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COMPACTNESS PROPERTIES OF THE p-HARMONIC FLOW INTO HOMOGENEOUS SPACES*

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1. INTRODUCTION

Let M and N be compact smooth Riemannian manifolds without boundaries with metrics γ and g, respectively. Let m and n denote the dimensions of M and N. For a C^1-map $f: M \to N$ the p-energy density is defined by

$$e(f)(x) := \frac{1}{p}|\mathrm{d}f_x|^p \tag{1}$$

and the p-energy by

$$E(f) := \int_M e(f)\,\mathrm{d}\mu. \tag{2}$$

Here, p denotes a real number in $[2, \infty[$, $|\mathrm{d}f_x|$ is the Hilbert-Schmidt norm with respect to γ and g of the differential $\mathrm{d}f_x \in T_x^*(M) \otimes T_{f(x)}(N)$ and μ is the measure on M which is induced by the metric. For concrete calculations we need an expression for $E(f)$ in local coordinates:

$$E_U(f) = \frac{1}{p}\int_\Omega (\gamma^{\alpha\beta}(g_{ij}\circ f)\partial_\alpha f^i \partial_\beta f^j)^{p/2}\sqrt{\gamma}\,\mathrm{d}x.$$

Here, $U \subset M$ and $\Omega \subset \mathbb{R}^m$ denote the domain and the range of the coordinates on M and it is assumed that $f(U)$ is contained in the domain of the coordinates chosen on N. Upper indices denote components, whereas ∂_α denotes the derivative with respect to the coordinate variable x^α. We use the usual summation convention.

Variation of the energy-functional yields the Euler-Lagrange equations of the p-energy which are

$$\Delta_p f = -(\gamma^{\alpha\beta}g_{ij}\partial_\alpha f^i \partial_\beta f^j)^{p/2-1}\gamma^{\alpha\beta}\Gamma^l_{ij}\partial_\alpha f^i \partial_\beta f^j \tag{3}$$

in local coordinates. The operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}}\partial_\beta(\sqrt{\gamma}(\gamma^{\alpha\beta}g_{ij}\partial_\alpha f^i \partial_\beta f^j)^{p/2-1}\gamma^{\alpha\beta}\partial_\alpha f^l)$$

is called p-Laplace operator (for $p = 2$ this is just the Laplace–Beltrami operator and does not depend on N). On the right-hand side of (3) the Γ^l_{ij} denote the Christoffel-symbols related to

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the manifold N. According to Nash's embedding theorem we can think of N as being isometrically embedded in some Euclidean space $\mathbb{R}^k$ since N is compact. Then, if we denote by F the function f regarded as a function into $N \subset \mathbb{R}^k$, equation (3) admits a geometric interpretation, namely

$$\Delta_p F \perp T_F N$$

with Δ_p being the p-Laplace operator with respect to the manifolds M and $\mathbb{R}^k$.

For $p > 2$ the p-Laplace operator is degenerate elliptic. (Weak) solutions of (3) are called (weakly) p-harmonic maps. Compactness results for weakly p-harmonic maps have been obtained by Luckhaus [1] and Toro and Wang [2] (see also Section 4). One possibility to generate p-harmonic maps is to investigate the heat flow related to the p-energy, i.e. to look at the heat-flow equation

$$\partial_t f - \Delta_p f \perp T_f N \tag{4}$$

$$f|_{t=0} = f_0 \tag{5}$$

or explicitly for (4)

$$\partial_t f - \Delta_p f = (pe(f))^{1-2/p} A(f)(\nabla f, \nabla f) \tag{6}$$

where $A(f)(\cdot, \cdot)$ is the second fundamental form on N. For $p = 2$ Eells and Sampson showed in their famous work [3] of 1964, that there exist global solutions of (4) provided N has nonpositive sectional curvature and that the flow tends for suitable $t_k \to \infty$ to a harmonic map. For $p = 2$ the general problem has been solved by Chen and Struwe in [4]. For $p > 2$ only little is known about existence and regularity of solutions of (4): If the target manifold N is a sphere there exists a global weak solution f of the p-harmonic flow with $f \in L^\infty(0, \infty; W^{1,p}(M, N))$ and $f_t \in L^2(0, \infty; L^2(M))$ for arbitrary initial data in $W^{1,p}(M, N)$ (see [5]). Here $W^{1,p}(M, N)$ denotes the nonlinear Sobolev space of functions $g \in W^{1,p}(M, \mathbb{R}^k)$ with $g(x) \in N$ for almost every $x \in M$. In the conformal case, i.e. if $p = \dim(M)$, there exists (again for initial data in $W^{1,p}(M, N)$) a global weak solution which is partially regular in the sense that $\nabla f \in C^{0,\alpha}$ in space-time up to finitely many singular times $T_1 < T_2 < \cdots < T_K \leq \infty$ (and K is a priori bounded in terms of the initial p-energy). Moreover, solutions in $L^\infty(0, T; W^{1,\infty}(M, N))$ are known to be unique (see [6]).

Recently considerable progress has been achieved in different geometric motivated problems if homogeneous spaces are assumed as targets: e.g. [2, 7 or 8]. Here, inspired by Toro and Wang [2], we want to consider the p-harmonic flow for compact targets N which are homogeneous spaces, i.e. $N = G/H$ is the quotient of a connected Lie group G by a closed discrete subgroup H. We will investigate weak solutions of (4)–(5) in the space

$$V(M, N) := \{g \in L^\infty(0, T; W^{1,p}(M, N)) : \partial_t g \in L^2(0, T; L^2(M))\}$$

equipped with the norm

$$\|g\|_{V(M,N)} := \operatorname*{ess\,sup}_{0 \leq t} \|g(t, \cdot)\|_{W^{1,p}(M,N)} + \|g_t\|_{L^2(M \times (0,\infty))}.$$

Recall that $L^\infty(0, T; W^{1,p}(M))$ is the dual space of $L^1(0, T; W^{-1,p'}(M))$.

2. THE FLOW EQUATION FOR HOMOGENEOUS TARGET

Let X be a killing field on N, that is the generator of an isometry of N, satisfying

$$\langle \nabla_v X(p), v \rangle = 0 \qquad \text{for all } p \in N,\ v \in T_pN, \tag{7}$$

where ∇_v denotes the covariant derivative in direction v and $\langle \cdot, \cdot \rangle$ is the inner product on $T_pN \subset T_p\mathbb{R}^k$, that is the scalar product in $\mathbb{R}^k$ restricted to T_pN.

For the sake of simplicity we assume in the sequel that M is the flat torus $\mathbb{R}^m/\mathbb{Z}^m$ (a justification will be given in the remark towards the end of this section). Hence, if $f \in V$ is a weak solution of the p-flow, we have

$$-\int_0^\infty \int_M \langle \partial_\alpha(\zeta X(f)), |\mathrm{d}f|^{p-2}\partial_\alpha f \rangle \,\mathrm{d}\mu\,\mathrm{d}t = \int_0^\infty \int_M \langle \partial_t f, \zeta X(f) \rangle \,\mathrm{d}\mu\,\mathrm{d}t$$

with ζ being a smooth cutoff function. Differentiating the product on the left-hand side, we obtain

$$\begin{aligned}
&\int_0^\infty \int_M \zeta \langle \partial_t f, X(f) \rangle \,\mathrm{d}\mu\,\mathrm{d}t \\
&= -\int_0^\infty \int_M (\zeta \langle \partial_\alpha X(f), |\mathrm{d}f|^{p-2}\partial_\alpha f \rangle + (\partial_\alpha \zeta)\langle X(f), |\mathrm{d}f|^{p-2}\partial_\alpha f \rangle) \,\mathrm{d}\mu\,\mathrm{d}t \\
&= -\int_0^\infty \int_M (\zeta |\mathrm{d}f|^{p-2} \underbrace{\langle \nabla_{\partial_\alpha f} X(f), \partial_\alpha f \rangle}_{=0 \text{ by } (7)} + (\partial_\alpha \zeta)\langle X(f), |\mathrm{d}f|^{p-2}\partial_\alpha f \rangle) \,\mathrm{d}\mu\,\mathrm{d}t \\
&= -\int_0^\infty \int_M (\partial_\alpha \zeta) |\mathrm{d}f|^{p-2} \langle X(f), \partial_\alpha f \rangle \,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Hence we have

$$\operatorname{div}(|\mathrm{d}f|^{p-2}\langle X(f), \nabla f \rangle) = \langle \partial_t f, X(f) \rangle \tag{8}$$

in the sense of distributions. Let n denote the dimension of N. Hélein [7] observed that on a homogeneous space of dimension n there exist n linearly independent Killing vector fields X_i. In fact, if N is represented as a quotient $N = G/H$ of a connected Lie group by a closed, discrete subgroup H, we may choose as $\{X_i\}_{1 \le i \le n}$ a basis of the Lie algebra $\mathfrak{g}$ of G. Thus, we conclude that (4) is equivalent to the system of equations

$$\operatorname{div}(|\mathrm{d}f|^{p-2}\langle X_i(f), \nabla f \rangle) = \langle \partial_t f, X_i(f) \rangle \qquad \text{for } i = 1, \ldots, n. \tag{9}$$

Notice that, since $f \in V(M, N)$, there are test functions $\varphi \in V(M, \mathbb{R})$ allowed in (9). The point is that (9) does, in contrast to (6), no longer contain a term of order $|\nabla f|^p$ and this will allow to prove the desired compactness result as an application of lemma 1 below. (This is very similar to the geometric argument in [5] in case $N = S^n$.)

Remark. There remains to justify that $\langle \nabla_{\partial_\alpha f} X(f), \partial_\alpha f\rangle = 0$ also in case of a general domain manifold M. In fact, for general M the corresponding expression is

$$\langle \nabla_{\partial_\alpha f} X(f), \partial^\alpha f\rangle = \langle \nabla_{\partial_\alpha f} X(f), \gamma^{\alpha\beta}\partial_\beta f\rangle.$$

Using orthogonal coordinates, we have that the vector fields $\partial^\alpha f$ and $\partial_\alpha f$ are parallel and the desired result follows by applying (7) for every α separately.

3. WEAK COMPACTNESS OF THE SOLUTIONS OF THE FLOW

Let $\{f_k\}$ be a bounded sequence in $V(M, N)$ of weak solutions of the p-harmonic flow from M to N. We may pass to a subsequence (again denoted $\{f_k\}$) of functions converging as distributions to $f\colon M \times (0, \infty) \to N$, i.e.

$$f_k \to f \qquad \text{in } \mathfrak{D}'(M \times (0, T)), \tag{10}$$

$$\nabla f_k \to \nabla f \qquad \text{in } \mathfrak{D}'(M \times (0, T)), \tag{11}$$

$$\partial_t f_k \rightharpoonup \partial_t f \qquad \text{weakly in } L^2(M \times (0, T)). \tag{12}$$

Let us recall a compactness lemma from [5].

LEMMA 1. For $k = 1, 2, \ldots$, let $f_k = (f_k^1, \ldots, f_k^{n+1})$ be vector functions of (x, t) on $M \times [0, T]$ satisfying the equation

$$\partial_t f_k - \Delta_p f_k = g_k, \qquad \text{on } M \times [0, T]$$

in the sense of distribution. Here, Δ_p denotes the p-Laplace operator related to the manifolds M and $\mathbb{R}^{n+1}$. Assume further that $\{f_k\}_{k \in \mathbb{N}}$ is bounded in $L^\infty(0, T; W^{1,p}(M, \mathbb{R}^{n+1}))$, $\{\partial_t f_k\}_{k \in \mathbb{N}}$ is bounded in $L^2(0, T; L^2(M, \mathbb{R}^{n+1}))$, and $\{g_k\}_{k \in \mathbb{N}}$ is bounded in $L^1(0, T; L^1(M, \mathbb{R}^{n+1}))$. Then, $\{f_k\}_{k \in \mathbb{N}}$ is precompact in $L^q(0, T; W^{1,q}(M, \mathbb{R}^{n+1}))$ for each $1 \le q < p$.

Notice that, since $\{f_k\}$ is bounded in $V(M, N)$, by the explicit form of (4), we actually have an L^1 bound for the right-hand side of (6). Thus, applying lemma 1 we conclude that after passing to a suitable subsequence

$$f_k \to f \qquad \text{a.e. on } M \times (0, T) \text{ and in } L^q(0, T; L^q(M, N)) \tag{13}$$

for $1 \le q < p$ and hence that

$$|\mathrm{d}f_k|^{p-2}\,\mathrm{d}f_k \rightharpoonup |\mathrm{d}f|^{p-2}\,\mathrm{d}f \qquad \text{in } L^{p'}(0, T; L^{p'}(M)). \tag{14}$$

Due to (12) and (14) it is now possible to pass to the limit in equation (9) if we use the following general fact.

LEMMA 2. If

$$f_k \rightharpoonup f \qquad \text{weakly in } L^q(\Omega)$$

$$g_k \to g \qquad \text{boundary a.e. on } \Omega \text{ (i.e. } \|g_k\|_{L^\infty(\Omega)} \le C \text{ for all } k)$$

then

$$f_k g_k \rightharpoonup fg \qquad \text{weakly in } L^q(\Omega).$$

Proof. The proof is a straightforward consequence of the theorems of Lebesgue and Egorov. ■

Now in fact, if φ is a smooth test function we have for each $i = 1, \ldots, n$:

$$0 = \lim_{k \to \infty} \int_0^\infty \int_M \varphi \langle \partial_t f_k, X_i(f_k) \rangle d\mu \, dt + \int_0^\infty \int_M (\partial_\alpha \varphi) |df_k|^{p-2} \langle X_i(f_k), \partial_\alpha f_k \rangle \, d\mu \, dt$$

$$= \int_0^\infty \int_M \varphi \langle \partial_t f, X_i(f) \rangle \, d\mu \, dt + \int_0^\infty \int_M (\partial_\alpha \varphi) |df|^{p-2} \langle X_i(f), \partial_\alpha f \rangle \, d\mu \, dt.$$

Thus, we have proved the following theorem.

THEOREM 1. If the target manifold N is a homogeneous space, then the space of weak solutions of (4) is weakly compact in the following sense: If $\{g_k\}$ is a bounded sequence in $V(M, N)$ of solutions of the p-harmonic flow then there exists a subsequence $\{f_k\}$ such that

$$\partial_t f_k \rightharpoonup \partial_t f \qquad \text{weakly in } L^2(0, T;\ L^2(M))$$

$$f_k \overset{*}{\rightharpoonup} f \qquad \text{weakly* in } L^\infty(0, T;\ W^{1,p}(M, N))$$

and then, f is a weak solution of (4).

4. THE STATIONARY CASE

If we apply theorem 1 to the stationary case (i.e. vanishing time derivative in (4)) we obtain as a corollary the following theorem.

THEOREM 2. If the target manifold N is a homogeneous space, and if $f_k \in W^{1,p}(M, N)$ is a sequence of weakly p-harmonic maps with $f_k \rightharpoonup f$ in $W^{1,p}(M, N)$, then f is a weakly p-harmonic map from M to N.

This result was proved by Toro and Wang for $M = \Omega \subset \mathbb{R}^m$ in [2, theorem 1] by using the observation that the right-hand side of (4) lies (due to the additional structure of the target manifold N) in the Hardy-space $\mathcal{H}^1_{loc}$, and then using the duality of $\mathcal{H}^1$ and BMO. Here, we could avoid this argument by directly using the geometrically motivated form (9) of equation (4).

However, unlike in the time independent situation where existence of weakly p-harmonic maps is easily derived by the direct method of the calculus of variation, existence of weak solutions of the p-harmonic flow into homogeneous spaces (which will be proved in a forthcoming paper) is not an automatic consequence of our compactness result.

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