Beitr Algebra Geom (2020) 61:681–693
https://doi.org/10.1007/s13366-020-00499-3

ORIGINAL PAPER

Generalized pencils of conics derived from cubics

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Received: 10 December 2019 / Accepted: 25 March 2020 / Published online: 15 April 2020

Abstract
Given a cubic K. Then for each point P there is a conic C_P associated to P. The conic C_P is called the *polar conic* of K with respect to the *pole* P. We investigate the situation when two conics C_0 and C_1 are polar conics of K with respect to some poles P_0 and P_1, respectively. First we show that for any point Q on the line P_0P_1, the polar conic C_Q of K with respect to Q belongs to the linear pencil of C_0 and C_1, and vice versa. Then we show that two given conics C_0 and C_1 can always be considered as polar conics of some cubic K with respect to some poles P_0 and P_1. Moreover, we show that P_1 is determined by P_0, but neither the cubic nor the point P_0 is determined by the conics C_0 and C_1.

Keywords Pencils · Conics · Polars · Polar conics of cubics

Mathematics Subject Classification 51A05 · 51A20

1 Terminology

We will work in the real projective plane $\mathbb{RP}^2 = \mathbb{R}^3\backslash\{0\}/\sim$, where $X \sim Y \in \mathbb{R}^3\backslash\{0\}$ are equivalent, if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3\backslash\{0\}$ will be denoted by capital letters, the components with the corresponding small letter, and the equivalence class by $[X]$. However, since we mostly work with representatives, we often omit the square brackets in the notation.

Let f be a non-constant homogeneous polynomial in the variables x_1, x_2, x_3 of degree n. Then f defines a projective algebraic curve

$$C_f := \{[X] \in \mathbb{RP}^2 \mid f(X) = 0\}$$

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of degree n. For a point $P \in \mathbb{RP}^2$,

$$Pf(X) := \langle P, \nabla f(X) \rangle$$

is also a homogeneous polynomial in the variables x_1, x_2, x_3. If the homogeneous polynomial f is of degree n, then C_{Pf} is an algebraic curve of degree $n-1$. The curve C_{Pf} is called the *polar curve* of C_f with respect to the *pole* P; sometimes we call it the *polar curve* of P with respect to C_f. In particular, when C_f is a cubic curve (*i.e.*, f is a homogeneous polynomial of degree 3), then C_{Pf} is a conic, which we call the *polar conic* of C_f with respect to the *pole* P, and when C_f is a conic, then C_{Pf} is a line, which we call the *polar line* of C_f with respect to the *pole* P (see, for example, Wieleitner 1939). By construction, the intersections of a curve C_f and its polar curve C_{Pf} with respect to a point P give the points of contact of the tangents from P to C_f, as well as points on C_f where $\nabla f = 0$ (see Examples 3 and 4). The geometric interpretation of poles and polar lines (or polar surface in higher dimensions) goes back to Monge, who introduced them in 1795 (see Monge 1809, § 3). The names pole and polar curve (or polar surface) were coined by Bobillier (see Bobillier 1827–1828a, b, c, Bobillier 1828–1829a, b) who also iterated the construction and considered higher polar curves (polar curves of polar curves). Grassmann then developed the theory of the poles using cutting methods (see Grassmann 1842a, b, 1843, and Cremona 1866, p. 61). However, the analytical method generally used today—which we follow here—is due to Joachimsthal (see Joachimsthal 1846, p. 373). Note that C_{Pf} is defined and can be a regular curve even if C_f is singular or reducible. We will therefore not impose any further conditions on f in the following.

A regular, symmetric matrix

$$A := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

with eigenvalues of both signs defines a bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $(X, Y) \mapsto \langle Y, AX \rangle$. The corresponding quadratic form $f(X) = \langle X, AX \rangle$ is homogeneous of degree 2, and it is convenient to identify the matrix A or its projective equivalence class with the conic C_f. Then, a point $[X]$ is on the polar line of C_f with respect to the pole $[Y]$ if and only if $\langle Y, AX \rangle = 0$. It follows immediately that a point $[X]$ is on the polar line of C_f with respect to $[Y]$, if and only if $[Y]$ is on the polar line of C_f with respect to $[X]$. Moreover, a line $[g]$ given by the equation $\langle g, Y \rangle = 0$ is the polar line of C_f with respect to the pole $[X] = [A^{-1}g]$.

For a conic C_0 represented by a matrix A_0, the map $\varphi_{C_0} : \mathbb{RP}^2 \to \mathbb{RP}^2$, $[X] \mapsto [A_0 X]$, which associates the pole $[X]$ to its polar line $[A_0 X]$, is called a *polarity*. Vice versa, for a conic C_1 represented by a matrix A_1, the map $\varphi_{C_1} : [Y] \mapsto [A_1^{-1} Y]$, which associates to the polar line $[Y]$ its pole $[A_1^{-1} Y]$, is also called a polarity. The composition of the two polarities $\varphi_{C_1 C_0} : [X] \mapsto [A_1^{-1} A_0 X]$ is a projective map associated to the pair C_0, C_1 of conics. More generally, a cubic f defines a polarity $\mathbb{RP}^2 \to \mathbb{RP}^5$ by associating the point $P \in \mathbb{RP}^2$ to C_{Pf} interpreted as an element of the projective space $\mathbb{RP}^5$ of conics in $\mathbb{RP}^2$.

This point of view can be considered as a guiding concept in the following. It may also open the door to further research questions. For example, one may ask which projective maps from $\mathbb{RP}^2$ to $\mathbb{RP}^5$ can be realized in this way.

Let now f be a homogeneous polynomial of degree $n > 2$, and let C_{Pf} be the polar curve of C_f with respect to a point P. Moreover, let C_{QPf} be the polar curve of C_{Pf} with respect to a point Q. Then we have

$$C_{QPf} = \big\{[X] \mid \langle P, Hf(X)Q\rangle = 0\big\},$$

where $Hf := \big(\frac{\partial f^2}{\partial x_i \partial x_j}\big)_{ij}$ is the Hessian of f. If C_{Qf} denotes the polar curve of C_f with respect to Q and C_{PQf} is the polar curve of C_{Qf} with respect to P, then, obviously,

$$C_{PQf} = C_{QPf}. \tag{1}$$

For two given conics C_0 and C_1, represented as matrices A_0 and A_1 as indicated above, the *linear pencil* of C_0 and C_1 is defined as the set of conics represented by the linear pencil of matrices

$$A_{\lambda,\mu} = \lambda A_0 + \mu A_1 \quad \text{where } \lambda, \mu \in \mathbb{R}, (\lambda, \mu) \neq (0, 0).$$

In the next section we will find for a fixed pair of conics C_0, C_1 points P_0, P_1 and a cubic E such that C_i is the polar conic of E with respect to P_i, and each conic in the linear pencil of C_0, C_1 is the polar conic of E with respect to a point on the line through P_0 and P_1.

2 Conics as polar conics of cubics

We investigate now the situation when two conics C_{Pf} and C_{Qf} are polar conics of some cubic C_f with respect to some poles P and Q, respectively. First we show that for any point R on the line PQ, the polar conic C_{Rf} of C_f with respect to R belongs to the linear pencil of C_{Pf} and C_{Qf}, and vice versa (see Fact 1). A necessary condition for $C_0 = C_{Pf}$ and $C_1 = C_{Qf}$ is, as we have seen in (1), that the polar line of C_0 with respect to Q coincides with the polar line of C_1 with respect to P. A general solution to this problem is given in Proposition 3. Finally, we show how to construct a cubic C_f and two points P and Q, such that C_0 and C_1 are the polar conics of C_f with respect to P and Q, respectively (see Theorem 5).

Fact 1 *Let C_f be a cubic, and let P and Q be two distinct points. Furthermore, let C_{Pf} and C_{Qf} be the polar conics of C_f with respect to P and Q, respectively. Then every conic in the linear pencil of C_{Pf} and C_{Qf} is the polar conic of C_f with a pole on PQ; and vice versa, for every point R on PQ, the polar conic of C_f with respect to R is a conic in the linear pencil of C_{Pf} and C_{Qf}.*

Proof Note that for any R on the line PQ, there exist $\lambda, \mu \in \mathbb{R}$ such that $R = \lambda P + \mu Q$. Hence, C_{Rf} is given by the equation

$$\langle R, \nabla f(X)\rangle = \lambda\langle P, \nabla f(X)\rangle + \mu\langle Q, \nabla f(X)\rangle = 0,$$

which shows that C_{Rf} belongs to the linear pencil of C_{Pf} and C_{Qf}. On the other hand, the conic in the linear pencil of C_{Pf} and C_{Qf} with this equation is the polar conic of C_f with respect to the pole $R = \lambda P + \mu Q$ on the line PQ. □

So, in the case when two given conics C_0, C_1 are polar conics of a cubic C_f with respect to two points P, Q, we can interpret the linear pencil of C_0, C_1 in a new way: namely as the polar conics of C_f with respect to points on the straight line joining P, Q. We will see in Theorem 5, that it is indeed always possible to interpret two conics C_0, C_1 as polar conics of a cubic C_f with respect to two points P, Q. Therefore, by Fact 1, we can generalize the notion of the pencil of two conics C_0, C_1 in the following way.

Definition 2 Let C_f be a cubic, let P and Q be two distinct points, and let C_{Pf} and C_{Qf} be the polar conics of C_f with respect to P and Q, respectively. Furthermore, let Γ be a curve which contains P and Q. Then the set of conics

$$\{C_{Rf} : R \in \Gamma\}$$

is the Γ-*pencil* of C_{Pf} and C_{Qf} with respect to C_f.

Hence, by Fact 1, if Γ is the straight line joining P and Q, then the Γ-pencil coincides with the linear pencil. However, if Γ is not a straight line, then the Γ-pencil shows, depending on the curve Γ, a very rich geometry which can be quite different from that of the linear pencil. Below, two examples of Γ-pencils are given where Γ is not a straight line.

Example 1 Figure 1 shows the Γ-pencil of the two hyperbolas $3x^2 - y^2 - 2y + 3 = 0$ and $3x^2 - y^2 + 2y + 3 = 0$ with respect to the cubic

$$x^3 + 3x^2 - y^2 + 1 = 0,$$

where $P_0 = (0, 1)$, $P_1 = (0, -1)$, and Γ is the circle $x^2 + y^2 = 1$.

Example 2 Figure 2 shows the Γ-pencil of the two circles $x^2 + y^2 = 1$ and $x^2 - 4x + y^2 = \frac{561}{100}$ with respect to the cubic

$$\frac{461x^3}{600} + x^2 + y^2 + \frac{461xy^2}{200} - \frac{1}{3} = 0,$$

where $P_0 = (0, 0)$, $P_1 = (-\frac{200}{561}, 0)$, and Γ is the ellipse

$$\frac{314721\,x^2}{10000} + \frac{561\,x}{50} + \frac{314721\,y^2}{6400} = 0.$$

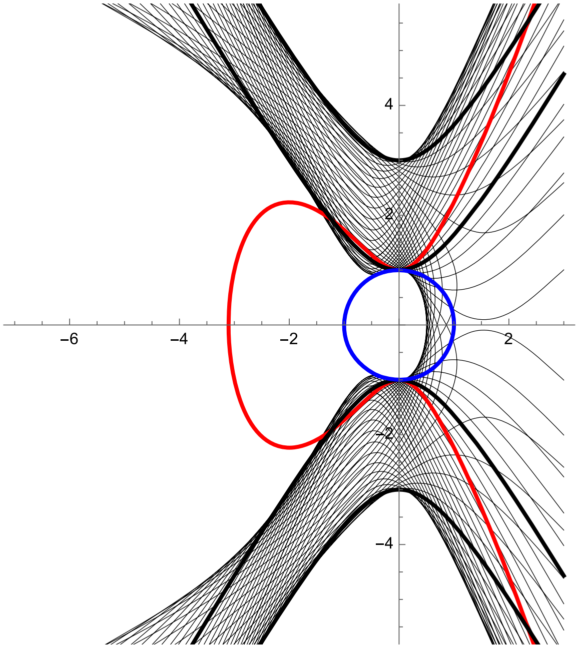

Fig. 1 The Γ-pencil (thin black lines) of the black hyperbolas with respect to the red cubic curve. Γ is the blue circle joining $P_0 = (0, 1)$ and $P_1 = (0, -1)$ (color figure online)

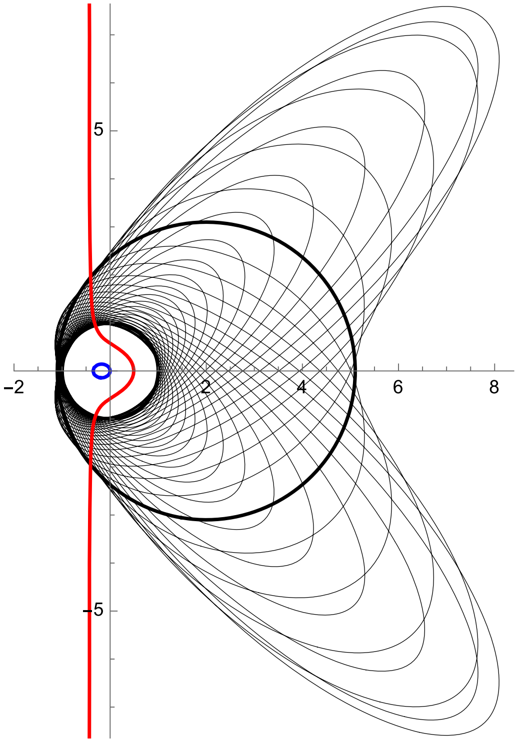

Fig. 2 The Γ-pencil (thin black lines) of the black circles with respect to the red cubic curve. Γ is the blue ellips joining $P_0 = (0, 0)$ and $P_1 = (-\frac{200}{561}, 0)$ (color figure online)

Remark 1 There is also another type of pencils of conics, called *exponential pencils* (introduced and investigated in Halbeisen and Hungerbühler 2018). It would be interesting to study the relation between Γ-pencils and exponential pencils.

The next result shows how we can find points P and Q on a given line g, such that for given conics C_0 and C_1, the polar line of P with respect to C_1 is the same as the polar line of Q with respect to C_0.

Proposition 3 *Given two conics C_0 and C_1 and a line g. Then we are in one of the following cases:*

(a) *There is exactly one pair of points P_0 and P_1 on g, such that the polar line of P_0 with respect to C_0 is the same as the polar line of P_1 with respect to C_1.*
(b) *For any $P_0 \in g$, there exists a unique P_1 on g such that the polar lines of C_0 with respect to P_0 and of C_1 with respect to P_1 coincide.*

In both cases, $P_1 = \varphi_{C_1C_0}(P_0)$ is the image of P_0 under the composition of the polarities associated to C_0 and C_1.

Proof Let A_0 and A_1 be the matrices corresponding to the conics C_0 and C_1. Let P_0 be a point on the given line g, *i.e.*, $\langle P_0, g\rangle = 0$. The polar line of C_0 with respect to P_0 is given by $\langle X, A_0P_0\rangle = 0$. The pole of this line with respect to C_1 is $A_1^{-1}A_0P_0$. We consider the projective map $\varphi_{C_1C_0} : P_0 \mapsto A_1^{-1}A_0P_0$ which is the composition of the two polarities induced by the conics C_0 and C_1. The image of g under $\varphi_{C_1C_0}$ is the line $\langle X, A_1A_0^{-1}g\rangle = 0$.

Suppose first that g is not an eigenvector of $A_1A_0^{-1}$. We want to show that points P_0 and P_1 exist on g such that $P_1 = \varphi_{C_1C_0}P_0$. Necessarily, P_1 is the intersection of g and $\varphi_{C_1C_0}(g)$, *i.e.*, $P_1 = g \times A_1A_0^{-1}g \neq 0$, and then $P_0 = A_0^{-1}A_1P_1 = g \times A_0A_1^{-1}g$.

The second case occurs if g is an eigenvector of $A_1A_0^{-1}$, *i.e.*, if the poles of g with respect to C_0 and C_1 coincide: Then, g and $\varphi_{C_1C_0}(g)$ coincide. Hence one can choose any point P_0 on g, and $P_1 = A_1^{-1}A_0P_0$ is the corresponding point on g such that the polar lines of C_0 with respect to P_0 and of C_1 with respect P_1 agree. □

Remark 2 In the previous proposition we could also fix the line g together with a projective map $\psi : \mathbb{RP}^2 \to \mathbb{RP}^2$ and ask the following question: Are there two conics C_0, C_1 and two points $P_0, P_1 \in g$ with $P_1 = \psi(P_0)$ such that the polar lines of P_i with respect to C_i coincide? This is indeed the case, since every projective map ψ can be written as the composition of two polarities (see, *e.g.* Dolgachev 2012, Theorem 1.1.9).

To motivate the main result of this section (which is Theorem 5), let us consider the following problem: take two lines g_0, g_1 and two points P_0, P_1 in the projective plane. Is there a conic C such that g_i is the polar line of P_i with respect to C? Recall that by von Staudt's Theorem any pair of Desargues triangles are polar triangles in a certain polarity (see, *e.g.* Coxeter 1993, Section 5.7). Hence, there must be many solutions in case of only two prescribed points and two prescribed lines. The interesting feature is, that these solutions form a pencil:

Proposition 4 *Let P_0, P_1 be two different points and g_0, g_1 two different lines in $\mathbb{RP}^2$, both points not incident with the lines. Then there is a linear pencil of real symmetric 3×3 matrices $A_0 + \lambda A_1$, $\lambda \in \mathbb{R}$, such that the corresponding conics C_λ and only those, have the property that g_i is the polar line of P_i with respect to C_λ. Moreover, if P is a point on the line through P_0, P_1, then there is a line g in the linear pencil of g_0, g_1, such that for all λ, the polar line of P with respect to C_λ is g.*

Proof By a suitable projective map we may assume without loss of generality that $P_0 = (1, 0, 0)^T$ and $P_1 = (0, 0, 1)^T$. Then, $g_0 = (g_{01}, g_{02}, 1)^T$ and $g_{01} \neq 0$ since P_0 is not incident with g_0 and g_1, and $g_1 = (1, g_{12}, g_{13})^T$ and $g_{13} \neq 0$ since P_1 is not incident with g_0 and g_1. The matrix A of a conic C with the property that g_i is the polar line of P_i with respect to C must then satisfy $AP_0 = g_0$ and $AP_1 = \mu g_1$ for some $\mu \neq 0$. Hence

$$A_\lambda = \begin{pmatrix} g_{01} & g_{02} & 1 \\ g_{02} & 0 & g_{12} \\ 1 & g_{12} & g_{13} \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\det(A_\lambda)$ cannot vanish identically in λ since g_0 and g_1 are not the same line, therefore $\det(A_\lambda) = 0$ for at most one value $\lambda = \lambda_0$. With the criterion of Hurwitz it follows that A_λ has eigenvalues of both signs for $g_{01}\lambda < g_{02}^2$ and is regular for $\lambda \neq \lambda_0$. Hence A_λ corresponds to a real, nondegenerate conic C_λ. The fact that the polar line of a point P on P_0P_1 with respect to C_λ is independent of λ follows now by a simple calculation. □

It is now natural to ask whether two conics can always be considered as polar conics of a cubic with respect to two poles, and if so, to what extent the cubic and the poles are determined by the conics. The following theorem gives a complete answer to these questions.

Theorem 5 *Let C_0 and C_1 be any two different conics given by matrices A_0 and A_1, respectively. Then there are infinitely many pairs of points P_0, P_1, where $P_1 = \varphi_{C_1C_0}(P_0)$ is the image of P_0 under the composition of the polarities associated to C_0, C_1, and there is a linear pencil of cubics given by $F_\lambda(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$, $\lambda \in \mathbb{R}$, such that C_i are the polar conics of C_{F_λ} with respect to P_i.*

Proof Given two conics C_0 and C_1. We have to find a cubic C_F and two points P_0 and P_1, such that C_0 and C_1 are the polar conics of C_F with respect to P_0 and P_1, respectively. It is convenient to consider the embedding of the affine plane $\mathbb{R}^2$ in $\mathbb{RP}^2$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \left[\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \right].$$

Depending on the position of C_0 and C_1 we may apply a suitable projective transformation, such that a standard situation results (see Halbeisen and Hungerbühler 2017):

Case A: Suppose that C_0 and C_1 have one of the following properties:

- four intersections
- no common point or two intersections
- two intersections and one first order contact
- one first order contact
- two first order contacts
- one third order contact

In these cases, we may assume that C_0 is the unit circle given by the matrix

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and that C_1 is given by

$$A_1 = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & \gamma \end{pmatrix}.$$

As we have seen in the introduction, the polar line of C_0 with respect to P_1 and the polar line of C_1 with respect to P_0 must agree. Hence $[A_0\, P_1] = [A_1\, P_0]$, or equivalently $[P_1] = [A_0^{-1}\, A_1\, P_0]$. Let us first consider the case where P_0 is on the symmetry axis of C_0 and C_1, *i.e.*, $P_0 = (x_0, 0, 1)$. In this case we obtain $P_1 = (x_0 + \alpha, 0, -x_0\alpha - \gamma)$.

It is from now on a bit more convenient to write x, y, z instead of x_1, x_2, x_3. The cubic curve C_F we are looking for is given by a homogeneous polynomial F of degree 3:

$$\begin{aligned} F(x, y, z) = \; & a_1\, x^3 + a_2\, y^3 + a_3\, z^3 + a_4\, x^2 y + a_5\, x^2 z \\ & + a_6\, x y^2 + a_7\, y^2 z + a_8\, x z^2 + a_9\, y z^2 + a_{10}\, x y z. \end{aligned} \tag{2}$$

We need that $C_{P_0 F} = A_0$ and $C_{P_1 F} = A_1$, where $P_0 F(X) = \langle P_0, \nabla F(X)\rangle$, and $P_1 F(X) = \langle P_1, \nabla F(X)\rangle$. The quadratic forms of $P_0 F(X)$ and $P_1 F(X)$ are given by

$$\begin{pmatrix} 3a_1 x_0 + a_5 & a_4 x_0 + \frac{a_{10}}{2} & a_5 x_0 + a_8 \\ a_4 x_0 + \frac{a_{10}}{2} & a_6 x_0 + a_7 & a_9 + \frac{a_{10} x_0}{2} \\ a_5 x_0 + a_8 & a_9 + \frac{a_{10} x_0}{2} & 3a_3 + a_8 x_0 \end{pmatrix}$$

and

$$\begin{pmatrix} 3a_1 p + a_5 q & a_4 p + \frac{1}{2} a_{10} q & a_5 p + a_8 q \\ a_4 p + \frac{1}{2} a_{10} q & a_6 p + a_7 q & a_9 q + \frac{1}{2} a_{10} p \\ a_5 p + a_8 q & a_9 q + \frac{1}{2} a_{10} p & 3a_3 q + a_8 p \end{pmatrix}$$

where $p := x_0 + \alpha$ and $q := -\alpha x_0 - \gamma$. The first of these two matrices has to be a multiple of A_0, the second a multiple of A_1. If we solve the resulting linear system of equations, we find: If $\alpha(1 + x_0^2) + x_0(1 + \gamma) \neq 0$ then

$$F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$$

is the linear pencil spanned by

$$
\begin{aligned}
f_1(x,y,z) &= (1+x_0\alpha+\gamma)x^3-(\alpha+x_0(1+\gamma))z^3+3\alpha x^2z+\\
&\quad +3(\beta+\gamma+x_0\alpha)xy^2+3(x_0+\alpha-x_0\beta)y^2z-3x_0\alpha xz^2\\
f_2(x,y,z) &= y^3.
\end{aligned}
$$

Observe, that if $\alpha(1+x_0^2)+x_0(1+\gamma)=0$ then the Hessian of F vanishes identically, and hence, the cubic curve C_F is reducible, which is precisely the case when $P_0=P_1$.

Now we consider a general point $P_0=(x_0,y_0,1)$ with $y_0\neq 0$. In this case, we obtain $P_1=(x_0+\alpha, y_0\beta, -x_0\alpha-\gamma)$. The matrix of the quadratic form $P_0F(X)$ is given by

$$
\begin{pmatrix}
3a_1x_0+a_4y_0+a_5 & a_4x_0+a_6y_0+\frac{1}{2}a_{10} & a_5x_0+a_8+\frac{1}{2}a_{10}y_0\\
a_4x_0+a_6y_0+\frac{1}{2}a_{10} & 3a_2y_0+a_6x_0+a_7 & a_7y_0+a_9+\frac{1}{2}a_{10}x_0\\
a_5x_0+a_8+\frac{1}{2}a_{10}y_0 & a_7y_0+a_9+\frac{1}{2}a_{10}x_0 & 3a_3+a_8x_0+a_9y_0
\end{pmatrix}
$$

and the matrix for $P_1F(X)$ can now be written as

$$
\begin{pmatrix}
3a_1p+a_4r+a_5q & a_4p+a_6r+\frac{1}{2}a_{10}q & a_5p+a_8q+\frac{1}{2}a_{10}r\\
a_4p+a_6r+\frac{1}{2}a_{10}q & 3a_2r+a_6p+a_7q & a_7r+a_9q+\frac{1}{2}a_{10}p\\
a_5p+a_8q+\frac{1}{2}a_{10}r & a_7r+a_9q+\frac{1}{2}a_{10}p & 3a_3q+a_8p+a_9r
\end{pmatrix}
$$

with $p:=x_0+\alpha, q:=-x_0\alpha-\gamma$, as above, and $r:=y_0\beta$. If $\alpha(1+x_0^2)+x_0(1+\gamma)\neq 0$, we find the following solution of the resulting linear system:

$$
F(x,y,z)=f_1(x,y,z)+\lambda f_2(x,y,z)
$$

where

$$
\begin{aligned}
f_1(x,y,z) &= (1+\alpha x_0+\gamma)x^3+\tfrac{1}{y_0}(\alpha(1+x_0^2)+x_0(1+\gamma))y^3\\
&\qquad -(x_0(1+\gamma)+\alpha)z^3+3\alpha x^2z-3\alpha x_0xz^2\\
f_2(x,y,z) &= \Big(x_0y_0(\alpha x+(1-\beta)z)+y_0(\beta+\gamma)x-(\alpha(1+x_0^2)+(\gamma+1)x_0)y\\
&\qquad +\alpha y_0z\Big)^3.
\end{aligned}
$$

Case B: Suppose that C_0 and C_1 have one second order contact and one intersection. In this case we may assume that C_0 is again the unit circle, given by the matrix A_0 above, and that C_1 is given by the matrix

$$
A_1=\begin{pmatrix}1 & -\nu & 0\\ -\nu & 1 & \nu\\ 0 & \nu & -1\end{pmatrix}
$$

with $\nu \neq 0$ (see Halbeisen and Hungerbühler 2017). Let $P_0 = (x_0, y_0, 1)$. Then we get this time $P_1 = A_0^{-1} A_1 P_0 = (x_0 - y_0\nu, y_0 + \nu(1 - x_0), 1 - y_0\nu)$. We make the same general Ansatz for F as above in (2). Then, the quadratic forms $P_0 F(X)$ and $P_1 F(X)$ are

$$\begin{pmatrix} 3a_1x_0 + a_4y_0 + a_5 & a_4x_0 + a_6y_0 + \frac{a_{10}}{2} & a_5x_0 + a_8 + \frac{a_{10}}{2}y_0 \\ a_4x_0 + a_6y_0 + \frac{a_{10}}{2} & 3a_2y_0 + a_6x_0 + a_7 & a_7y_0 + a_9 + \frac{a_{10}}{2}x_0 \\ a_5x_0 + a_8 + \frac{a_{10}}{2}y_0 & a_7y_0 + a_9 + \frac{a_{10}}{2}x_0 & 3a_3 + a_8x_0 + a_9y_0 \end{pmatrix}$$

and

$$\begin{pmatrix} 3a_1p + a_4r + a_5q & a_4p + a_6r + \frac{1}{2}a_{10}q & a_5p + a_8q + \frac{1}{2}a_{10}r \\ a_4p + a_6r + \frac{1}{2}a_{10}q & 3a_2r + a_6p + a_7q & a_7r + a_9q + \frac{1}{2}a_{10}p \\ a_5p + a_8q + \frac{1}{2}a_{10}r & a_7r + a_9q + \frac{1}{2}a_{10}p & 3a_3q + a_8p + a_9r \end{pmatrix}$$

where $p = x_0 - y_0\nu$, $q = 1 - y_0\nu$, and $r = y_0 + \nu(1 - x_0)$. The first of these two matrices has to be a multiple of A_0, the second a multiple of A_1. Solving the linear system of equations yields the following:

If $y_0 = 0$ and $x_0 \neq 1$, then the cubic function F is the linear pencil $F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$, spanned by

$$\begin{aligned} f_1(x, y, z) &= (x_0 - 1)x^2z - y^2z + xy^2 - (x_0 - 1)x_0xz^2 + \tfrac{1}{3}(x_0 - 1)^2(x_0 + 1)z^3 \\ f_2(x, y, z) &= (x - x_0z)^3. \end{aligned}$$

If $y_0 \neq 0$ and $x_0 = 1$, then $F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$, with

$$\begin{aligned} f_1(x, y, z) &= y(y_0x^2 - xy + \tfrac{1}{3}y_0y^2 + yz - y_0z^2) \\ f_2(x, y, z) &= (x - z)^3. \end{aligned}$$

If $y_0 \neq 0$ and $x_0 = 1 + y_0^2$, then $F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$, with

$$\begin{aligned} f_1(x, y, z) &= (x - z)(y_0x^2 - 3xy + 3y_0y^2 + 3yz + y_0xz - 2y_0z^2) \\ f_2(x, y, z) &= (y - y_0z)^3. \end{aligned}$$

Finally, if $y_0 \neq 0$ and $x_0 \neq 1$ and $x_0 \neq 1 + y_0^2$, then $F(x, y, z) = f_1(x, y, z) + \lambda f_2(x, y, z)$, with

$$\begin{aligned} f_1(x, y, z) = {} & u(1 + x_0(y_0^2 - 1))x^3 + v^2y_0^3y^3 + \\ & +(v(3x_0^2 - x_0 - v)y_0^2 - v^2x_0^3 - x_0y_0^4)z^3 \\ & -3uvy_0x^2y - 3x_0u^2x^2z - 3v^3y_0^2y^2z + 3x_0u(y_0^2 - vx_0)z^2x \\ & +3vy_0(x_0^2v - y_0^2(v + x_0))z^2y + 6uvx_0y_0xyz \\ f_2(x, y, z) = {} & (ux - y_0vy - wz)^3 \end{aligned}$$

where $u = 1 - x_0 + y_0^2$, $v = x_0 - 1$, and $w = x_0(1 - x_0) + y_0^2$. □

It is remarkable that in *Case B*, the pencil of cubics does not depend on ν.

Observe, that the situation in Proposition 4 and in Theorem 5 is somewhat different in that the point P_1 cannot be chosen independently of P_0 in Theorem 5. However, we have the following common feature:

Proposition 6 *For each point P on the line through P_0 and P_1 in Theorem 5, the polar conic of P with respect to the pencil C_{F_λ} does not depend on λ.*

Proof For P_0, P_1 we have that the polar conic $\langle P_i, \nabla F_\lambda(X)\rangle = 0$ is independent of λ. This equation written out in full is

$$\langle P_i, \nabla f_1(X)\rangle + \lambda\langle P_i, \nabla f_2(X)\rangle = 0.$$

Direct inspection of all cases in the proof of Theorem 5 shows that $\langle P_i, \nabla f_2(X)\rangle$ vanishes identically in X, and the claim follows. □

Remark 3 In order to obtain a cubic with respect to two given conics and a pole, we had to solve an over-constrained system of linear equations. Thus, it is somewhat surprising that this system is not just solvable, but has infinitely many solutions, and that the solutions lead to a linear pencil of cubics with only "few" singular or reducible cubics (see also Examples 3 and 4).

We conclude this paper by providing two linear pencils of cubics which belong to two given conics C_0 and C_1 and a point P_0 (see Theorem 5).

Example 3 Figure 3 shows the linear pencil of cubics which belong to the conics

$$C_0 : x^2 + y^2 = 1 \qquad \text{and} \qquad C_1 : x^2 + 4x + 5y^2 + 2 = 0$$

and the points $P_0 = (0, 0)$, $P_1 = (-1, 0)$: tangents to the red cubics in points of C_0 meet in P_0, and tangents to the red cubics in points of C_1 meet in P_1. At the intersections of C_0 and C_1 the gradient of the corresponding cubic vanishes. This examples belongs to *Case A* in the proof of Theorem 5 since C_0 and C_1 have two intersections.

Example 4 Figure 4 shows the linear pencil of cubics which belong to the conics

$$C_0 : x^2 + y^2 = 1 \qquad \text{and} \qquad C_1 : x^2 - 4xy + 4y + y^2 = 1$$

and the points $P_0 = (1, -2)$, $P_1 = (1, -\frac{2}{5})$: tangents to the red cubics in points of C_0 meet in P_0, and tangents to the red cubics in points of C_1 meet in P_1. At the intersections of C_0 and C_1 the gradient of the corresponding cubic vanishes. This examples belongs to *Case B* in the proof of Theorem 5 since C_0 and C_1 have one second order contact and one intersection.

Acknowledgements We would like to thank the referee for his or her comments and suggestions, which helped to improve the quality of the article.

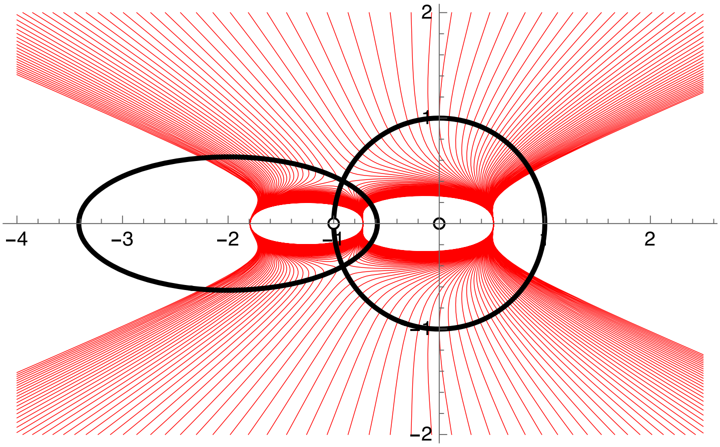

Fig. 3 The linear pencil of cubics (thin red lines) which belongs to the two conics (thick black lines), and the points $P_0 = (0, 0)$, $P_1 = (-1, 0)$ (small black circles). In this example, all members of the linear pencil given by Theorem 5 are irreducible cubic curves, and only two curves of the pencil have a singular point, namely a double point at the intersections of C_0 and C_1 (color figure online)

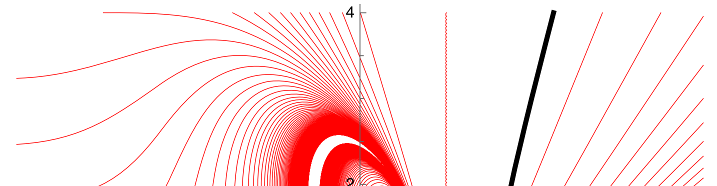

Fig. 4 The linear pencil of cubics (thin red lines) which belongs to the two conics (thick black lines), and the points $P_0 = (1, -2)$, $P_1 = (1, -\frac{2}{5})$ (small black circles). In this example, all cubics in the pencil have a singular point in $(1, 0)$. One cubic of the pencil is reducible and decomposes into the line and an ellipse trough the two intersections of C_0 and C_1 (color figure online)

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