional curvature and that the flow tends for suitable $t_k \to \infty$ to a harmonic map. Existence and uniqueness of partially regular solutions of the harmonic flow on Riemannian sufaces (i.e., $p = m = 2$) has been shown by Struwe in [10]. For $p = 2$ the higher dimensional problem (i.e., $m > 2$) has been solved by Chen and Struwe in [4]. For $p > 2$ only little is known about existence and regularity of solutions of (1.4): If the target manifold N is a sphere there exists a global weak solution f of the p-harmonic flow with $f \in L^\infty(0,\infty; W^{1,p}(M,N))$ and $f_t \in L^2(0,\infty; L^2(M))$ for arbitrary initial data in $W^{1,p}(M,N)$ (see [3]). Here, $W^{1,p}(M,N)$ denotes the nonlinear Sobolev space of functions $g \in W^{1,p}(M,\mathbb{R}^k)$ with $g(x) \in N$ for almost every $x \in M$. In the conformal case, i.e., if $p = \dim(M)$, there exists (again for initial data in $W^{1,p}(M,N)$) a global weak solution which is partially regular in the sense that $\nabla f \in C^{0,\alpha}$ in space-time up to finitely many singular times $T_1 < T_2 < \ldots < T_K \leq \infty$ (and K is a priori bounded in terms of the initial p-energy). Moreover solutions in $L^\infty(0,T; W^{1,\infty}(M,N))$ are known to be unique (see [8]).

Recently considerable progress has been achieved in different geometrically motivated problems if homogeneous spaces with left invariant metric are assumed as targets: see e.g. [7], [12] or [11]. Here, inspired by Toro and Wang [12], we consider the p-harmonic flow for compact targets N which are homogeneous spaces (i.e., $N = G/H$ is the quotient of a connected Lie group G by a closed discrete subgroup H) with a left invariant metric. It is known that then the space of weak solutions of (1.4) is "weakly compact" (see [9]). However, as far as the p-harmonic flow is concerned it is usually quite difficult to find approximating problems for which one can establish existence and then use the additional compactness properties to pass to the limit. In particular the techniques used in [3] for $N = S^n$ and in [8] for the conformal case do not seem to work here. Nevertheless a (time-) discrete scheme will provide the desired approximate solutions. Similar techniques have been applied by Bethuel, Coron, Ghidaglia, and Soyeur in [1] and recently by Freire in [6]. We restrict ourselves to the range $2 < p < \dim(M)$, since the cases $p = 2$ and $p = \dim(M)$ are known (see [4] and [8]) and existence theory for $p > \dim M$ offers fewer difficulties. We will establish the following theorem:

Theorem 1.1. *For $2 < p < \dim(M)$ there exists a global weak solution of the p-harmonic flow between Riemannian manifolds M and N for arbitrary initial data having finite p-energy in the case when the target N is a homogeneous space with a left invariant metric. The solution $f : M \times [0,\infty) \to N$ satisfies the energy inequality*

$$\frac{1}{2}\int_0^T \int_M |\partial_t f|^2 dt\, d\mu + \frac{1}{p}\int_M |df(T)|^p d\mu \leq \frac{1}{p}\int_M |df(0)|^p d\mu \tag{1.7}$$

for all $T > 0$.

We finish this introduction by fixing some more notations:

$$V(M,N) := \{g \in L^\infty(0,T;W^{1,p}(M,N)) \,:\, \partial_t g \in L^2(0,T;L^2(M))\}$$

be equipped with the norm

$$\|g\|_{V(M,N)} := \operatorname*{ess\,sup}_{0\le t} \|g(t,\cdot)\|_{W^{1,p}(M,N)} + \|\partial_t g\|_{L^2(M\times(0,\infty))}$$

and recall that $L^\infty(0,T;W^{1,p}(M))$ is the dual space of $L^1(0,T;W^{-1,p'}(M))$.

2. The flow equation for homogeneous target. Let X be a Killing field on N, that is the generator of an isometry of N, satisfying

$$\langle \nabla_v X(p), v\rangle = 0 \quad \text{for all } p\in N,\ v\in T_pN, \tag{2.8}$$

where ∇_v denotes the covariant derivative in direction v and $\langle\cdot,\cdot\rangle$ is the inner product on $T_pN \subset T_p\mathbb{R}^k$, that is the scalar product in $\mathbb{R}^k$ restricted to T_pN.

For the sake of simplicity we assume in this section that M is the flat torus $\mathbb{R}^m/\mathbb{Z}^m$ (a justification will be given in the remark towards the end of this section). Hence, if $f \in V(M,N)$ is a weak solution of the p-flow, we have

$$-\int_0^\infty \int_M \langle \partial_\alpha(\zeta X(f)), |df|^{p-2}\partial_\alpha f\rangle d\mu\, dt = \int_0^\infty \int_M \langle \partial_t f, \zeta X(f)\rangle d\mu\, dt$$

for all smooth cutoff functions ζ. Differentiating the product on the left-hand side, we obtain

$$\begin{aligned}
&\int_0^\infty \int_M \zeta\langle \partial_t f, X(f)\rangle d\mu\, dt\\
&\qquad = -\int_0^\infty \int_M (\zeta\langle \partial_\alpha X(f), |df|^{p-2}\partial_\alpha f\rangle + (\partial_\alpha\zeta)\langle X(f), |df|^{p-2}\partial_\alpha f\rangle) d\mu\, dt\\
&= -\int_0^\infty \int_M (\zeta |df|^{p-2} \underbrace{\langle \nabla_{\partial_\alpha f} X(f), \partial_\alpha f\rangle}_{=0 \text{ by } (2.8)} + (\partial_\alpha\zeta)\langle X(f), |df|^{p-2}\partial_\alpha f\rangle) d\mu\, dt\\
&= -\int_0^\infty \int_M (\partial_\alpha\zeta)|df|^{p-2}\langle X(f), \partial_\alpha f\rangle d\mu\, dt\,.
\end{aligned}$$

Hence we have

$$\operatorname{div}(|df|^{p-2}\langle X(f), \nabla f\rangle) = \langle \partial_t f, X(f)\rangle \tag{2.9}$$

in the sense of distributions. Notice that, since $f \in V(M,N)$, there are test functions $\varphi \in V(M,\mathbb{R})$ allowed in (2.9). Let n denote the dimension of N. Hélein [7] observed that on a homogeneous space with a left invariant metric of dimension n there exist n linearly independent Killing vector fields X_i and n

linearly independent tangent vector fields Y_i such that any tangent vector field W on N can be written as

$$W = \langle W, X_i \rangle Y_i \,. \tag{2.10}$$

In fact, if N is represented as a quotient $N = G/H$ of a connected Lie group by a closed, discrete subgroup H, we may choose as $\{X_i\}_{1\le i\le n}$ a basis of the Lie algebra $\mathfrak{g}$ of G and Y_j a suitable linear combination of the vector fields X_i, $1 \le i \le n$. In general $Y_i \ne X_i$ as the vector fields X_i need not to be orthonormal.

Applying (2.10) to the vector fields $\partial_\alpha f$, $1 \le \alpha \le m$, we get

$$|df|^{p-2}\partial_\alpha f = |df|^{p-2}\langle \partial_\alpha f, X_i(f)\rangle Y_i(f)$$

and hence, by taking the divergence on both sides of this equation and using (2.9) and (2.10) we obtain (weakly)

$$\begin{aligned} &\operatorname{div}(|df|^{p-2}\nabla f) \\ &\quad= \operatorname{div}(|df|^{p-2}\langle \nabla f, X_i(f)\rangle)Y_i(f) + |df|^{p-2}\langle \partial_\alpha f, X_i(f)\rangle \partial_\alpha(Y_i(f)) \\ &\quad= \partial_t f + |df|^{p-2}\langle \partial_\alpha f, X_i(f)\rangle \partial_\alpha(Y_i(f)) \,. \end{aligned}$$

Thus (1.4) implies

$$\partial_t f - \Delta_p f = -|df|^{p-2}\langle \partial_\alpha f, X_i(f)\rangle \partial_\alpha(Y_i(f)). \tag{2.11}$$

In fact (2.11) is equivalent to (1.4) as can easily be seen. Now, the product on the right-hand side of (2.11) has a very special structure: We have seen, that the divergence of the first factor $|df|^{p-2}\langle \partial_\alpha f, X_i(f)\rangle$ is $\langle \partial_t f, X_i(f)\rangle$ (this was (2.9)), whereas the second factor $\partial_\alpha(Y_i(f))$ is a gradient and hence has vanishing curl. Thus having control on the time derivative of the solution a variant of the div-curl Lemma should apply as we consider sequences of approximate solutions in the next section.

Remark 2.1. It remains to justify that $\langle \nabla_{\partial_\alpha f} X(f), \partial_\alpha f\rangle = 0$ also in case of a general domain manifold M. In fact, for general M the corresponding expression is

$$\langle \nabla_{\partial_\alpha f} X(f), \partial^\alpha f\rangle = \langle \nabla_{\partial_\alpha f} X(f), \gamma^{\alpha\beta}\partial_\beta f\rangle.$$

Using orthogonal coordinates on M, we have that the vector fields $\partial^\alpha f$ and $\partial_\alpha f$ are parallel and the desired result follows by applying (2.8) for every α separately.

3. Existence of a global weak solution. For $g \in W^{1,p}(M,N)$ fixed, $f \in W^{1,p}(M,\mathbb{R}^k)$ and $h > 0$ let

$$E_g(f) := \int_M \left(\frac{1}{p}\,|df|^p + \frac{1}{2h}\,|f-g|^2\right) d\mu \,.$$

By the direct method of the calculus of variations we find a function $w \in W^{1,p}(M,N)$ such that

$$E_g(w) = \inf_{f \in W^{1,p}(M,N)} E_g(f)$$

and we write $w \in \arg\min E_g$. Now we define recursively a family $f_i \in W^{1,p}(M,N)$ by

$$f_{i+1} \in \arg\min E_{f_i} \qquad \text{for } i = 0, 1, \ldots$$

where f_0 is the initial value in (1.5). Notice that f_i is a weak solution of the Euler-Lagrange equation to energy E_{f_i}, i.e., there holds for every $i = 1, 2, \ldots$

$$\Pi_{T_{f_i}N}\left(\frac{1}{h}(f_i - f_{i-1})\right) + \Delta_p f_i = (p\, e(f_i))^{1-2/p} A(f_i)(\nabla f_i, \nabla f_i) \tag{3.12}$$

in distributional sense. In (3.12) $\Pi_{T_f N}$ denotes the orthogonal projection onto the tangent space $T_f N$.

Since f_i minimizes $E_{f_{i-1}}$ we have in particular $E_{f_{i-1}}(f_i) \le E_{f_{i-1}}(f_{i-1})$, i.e.,

$$\int_M \left(\frac{1}{p}\,|df_i|^p + \frac{1}{2h}\,|f_i - f_{i-1}|^2\right) d\mu \le \int_M \frac{1}{p}\,|df_{i-1}|^p d\mu\,. \tag{3.13}$$

Now we define the function $f^{(h)} : M \times [0,\infty) \to N$ by

$$f^{(h)}(t,\cdot) := f_i \qquad \text{for } t \in [ih, (i+1)h).$$

Thus, rewriting (3.12) in view of Section 2 we get the analogues of (2.9) and (2.11):

$$(2.9)' \quad \operatorname{div}(|df^{(h)}|^{p-2}\langle X(f^{(h)}), \nabla f^{(h)}\rangle) = \langle \Pi_{T_{f^{(h)}}N}\partial^{(-h)} f^{(h)}, X(f^{(h)})\rangle$$

$$(2.11)' \quad \Pi_{T_{f^{(h)}}N}\partial^{(-h)} f^{(h)} - \Delta_p f^{(h)} = -|df^{(h)}|^{p-2}\langle \partial_\alpha f^{(h)}, X_i(f^{(h)})\rangle \partial_\alpha(Y_i(f^{(h)}))$$

weakly on $(h,\infty) \times M$. Here, $\partial^{(h)}$ denotes the forward difference quotient in time with step length h, i.e., $(\partial^{(h)} f)(t,x) = \frac{1}{h}(f(t+h,x) - f(t,x))$. Summing up (3.13), we obtain

$$\frac{1}{2}\int_0^{kh}\int_M |\partial^{(h)} f^{(h)}|^2 d\mu\, dt + \frac{1}{p}\int_M |df^{(h)}(kh)|^p d\mu \le \frac{1}{p}\int_M |df_0|^p d\mu\,. \tag{3.14}$$

So in particular we see that $\{f^{(h)}\}_{h>0}$ is a bounded set in $L^\infty(0,\infty;W^{1,p}(M,N))$ and hence every sequence in $\{f^{(h)}\}_{h>0}$ has a subsequence $f_i := f^{(h_i)}$ such that for any fixed $T>0$

$$f_i \overset{*}{\rightharpoonup} f \quad \text{weakly}^* \text{ in } L^\infty(0,\infty;W^{1,p}(M)). \tag{3.15}$$

Now we show that the difference quotients for *fixed* step length $H>0$ of the sequence $\{f^{(h)}\}$ are bounded in $L^2(0,\infty;L^2(M))$.

Lemma 3.1. *Let $H = ih+\sigma$ and $T = kh+\tau$, k, $i \in \mathbb{N}$, $0 \le \sigma, \tau < h$. Then*

$$\int_0^T \int_M |\partial^{(H)} f^{(h)}|^2 d\mu\, dt \le c \int_0^{h(k+i+1)} \int_M |\partial^{(h)} f^{(h)}|^2 d\mu\, dt$$

for a constant $c>0$.

Proof. We have

$$\begin{aligned} I &:= \int_0^T \int_M |\partial^{(H)} f^{(h)}|^2 d\mu\, dt \\ &\le \sum_{j=0}^{k} \int_{jh}^{jh+h-\sigma} \int_M |\partial^{(H)} f^{(h)}|^2 d\mu\, dt \\ &\qquad + \int_{jh+h-\sigma}^{(j+1)h} \int_M |\partial^{(H)} f^{(h)}|^2 d\mu\, dt \end{aligned} \tag{3.16}$$

and the integrand in each term does not depend on time:

$$\begin{aligned} &\int_{jh}^{jh+h-\sigma} \int_M |\partial^{(H)} f^{(h)}|^2 d\mu\, dt + \int_{jh+h-\sigma}^{(j+1)h} \int_M |\partial^{(H)} f^{(h)}|^2 d\mu\, dt \\ &\quad = (h-\sigma)\int_M \left|\frac{f_{j+i}-f_j}{H}\right|^2 d\mu + \sigma \int_M \left|\frac{f_{j+i+1}-f_j}{H}\right|^2 d\mu \\ &\quad \le \frac{h^2(h-\sigma)i}{H^2} \int_M \sum_{l=j+1}^{j+i} \left|\frac{f_l - f_{l-1}}{h}\right|^2 d\mu \\ &\qquad + \frac{h^2\sigma(i+1)}{H^2} \int_M \sum_{l=j+1}^{j+i+1} \left|\frac{f_l - f_{l-1}}{h}\right|^2 d\mu \end{aligned} \tag{3.17}$$

where we used the fact that the arithmetic mean is less or equal than the quadratic mean. Using (3.17) in (3.16) the claim follows after a short calculation. □

Now we prove a "discrete" Aubin type lemma:

Lemma 3.2. *The set $\{f^{(h)}\}_{h>0}$ is precompact in the space $L^r(0,T;L^r(M))$ for all $r<\infty$.*

Proof. The sequence $\{f^{(h)}\}_{h>0}$ is bounded in $L^\infty(0,T;L^\infty(M))$. Hence it is sufficient to show that $\{f^{(h)}\}_{h>0}$ is totally bounded in $L^1(0,T;L^1(M))$: For simplicity we consider $M=\mathbb{R}^m/\mathbb{Z}^m$ (in the general case consider the set $\{f^{(h)}\varphi_j\}$ for a partition of unity $\varphi_j: M\to\mathbb{R}$ with each support contained in a single coordinate chart of M). Let $\psi:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$ be a smooth non-negative function with support in $B_1(0)\subset\mathbb{R}^m\times\mathbb{R}$ and with L^1-norm equal to 1. Furthermore, let $\psi_\delta(z)=(1/\delta^{m+1})\psi(z/\delta)$ (where $z=(x,t)$ denotes a variable in space-time) and $f_\delta^{(h)}=\psi_\delta * f^{(h)}$ be the standard mollification in every component of $f^{(h)}$ (we extend $f^{(h)}$ with value 0 constant in time for $t<0$ and $t\geq T$). Here, '$*$' denotes the convolution in space-time. Then of course $\{f_\delta^{(h)}\}_{h>0}$ (for $\delta>0$ fixed) is pointwise bounded and equicontinuous. In fact, if $\sup_{h>0}\|f^{(h)}\|_{L^1((0,T)\times M)}\leq c$, we have

$$\begin{aligned}|f_\delta^{(h)}(z)| &\leq \int_{B_1(0)}\psi(\zeta)|f^{(h)}(z-\delta\zeta)|\,d\zeta\\ &\leq \frac{c}{\delta^{m+1}}\sup\psi\,,\quad\text{and}\end{aligned}$$

$$\begin{aligned}|Df_\delta^{(h)}(z)| &\leq \frac{1}{\delta}\int_{B_1(0)}|D\psi(\zeta)||f^{(h)}(z-\delta\zeta)|\,d\zeta\\ &\leq \frac{c}{\delta^{m+2}}\sup|D\psi|,\end{aligned}$$

where D denotes time or spatial derivative. Thus, $\{f_\delta^{(h)}\}_{h>0}$ is a bounded, equicontinuous subset of $C^0([0,T]\times M)$ and hence precompact in $C^0([0,T]\times M)$ by Arzela-Ascoli's theorem, and consequently also precompact in $L^1(0,T;L^1(M))$. It remains to show that $f_\delta^{(h)}$ is L^1-close to $f^{(h)}$. To simplify the notation in the following calculation we drop the h (which is fixed for the moment) and just write f for $f^{(h)}$. We assume that $\delta\leq h/2$. Then we have

$$\begin{aligned}(3.18)\qquad &|f(x,t)-f_\delta(x,t)|\\ &\leq \int_{B_1(0)}\psi(P,Q)|f(x,t)-f(x-\delta P,t-\delta Q)|\,dP\,dQ\\ &\leq \int_{B_1(0)}\psi(P,Q)\int_0^{\delta|P|}\left|D_rf\left(x-r\frac{P}{|P|}\,,t-\delta Q\right)\right|\,dr\,dP\,dQ\\ &\qquad+\int_{B_1(0)}\psi(P,Q)\sum_{i\in\mathbb{Z}}|f_{i+1}(x)-f_i(x)|\chi_{i,\delta Q}(t)\,dP\,dQ\\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

where $\chi_{i,\delta Q}$ denotes the characteristic function of the interval $[(i+1)h, (i+1)h+\delta Q)$ (by $[a,b]$ we always mean $[\min(a,b),\max(a,b)]$) and where we set $f_i = 0$ for negative i. Integrating the term I with respect to x and t, we obtain by estimating the directional derivative $|D_r f|$ by the full derivative $|df|$ and using Hölder's inequality

$$\int_0^T \int_M \mathrm{I}\, d\mu\, dt \le \delta T |M|^{1/p'} \Big(\int_M |df_0|^p\, d\mu \Big)^{1/p}. \tag{3.19}$$

Furthermore, integrating the second term II, we arrive at

$$\int_0^T \int_M \mathrm{II}\, d\mu\, dt \le \delta \sum_{ih\le T} \int_M |f_{i+1} - f_i|\, d\mu \tag{3.20}$$

and by applying Hölder's inequality and (3.14) we see that the sum of the terms in (3.19) and (3.20) is smaller than a constant $c = c(T)$ not depending on h.

Since we have shown above that $\{f_\delta^{(h)}\}_{h>0}$ is totally bounded in $L^1(0,T;L^1(M))$ for all $\delta > 0$, it follows that $\{f^{(h)}\}_{h>0}$ is also totally bounded in $L^1(0,T;L^1(M))$ and this completes the proof. □

Remark 3.3. An alternative proof is possible by using the compactness Theorem of Fréchet-Kolmogorov (see e.g. [2]).

Combining Lemma 3.1 and (3.14), it follows that $\{\partial^{(H)} f\}_{H>0}$ is bounded in $L^2(0,\infty;L^2(M))$. Hence by Lemma (3.2), f has a distributional time derivative in $L^2(0,\infty;L^2(M))$. Furthermore, we have (for a suitable subsequence $h \to 0$) that

$$\partial^{(h)} f^{(h)} \rightharpoonup \partial_t f \quad \text{weakly in } L^2(0,\infty;L^2(M))$$

as a consequence of the partial integration rule for the discrete operator $\partial^{(h)}$. Moreover, we have

$$\Pi_{T_{f^{(h)}}N} \partial^{(-h)} f^{(h)} \rightharpoonup \partial_t f \quad \text{weakly in } L^2(\varepsilon,\infty;L^2(M)) \tag{3.21}$$

for arbitrary $\varepsilon > 0$ and hence we can pass to the limit in the first term of $(2.11)'$. To see this observe that (after possible extraction of a further subsequence) $f^{(h)} \to f$ boundedly a.e. on M (i.e., the sequence $f^{(h)}$ is bounded in $L^\infty(M)$ and converges a.e.) and use the following lemma:

Lemma 3.4. *If*

- $f_k \rightharpoonup f$ *weakly in* $L^q(\Omega)$
- $g_k \to g$ *boundedly a.e. on* Ω *(i.e.,* $\|g_k\|_{L^\infty(\Omega)} \le C$ *for all* k*),*

then

$$f_k g_k \rightharpoonup fg \quad \textit{weakly in } L^q(\Omega).$$

Proof. The proof is a straightforward consequence of the Lebesgue dominated convergence theorem and Egoroff's theorem. □

In order to pass to the limit in the p-Laplace term in $(2.11)'$ we need some more compactness for $\{f^{(h)}\}_{h>0}$:

Lemma 3.5. *The set* $\{f^{(h)}\}_{h>0}$ *is precompact in* $L^q(0,T;W^{1,q}(M))$ *for each* q *satisfying* $1 \le q < p$.

Proof. We use a technique from [3]: Using the facts we have proved so far we start by extracting a subsequence f_i of $\{f^{(h_j)}\}_j \subset \{f^{(h)}\}_{h>0}$ such that

$$\begin{aligned} f_i \rightharpoonup f \quad & \text{weakly in } L^p(0,T;W^{1,p}(M)) \quad \text{and} \\ f_i \to f \quad & \text{strongly in } L^p(0,T;L^p(M)). \end{aligned}$$

Take K such that

$$\|df_i\|_{L^p(0,T;L^p(M))} \le K \tag{3.22}$$

$$\|\partial^{(-h_i)} f_i\|_{L^1(0,T;L^1(M))} \le K \tag{3.23}$$

for all i and with the convention $f_i = 0$ for negative time t.

For $\delta \in \,]0,1]$, let $E^i_\delta = \{(x,t) \in M \times [0,T];\ |f_i(x,t) - f(x,t)| \ge \delta\}$. Then

$$\int_{E^i_\delta} |df_i - df|^q \, d\mu \, dt \le (2K)^q |E^i_\delta|^{(p-q)/p}. \tag{3.24}$$

On the other hand, we define the cutoff function

$$\eta(y) = y \, \min\left\{\frac{\delta}{|y|}, 1\right\} \; : \; \mathbb{R}^k \to \mathbb{R}^k$$

which cuts every vector longer than δ at length δ. Thus the strong monotonicity of the p-Laplace operator implies that for some constant C there holds (as in [3]) the estimate

$$\begin{aligned} & C \int_{M\times[0,T]\setminus E^i_\delta} |df_i - df|^p \, d\mu \, dt \\ & \quad \le \int_{M\times[0,T]\setminus E^i_\delta} \operatorname{trace}\big((|df_i|^{p-2} df_i - |df|^{p-2} df)^* (df_i - df)\big) \, d\mu \, dt \quad = \end{aligned}$$

$$\begin{aligned}
&= \int_{M\times[0,T]} |df_i|^{p-2}\operatorname{trace}\left(df_i^* d\eta(f_i-f)\right)\, d\mu\, dt \\
&\qquad - \delta \int_{E_\delta^i} |df_i|^{p-2}\operatorname{trace}\left(df_i^* d\left(\frac{f_i-f}{|f_i-f|}\right)\right) d\mu\, dt \\
&\qquad - \int_{M\times[0,T]\setminus E_\delta^i} |df|^{p-2}\operatorname{trace}\left(df^*(df_i-df)\right) d\mu\, dt \\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}\,.
\end{aligned}$$

Now,

$$|\mathrm{I}| \le \int_{M\times[0,T]} (c|df_i|^p + |\partial^{(-h_i)} f_i|)|\eta(f_i-f)|\, d\mu\, dt \le c'\delta K$$

where one uses Equation $(2.11)'$. In local coordinates the term II may be written in the following way:

$$\begin{aligned}
\mathrm{II} &= -\int_{E_\delta^i} \frac{\delta}{|f_i-f|^3}|df_i|^{p-2}\gamma^{\alpha\beta}\partial_\alpha f_i^m \big(|f_i-f|^2\partial_\beta(f_i^m-f^m) \\
&\qquad - (f_i^m-f^m)(f_i^n-f^n)\partial_\beta(f_i^n-f^n)\big)\, d\mu\, dt \\
&= -\int_{E_\delta^i} \frac{\delta}{|f_i-f|^3}|df_i|^{p-2}\gamma^{\alpha\beta}\partial_\alpha f_i^m \big(|f_i-f|^2\partial_\beta f_i^m \\
&\qquad - (f_i^m-f^m)(f_i^n-f^n)\partial_\beta f_i^n\big) \\
&\quad + \int_{E_\delta^i} \frac{\delta}{|f_i-f|^3}|df_i|^{p-2}\gamma^{\alpha\beta}\partial_\alpha f_i^m \big(|f_i-f|^2\partial_\beta f^m \\
&\qquad - (f_i^m-f^m)(f_i^n-f^n)\partial_\beta f^n\big) \\
&= \mathrm{II}'+\mathrm{II}''.
\end{aligned}$$

Note that $\mathrm{II}' \le 0$ and, by Hölder's inequality,

$$|\mathrm{II}''| \le 2\int_{E_\delta^i} |df_i|^{p-1}|df|\, d\mu\, dt \le 2K^{p-1}\left(\int_{E_\delta^i} |df|^p\, d\mu\, dt\right)^{1/p}.$$

For III, we use the weak convergence of f_i and Hölder's inequality again to get

$$\begin{aligned}
|\mathrm{III}| &\le \left|\int_{M\times[0,T]} |df|^{p-2}\operatorname{trace}(df^* d(f_i-f))\, d\mu\, dt\right| \\
&\qquad + \left|\int_{E_\delta^i} |df|^{p-2}\operatorname{trace}(df^* d(f_i-f))\, d\mu\, dt\right| \\
&\le o(1) + 2K\left(\int_{E_\delta^i} |df|^p\, d\mu\, dt\right)^{1/p'}.
\end{aligned}$$

Thus, we obtain

$$(3.25)\qquad C\int_{M\times[0,T]\setminus E_\delta^i}|df_i-df|^p\,d\mu\,dt \;\le\; \mathrm{I}+\mathrm{II}''+\mathrm{III} \;\le\; |\mathrm{I}|+|\mathrm{II}''|+|\mathrm{III}|\,.$$

Choosing δ to be small, using the facts that

$$df\in L^p(0,T;L^p(M)),$$

$|E_\delta^i|\to 0$ as $i\to\infty$ and (3.24)–(3.25), the assertion follows. □

Now, by Lemma 3.5 (and by a standard diagonalizing argument) we may assume that for a suitable sequence $h\to 0$

$$df^{(h)}\to df\quad\text{strongly in } L^q(0,T;L^q(M)) \text{ for all } q<p$$

and hence (since $\{df^{(h)}\}_{h>0}$ is bounded in $L^p(0,T;L^p(M))$)

$$(3.26)\quad |df^{(h)}|^{p-2}df^{(h)}\rightharpoonup |df|^{p-2}\,df\quad\text{weakly in } L^{p'}(0,T;L^{p'}(M)).$$

This allows to pass to the limit in the p-Laplace term of Equation $(2.11)'$ and we are left with the problem to do this also on the right-hand side of $(2.11)'$. Notice, that in this section we did not yet make use of the fact, that N is a homogeneous space. But in the following argument this is crucial.

First, notice that in the limit we get (2.9) from (3): In fact, if φ is a smooth test function which vanishes on $[0,\varepsilon]\times M$ for some positive ε, we have

$$(3.27)\qquad \int_0^\infty\int_M\big(|df|^{p-2}\langle\partial_\alpha f,X(f)\rangle\partial_\alpha\varphi-\langle\partial_t f,X(f)\rangle\varphi\big)d\mu\,dt$$
$$=\lim_{h\to 0}\int_0^\infty\int_M\big(|df^{(h)}|^{p-2}\langle\partial_\alpha f^{(h)},X(f^{(h)})\rangle\partial_\alpha\varphi$$
$$-\langle\partial^{(-h)}f^{(h)},X(f^{(h)})\rangle\varphi\big)\,d\mu\,dt=0$$

for suitable $h\to 0$. Here we used (3.15), (3.26) and Lemma 3.4. Now we let $h\to 0$ on the right-hand side of the distributional form of $(2.11)'$ and obtain

$$\int_0^\infty \int_M - \underbrace{|df^{(h)}|^{p-2}\langle \partial_\alpha f^{(h)}, X_k(f^{(h)})\rangle \partial_\alpha Y_k(f^{(h)})}_{= \operatorname{div}\left(|df^{(h)}|^{p-2}\langle \nabla f^{(h)}, X_k(f^{(h)})\rangle Y_k(f^{(h)})\right) - \underbrace{\operatorname{div}\left(|df^{(h)}|^{p-2}\langle \nabla f^{(h)}, X_k(f^{(h)})\rangle\right)}_{= \langle \partial^{(-h)} f^{(h)}, X_k(f^{(h)})\rangle} Y_k(f^{(h)})} \varphi \, d\mu \, dt$$

$$= \int_0^\infty \int_M \Big(|df^{(h)}|^{p-2}\langle \partial_\alpha f^{(h)}, X_k(f^{(h)})\rangle Y_k(f^{(h)}) \partial_\alpha \varphi$$
$$- \langle \partial^{(-h)} f^{(h)}, X_k(f^{(h)})\rangle Y_k(f^{(h)}) \varphi \Big) \, d\mu \, dt$$

$$\to \int_0^\infty \int_M \Big(|df|^{p-2}\langle \partial_\alpha f, X_k(f)\rangle Y_k(f) \partial_\alpha \varphi - \langle \partial_t f, X_k(f)\rangle Y_k(f) \varphi \Big) \, d\mu \, dt$$

$$= \int_0^\infty \int_M -|df|^{p-2}\langle \partial_\alpha f, X_k(f)\rangle \partial_\alpha Y_k(f) \varphi \, d\mu \, dt$$

where we used (3), (3.27) and Lemma 3.4 again. Thus, we have proved Theorem (1.1).

Remark 3.6. (1) The energy inequality (1.7) follows from (3.14) after passing to the limit.

(2) Observing that (1.4) is equivalent to the system of equations

$$\operatorname{div}(|df|^{p-2}\langle X_i(f), \nabla f\rangle) = \langle \partial_t f, X_i(f)\rangle \quad \text{for } i = 1, \dots, n, \tag{3.28}$$

it is possible to shorten the proof of Theorem 1.1 somewhat, since the claim follows allready from (3.27). However the proof we gave is closer to the general problem of an arbitrary target manifold N and hence might be generalized more easily.

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