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CLOSING THEOREMS FOR PERSPECTIVITIES IN SPACE

MARCO BRAMATO AND NORBERT HUNGERBÜHLER

ABSTRACT. We consider perspectivities $\varphi_P : E_1 \to E_2$ with fixed point P, mapping a plane E_1 in the projective space $\mathbb{RP}^3$ to another plane E_2. This map has a unique projective extension to $\mathbb{RP}^3$. If one chooses points $P_1, \ldots, P_n$ properly, closing theorems result, namely so that the composition $\varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_1}$ is the identity on E_1 or even on the whole $\mathbb{RP}^3$. We examine the conditions on the positions of the points P_i so that such theorems apply. This results in theorems for coplanar points P_i and in general position. The findings are also extended to perspectivities between more than two planes. We also prove similar results for closing theorems for perspectivities between lines in $\mathbb{RP}^3$.

1. INTRODUCTION

Closing theorems, also called porisms, of reversion maps have recently been studied quite intensively. However, their history goes back a long way. In fact, one can interpret for example Pappus's hexagon theorem as a closing theorem in the following way: Let $A_1, A_2, \ldots, A_6$ be a Pappus hexagon on the lines ℓ_1, ℓ_2 with Pappus points P_1, P_2, P_3 on the Pappus line ℓ. Then the hexagon with the same Pappus points closes for any choice of the initial point A_1' on ℓ_1 (see Figure 1).

The mapping which maps the point A_1 on ℓ_1 to the point A_2 on ℓ_2 via the point P_1 is a perspectivity (here also called a reversion) from ℓ_1 to ℓ_2. Exchanging the degenerate conic $\ell_1 \cup \ell_2$ by a non-degenerate conic C, one can in the same way interpret Pascal's hexagon theorem as a closing theorem, and define a reversion map on C. Closing theorems of this type have been studied in [6], [1], [5], [10], [9], [7] and [4]. We refer to [2] and [3] for a detailed account of such closing theorems and their generalizations. The aim of this article is to study closing theorems of perspectivities in the three dimensional projective space.

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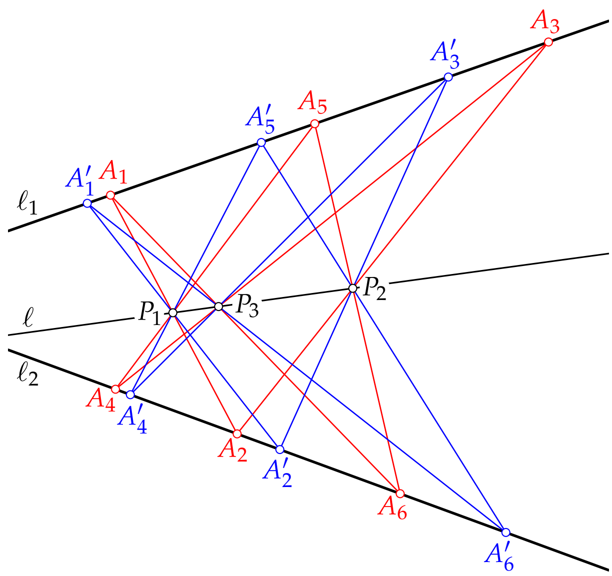

Figure 1. Pappus hexagons $A_1, A_2, \dots, A_6$ and $A_1', A_2', \dots, A_6'$.

2. Setup and Notation

We will work in the standard model of the real projective space. The set of points $\mathbb{P}$ is given by $\mathbb{RP}^3 = \mathbb{R}^4 \setminus \{0\} / \sim$, where $X \sim Y \in \mathbb{R}^4 \setminus \{0\}$ are equivalent if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Similarly, the set of planes $\mathbb{B}$ is also $\mathbb{R}^4 \setminus \{0\} / \sim$, where again $E \sim F \in \mathbb{R}^4 \setminus \{0\}$ are equivalent if $E = \lambda F$ for some $\lambda \in \mathbb{R}$. A point $[X]$ and a plane $[E]$ are incident if $\langle X, E \rangle = 0$, where we denoted equivalence classes by square brackets and the standard inner product in $\mathbb{R}^4$ by $\langle \cdot, \cdot \rangle$. Since we mostly work with representatives we will omit the square brackets in the notation of equivalence classes. Vectors will be written as rows or, for better readability, as columns. Lines are intersections of two different planes, or equivalently the linear span of two different points.

We will work with perspectivities between planes. Let E_1 be a plane given by the equation $\langle e_1, X \rangle = 0$, E_2 be a plane given by $\langle e_2, X \rangle = 0$, and P be a point incident neither with E_1 nor with E_2. Then, the perspectivity $\varphi_P : E_1 \to E_2$ with respect to P maps the point X on E_1 to the point $\varphi_P(X)$ on E_2 which is the intersection of the line through X and P with E_2 (see Figure 2).

Throughout this paper we will use the notation

$$X \xrightarrow[E_1 \quad E_2]{P} Y$$

for the situation shown in Figure 2. If it is clear from the context which planes are involved, we will omit them in the notation.

The map $\varphi_P : E_1 \to E_2$ defined in this way has a projective extension to the whole space $\mathbb{RP}^4$, which we still denote by φ_P. The first lemma gives an explicit formula for this extension.

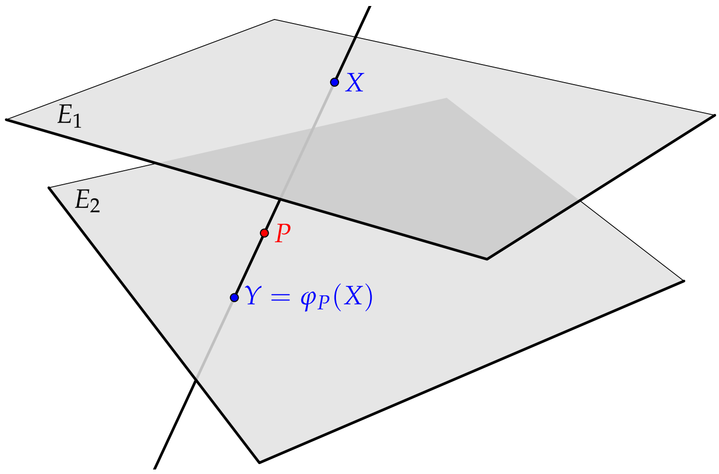

Figure 2. The perspectivitiy $\varphi_P : E_1 \to E_2, X \mapsto Y = \varphi_P(X)$.

Lemma 1. *The map*

$$\varphi_P : \mathbb{RP}^4 \to \mathbb{RP}^4, \quad X \mapsto \big(\langle X, e_1\rangle\langle P, e_2\rangle + \langle P, e_1\rangle\langle X, e_2\rangle\big)P - \langle P, e_1\rangle\langle P, e_2\rangle X \tag{2.1}$$

is the unique projective involution which extends the perspectivity $\varphi_P : E_1 \to E_2$.

Proof. Clearly, the map φ_P given by (2.1) is linear. Furthermore the points X, P and $\varphi_P(X)$ are collinear. It is easy to check that $\varphi_P(X)$ is a point on E_2 if X is a point on E_1. Finally, we have that $\varphi_P \circ \varphi_P = \langle e_1, P\rangle^2\langle e_2, P\rangle^2\,\mathrm{id}$, where id denotes the identity on $\mathbb{RP}^4$.

To show uniqueness, take two projective extensions φ'_P and φ''_P. Then, the composition $\varphi''_P \circ {\varphi'_P}^{-1}$ has P and all points on $E_1 \cup E_2$ as fixed points and is hence the identity. □

Notice that the image of a point X not on $E_1 \cup E_2$ under the projective extension of φ_P can easily be constructed as follows. Take two lines $\ell_1 \neq \ell_2$ passing through X. The four intersection points of ℓ_1 and ℓ_2 with E_1 and E_2 have well defined images by φ_P on the other plane. Hence the images of ℓ_1 and ℓ_2 are determined, and their intersection is $\varphi_P(X)$.

3. Closing theorems for two planes

We come to a first result.

Proposition 2. *Let*

$$E_1 : \langle X, e_1\rangle = 0, \quad E_2 : \langle X, e_2\rangle = 0, \quad F : \langle X, f\rangle = 0$$

be three different planes with common intersection line ℓ, *i.e.*, $f = \lambda_1 e_1 + \lambda_2 e_2$, *and let* $P_1, P_2, \dots, P_{2n}$ *be points on* $F \setminus \ell$. *Then the composition*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_2} \circ \varphi_{P_1} = \mathrm{id}$$

is the identity on $\mathbb{RP}^4$ *if and only if we have*

$$\sum_{k=1}^{2n} \frac{(-1)^k P_k}{\langle P_k, e\rangle} = 0 \tag{3.1}$$

for $e = \lambda_1 e_1 - \lambda_2 e_2$.

Proof. Consider the map $\alpha : \mathbb{RP}^4 \to \mathbb{RP}^4, X \mapsto AX$, for a regular 4×4-matrix A such that E_1 is mapped to $E_1' : \langle X, e_1' \rangle = 0$, E_2 is mapped to $E_2' : \langle X, e_2' \rangle = 0$, and F is mapped to $F' : \langle X, f' \rangle = 0$, with

$$e_1' = 2\lambda_1 A^{-\top} e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad e_2' = 2\lambda_2 A^{-\top} e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad f' = A^{-\top} f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

This is achieved by choosing the third row of A as f and the fourth row as e. Then the points $P_k' = AP_k$ lie on F' and have normalized coordinates $(p_{k1}, p_{k2}, 0, 1)$. By using (2.1) the map $\varphi_{P_k'}$ can be written as

$$\varphi_{P_k'}(X) = \begin{pmatrix} 1 & 0 & 0 & -2p_{k1} \\ 0 & 1 & 0 & -2p_{k2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

It is then easy to see that the map $\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2\sum_{k=1}^{2n}(-1)^{k+1} p_{k1} \\ 0 & 1 & 0 & 2\sum_{k=1}^{2n}(-1)^{k+1} p_{k2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is the identity matrix iff

$$\sum_{k=1}^{2n} \frac{(-1)^k P_k'}{\langle P_k', e_1' - e_2' \rangle} = 0.$$

For the original points P_k, this translates to (3.1). □

Remarks.

- If the points $P_1, \ldots, P_{2n-1}$ on $F \setminus \ell$ are given, then there exists a unique point P_{2n} on $F \setminus \ell$ such that (3.1) is satisfied.
- The sum in (3.1) is invariant under permutations of the points with odd indices and under permutations of the the points with even indices.

As a consequence we get the following result.

Corollary 3. *Let* E_1, E_2 *and* F *be three different planes with common intersection line* ℓ*, and let* $P_1, P_2, \ldots, P_{2n}$ *be points on* $F \setminus \ell$*. Assume that*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}(X_1) = X_1$$

for just one point X_1 *on* E_1*. Then,*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \text{id}$$

is the identity on $\mathbb{R}P^4$. In particular, if for just one X_1 on E_1 the chain of points

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_1]{P_2} X_3 \xrightarrow[E_1 \quad E_2]{P_3} \dots X_{2n} \xrightarrow[E_2 \quad E_1]{P_{2n}} X_1 \tag{3.2}$$

closes, then it closes for any other X_1 on E_1, and also the chain

$$X_1 \xrightarrow[E_2 \quad E_1]{P_1} X_2 \xrightarrow[E_1 \quad E_2]{P_2} X_3 \xrightarrow[E_2 \quad E_1]{P_3} \dots X_{2n} \xrightarrow[E_1 \quad E_2]{P_{2n}} X_1 \tag{3.3}$$

closes for every X_1 on E_2. See Figure 3.

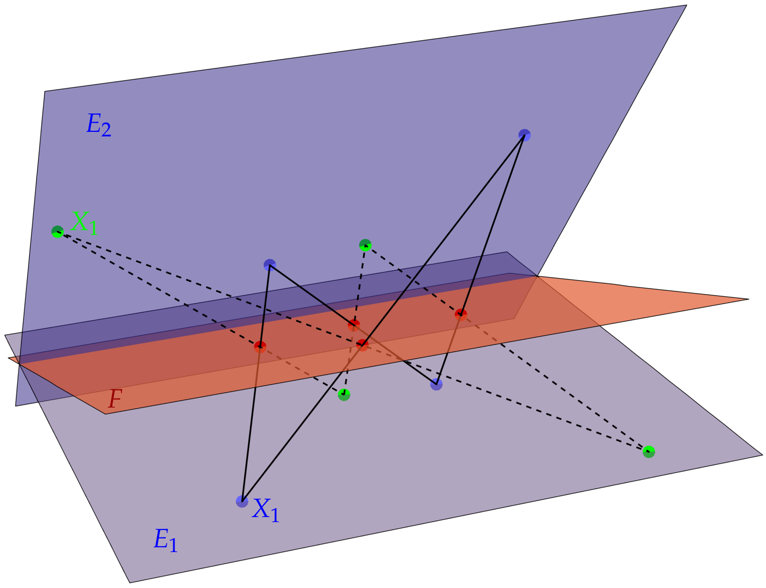

Figure 3. Illustration Corollary 3. The red points $P_1, \dots, P_4$ on the red plane F satisfy (3.1). Then the solid chain (3.2) closes for every starting point X_1 on the plane E_1, and the dashed chain (3.3) closes for every starting point X_1 on the plane E_2.

Proof. Let e be the vector from Proposition 2, and Q be the unique point on F such that

$$\frac{Q}{\langle Q, e\rangle} = -\sum_{k=1}^{2n-1} \frac{(-1)^k P_k}{\langle P_k, e\rangle}.$$

Then, according to Proposition 2, the map

$$\varphi_Q \circ \varphi_{P_{2n-1}} \circ \varphi_{P_{2n-2}} \circ \dots \circ \varphi_{P_1} = \mathrm{id}$$

on $\mathbb{R}P^4$. On the other hand, for a point X_1 on E_1, the point X_{2n} on E_2 is determined by X_1 and the points $P_1, \dots, P_{2n-1}$. Hence the point $P_{2n} = Q$ is the intersection of the plane F with the line through X_{2n} and X_1. □

The last point of Corollary 3, namely that the chain closes for every starting point on E_1 and also for every starting point on E_2 is in sharp contrast to our next result.

Proposition 4. *Let E_1, E_2 and F be three different planes which do not share a common intersection line, and let $P_1, P_2, \dots, P_{2n}$ be points on F but outside E_1 and E_2. Assume that*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_2} \circ \varphi_{P_1}(X_1) = X_1 \tag{3.4}$$

for just one point X_1 on $E_1 \setminus F$. Then,

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_2} \circ \varphi_{P_1}(X) = X \tag{3.5}$$

holds for all $X \in E_1$.

Notice that if (3.5) holds for all $X \in E_1$ then (3.5) is in general not true for $X \in E_2$.

Proof. By applying a suitable projective transformation $\alpha : \mathbb{RP}^4 \to \mathbb{RP}^4, X \mapsto AX$, we may assume that E_1 is mapped to $E_1' : \langle X, e_1' \rangle = 0$, E_2 is mapped to $E_2' : \langle X, e_2' \rangle = 0$, and F is mapped to $F' : \langle X, f' \rangle = 0$, with

$$e_1' = A^{-\top} e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2' = A^{-\top} e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad f' = A^{-\top} f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then the points $P_k' = AP_k$ lie on F' and have normalized coordinates $(p_{k1}, p_{k2}, 0, p_{k4})$. By using (2.1) the map $\varphi_{P_k'}$ can be written as

$$\varphi_{P_k'}(X) = \begin{pmatrix} p_{k1}^2 + p_{k4}^2 & 0 & 0 & -2p_{k1}p_{k4} \\ 2p_{k1}p_{k2} & -p_{k1}^2 + p_{k4}^2 & 0 & -2p_{k2}p_{k4} \\ 0 & 0 & -p_{k1}^2 + p_{k4}^2 & 0 \\ 2p_{k1}p_{k4} & 0 & 0 & -p_{k1}^2 - p_{k4}^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Using induction, we find for $X = (1, x_2, x_3, 1)$

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_2} \circ \varphi_{P_1}(X) = \begin{pmatrix} \lambda \\ \mu + \nu x_2 \\ \nu x_3 \\ \lambda \end{pmatrix}$$

with

$$\begin{aligned} \lambda &= \prod_{k=1}^{2n} (p_{k1} + (-1)^k p_{k4})^2, \\ \mu &= 2 \prod_{k=1}^{2n} (p_{k1} + (-1)^k p_{k4}) \sum_{k=1}^{2n} (-1)^k p_{k2} \prod_{i=1}^{k-1} (p_{i1} + (-1)^i p_{i4}) \prod_{i=k+1}^{2n} (p_{i1} - (-1)^i p_{i4}), \\ \nu &= \prod_{k=1}^{2n} (p_{k1}^2 - p_{k4}^2). \end{aligned}$$

Hence, if we have (3.4) for some X_1 on $E_1 \setminus F$ it follows that $\nu = \lambda$ and $\mu = 0$. (Here we used the assumption that $X_1 \notin F$, i.e., $x_3 \neq 0$.) But then, we have (3.5) for all X on E_1. □

Remarks.

- If (3.4) holds for some X_1 on $E_1 \cap F$, then (3.5) is in general not true for all X on E_2.
- Notice that for given $P_1, \ldots, P_{2n-1}$ on $F \setminus (E_1 \cup E_2)$ we always find a unique P_{2n} such that (3.4) holds for a given X_1 on $E_1 \setminus F$: The point P_{2n} is the intersection of the line through X_1 and $\varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_1}(X_1)$ with the plane F.

So far we have considered reversion points which are coplanar. Next, we investigate the case of arbitrary reversion points.

Proposition 5. *Let E_1, E_2 be two planes in the projective space and $P_1, \ldots, P_{2n-1}$ points neither on E_1 nor on E_2. Then there is a unique point P_{2n} such that*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_1}(X) = X \tag{3.6}$$

for all X on E_1.

Proof. We may assume that $n > 1$. According to Proposition 4 there is a point P_1' on the plane spanned by P_1, P_2, P_3 such that $\varphi_{P_1'} \circ \varphi_{P_3} \circ \varphi_{P_2} \circ \varphi_{P_1}(X) = X$ for all X on E_1. Hence we can replace $\varphi_{P_3} \circ \varphi_{P_2} \circ \varphi_{P_1}$ by $\varphi_{P_1'}$ in (3.6). In the same way, we can replace $\varphi_{P_5} \circ \varphi_{P_4} \circ \varphi_{P_1'}$ by $\varphi_{P_2'}$ for a point on the plane spanned by P_1', P_4, P_5. Continuing this way, we find $P_{2n} = P_{n-1}'$.

To see that P_{2n} is unique with this property, observe that P_{2n} must be the intersection of the line trough X_1 and $\varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_1}(X_1)$, and Y_1 and $\varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_1}(Y_1)$ for two points X_1, Y_1 on E_1. $\square$

The following theorem corresponds to [3, Theorem 12] in two dimensions. We thus obtain a statement for arbitrary reversion points, so that a corresponding closure figure even applies for all starting points A_1 on E_1 *and* E_2.

Theorem 6. *Let E_1 and E_2 be two arbitrary planes. Let the points $X_1, X_2, \ldots, X_{2n-1}$ lie alternately on E_1 and E_2, and let $P_1, P_2, \ldots, P_{2n-2}$ be reversion points that lie arbitrarily, but not on E_1 nor on E_2. If the composition*

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{P_3} \ldots X_{2n-2} \xrightarrow{P_{2n-2}} X_{2n-1} \tag{3.7}$$

closes neither for X_1 on E_1 nor for X_1 on E_2, then there exists a unique straight line ℓ on which a reversion point P_{2n-1} can be chosen arbitrarily and a corresponding unique point P_{2n} on ℓ, so that the closing figure

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{P_3} \ldots X_{2n-1} \xrightarrow{P_{2n-1}} X_{2n} \xrightarrow{P_{2n}} X_1 \tag{3.8}$$

holds for any initial point X_1 on E_1 and also for all X_1 on E_2.

Proof. We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that E_1 and E_2 are parallel planes. We can assume that planes E_1 and E_2 are orthogonal to the ground plane $x_3 = 0$ and to the elevation plane $x_1 = 0$. We draw the situation as depicted in Figure 4 according to the rules of descriptive geometry (see [8]). A point P in the ground plane is labeled P', and in the elevation plane P''. Now we can apply [3, Theorem 12] separately in the ground plane and in the elevation plane.

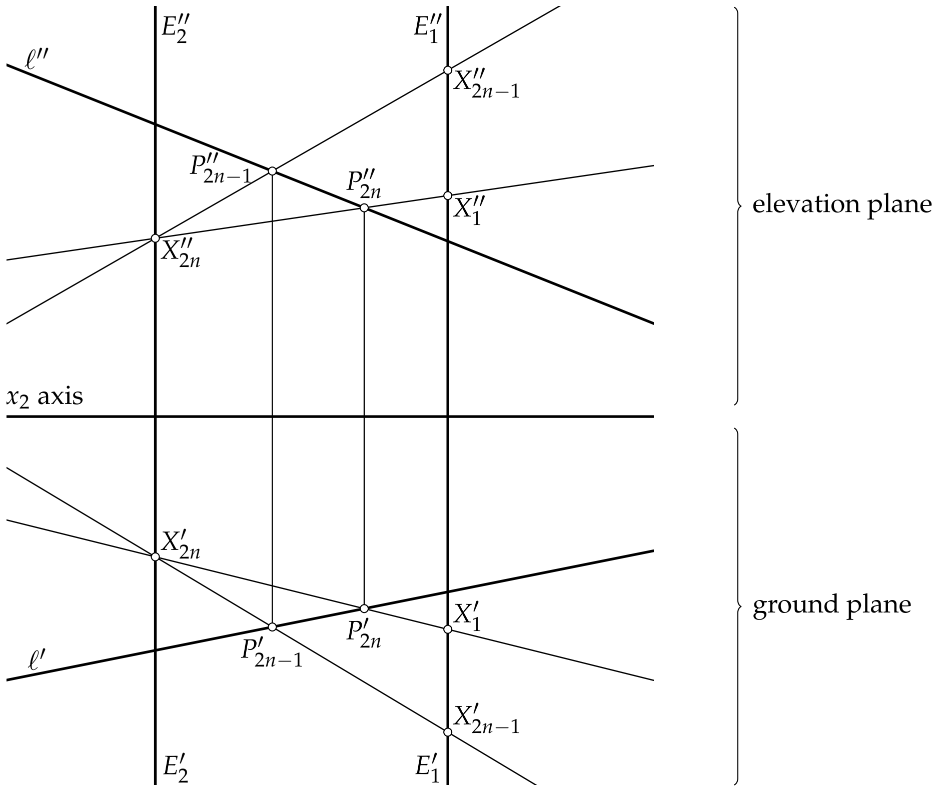

Figure 4. Theorem 6 in the ground plane and the elevation plane.

According to Theorem [3, Theorem 12] there is a unique line ℓ' in the ground plane with the following property. For an arbitrary point P'_{2n-1} on ℓ' there is a unique point $P_{2n'}$ on ℓ' such that the porism

$$X_1' \xrightarrow{P_1'} X_2' \xrightarrow{P_2'} X_3' \xrightarrow{P_3'} \dots X_{2n-1}' \xrightarrow{P_{2n-1}'} X_{2n}' \xrightarrow{P_{2n}'} X_1'$$

closes for any X_1' on E_1'. Similarly, there is a line ℓ'' in the elevation plane with the analogous property. Observe that ℓ' and ℓ'' are the ground and elevation projection of a line ℓ in space, and we will see now that this line has the desired property.

Let us choose a point P_{2n-1} on ℓ with projections P'_{2n-1} on ℓ' and P''_{2n-1} on ℓ''. Then there are points P'_{2n} on ℓ' and P''_{2n} on ℓ'' with the closing property in the ground plane and in the elevation plane respectively. We still need to show, that these points are the projections of the same point on ℓ.

According to Proposition 5, there is a unique point $\tilde{P}_{2n}$ with the closing property for all X_1 on E_1. But then, the corresponding projections in the ground plane and the elevation plane also close. Hence $\tilde{P}'_{2n} = P'_{2n}$ and $\tilde{P}''_{2n} = P''_{2n}$. □

With the following lemma we can replace arbitrary reversion points except for the last one by corresponding reversion points on a common plane. This will be useful later on.

Lemma 7. *Let E_1 and E_2 be two distinct planes and let $P_1, P_2, \ldots, P_n$ be arbitrary reversion points acting between E_1 and E_2. We can then choose any plane Π (different from E_1 and E_2) and find points $Q_1, Q_2, \ldots, Q_{n-1}$ on Π and a point Q'_n such that*

$$\varphi_{P_n} \circ \varphi_{P_{n-1}} \cdots \varphi_{P_1}(X_1) = \varphi_{Q'_n} \circ \varphi_{Q_{n-1}} \cdots \varphi_{Q_1}(X_1) \tag{3.9}$$

for any X_1 on E_1 or E_2.

Proof. Let l_{12} be the straight line through P_1 and P_2. We choose Q_1 to be the intersection of l_{12} with Π. Let X_1 be an arbitrary point on E_1 and X_2 on E_2 and X_3 on E_1 be such that

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3.$$

Then, let X'_2 be the intersection of the line through X_1 and Q_1 with E_2, and define Q'_2 as the intersection of the line through X_3 and X'_2 with l_{12}. Hence, we have

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{Q'_2} X'_2 \xrightarrow{Q_1} X_1. \tag{3.10}$$

The points $X_1, X_2, X_3, X'_2, P_1, P_2, Q_1, Q'_1$ all lie on a plane J. Because of the Scissors Theorem (see, e.g., [3, Theorem 2]), the closing property (3.10) holds for all X_1 on $E_1 \cap J$ and on $E_2 \cap J$. Hence we have

$$\varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_2} \circ \varphi_{Q_1} \tag{3.11}$$

on $(E_1 \cup E_2) \cap J$. Because of Corollary 3 and Proposition 4 the relation actually holds for all points X_1 on $E_1 \cup E_2$.

Now let l_{23} be the straight line through P_3 and Q'_2. As before, we choose Q_2 to be the intersection of l_{23} and Π. Repeating the above arguments, we find a uniquely determined point Q'_3 on l_{23} such that

$$\varphi_{P_3} \circ \varphi_{Q'_2} = \varphi_{Q'_3} \circ \varphi_{Q_2} \tag{3.12}$$

holds on $E_1 \cup E_2$. Together we infer from (3.11) and (3.12)

$$\varphi_{P_3} \circ \varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_3} \circ \varphi_{Q_2} \circ \varphi_{Q_1}.$$

Continuing in the same way, we arrive at (3.9). □

The next result shows how to construct a closing theorem when an odd number of prescribed reversion points is given.

Theorem 8. *Let E_1 and E_2 be two different planes, let n be an odd number of points $P_1, P_2, \ldots, P_n$ that lie neither on E_1 nor E_2. Then there exists a plane Σ with the following property: Each point P_{n+1} on Σ defines a unique straight line ℓ on Σ, so that for each point P_{n+2} on ℓ there is a unique point P_{n+3} on ℓ, so that the closure figure*

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{n+2}} X_{n+3} \xrightarrow{P_{n+3}} X_1.$$

is valid for every X_1 on E_1 or E_2.

Proof. As in the proof of Lemma 7 we choose a plane Π, different from E_1 and E_2 and find points $Q_1, Q_2, \ldots, Q_{n-1}$ on Π, and an additional point Q'_n such that

$$\varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_n} \circ \varphi_{Q_{n-1}} \circ \ldots \circ \varphi_{Q_2} \circ \varphi_{Q_1}. \tag{3.13}$$

Then, we apply Theorem 12 in [3] to construct points X, Y on Π such that

$$\varphi_X \circ \varphi_Y \circ \varphi_{Q_{n-1}} \circ \ldots \circ \varphi_{Q_2} \circ \varphi_{Q_1} = \mathrm{id} \tag{3.14}$$

on $(E_1 \cup E_2) \cap \Pi$. Because of Corollary 3 and Proposition 4 this actually holds on $E_1 \cup E_2$. It follows from (3.13) and (3.14) that we have

$$\varphi_{P_n} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_n} \circ \varphi_Y \circ \varphi_X \tag{3.15}$$

on $E_1 \cup E_2$. Now we consider the plane Σ through the points Q'_n, X and Y. Let e_1 and e_2 be the intersection lines of X with E_1 and E_2 respectively. Then we choose P_{n+1} on $\Sigma \setminus (e_1 \cup e_2)$. Like in the proof of Theorem 6 we find a line ℓ in Σ on which we can choose a point P_{n+2}, which in turn determines the point P_{n+3} on ℓ such that

$$\varphi_{P_{n+3}} \circ \varphi_{P_{n+2}} \circ \varphi_{P_{n+1}} \circ \varphi_{Q'_n} \circ \varphi_Y \circ \varphi_X = \mathrm{id} \tag{3.16}$$

on $e_1 \cup e_2$. Then, because of Proposition 4, this actually holds on $E_1 \cup E_2$. The claim now follows from (3.15) and (3.16). □

4. Closing theorems for more than two planes

In this section we consider reversions between more than two planes and ask for closing theorems. We start with the situation, when the reversion points are coplanar.

Theorem 9. *Let $E_1, E_2, \ldots, E_n$ be planes in $\mathbb{R}P^3$ which share a common line g, and let F be another plane, different from the planes E_i such that $E_i \neq E_{i+1}$ and $E_n \neq E_1$. Let $P_1, P_2, \ldots, P_n$ be points on F such that $\varphi_{P_i} : E_i \to E_{i+1}$ is well defined. If*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_3]{P_2} X_3 \xrightarrow[E_3 \quad E_4]{P_3} \ldots X_{n-1} \xrightarrow[E_{n-1} \quad E_n]{P_{n-1}} X_n \xrightarrow[E_n \quad E_1]{P_n} X_1 \tag{4.1}$$

for one point X_1 on $E_1 \setminus F$, then this porism holds for all X on E_1. Moreover, for all X_1 on E_i we have

$$X_1 \xrightarrow[E_i \quad E_{i+1}]{P_i} X_2 \xrightarrow[E_{i+1} \quad E_{i+2}]{P_{i+1}} X_3 \xrightarrow[E_{i+2} \quad E_{i+3}]{P_{i+2}} \ldots X_{n-1} \xrightarrow[E_{i-2} \quad E_{i-1}]{P_{i-2}} X_n \xrightarrow[E_{i-1} \quad E_i]{P_{i-1}} X_1 \tag{4.2}$$

and

$$X_1 \xrightarrow[E_i \quad E_{i-1}]{P_{i-1}} X_2 \xrightarrow[E_{i-1} \quad E_{i-2}]{P_{i-2}} X_3 \xrightarrow[E_{i-2} \quad E_{i-3}]{P_{i-3}} \ldots X_{n-1} \xrightarrow[E_{i+2} \quad E_{i+1}]{P_{i+1}} X_n \xrightarrow[E_{i+1} \quad E_i]{P_i} X_1. \tag{4.3}$$

Notice that we do not require the planes E_i to be different.

Proof. We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{R}P^3$ such that the planes E_i are parallel. Observe that $\varphi_{P_i} : E_i \to E_{i+1}$ maps a line on E_i to a parallel line on E_{i+1}. In particular, the map $\varphi := \varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_1} : E_1 \to E_1$ is either a proper translation or a homothety. A proper translation does not have a fixed point, hence φ is a homothety with ratio $\lambda \neq 0$. We have to show that $\lambda = 1$.

1. case: F not parallel to the planes E_i. In this case, the line $\ell := F \cap E_1$ is a fixed line of φ. If $\lambda \neq 1$, then the center of the homothety lies on ℓ since all fixed lines of a homothety pass through its center. Since we have a second fixed point X_1 not on ℓ, we conclude that $\lambda = 1$, and hence φ is the identity map on E_1.

2. case: F parallel to the planes E_i. In this case the factor λ_i of the homothety φ_i is given by the oriented ratio $\varepsilon_i \operatorname{dist}(E_{i+1}, F) / \operatorname{dist}(E_i, F)$, where $\varepsilon_i = -1$ if F lies between E_i and E_{i+1}, and $\varepsilon_i = 1$ otherwise. The ratio λ of φ is the product $\lambda_1 \lambda_2 \ldots \lambda_n = 1$. Indeed, it is trivial to see that this product must be 1 or -1 since all numerators and denominators cancel. To see that the product of all ε_i is 1, we may argue as follows: If F lies on one side of all E_i, then all $\varepsilon_i = 1$. If F moves from one side of an E_i to the other side, an even number of the ε_i changes sign.

This finishes the proof of (4.1), and (4.2) and (4.3) follow immediately. □

We consider again an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes E_i are parallel. Notice that we can always find a point P_n on F satisfying (4.1) for given $P_1, P_2, \ldots, P_{n-1}$ on F. Indeed, choose P_n as the intersection of F with the line trough a point X_1 on $E_1 \setminus F$ and the point $X_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(X_1)$ on E_n. This leads to the following more general result.

Theorem 10. *Let $E_1, E_2, \ldots, E_n$ be planes in $\mathbb{RP}^3$ which share a common line g, such that $E_i \neq E_{i+1}$ and $E_n \neq E_1$. Let $P_1, P_2, \ldots, P_{n-1}$ be points such that $\varphi_{P_i} : E_i \to E_{i+1}$ is well defined. Then there is a unique point P_n such that*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_3]{P_2} X_3 \xrightarrow[E_3 \quad E_4]{P_3} \ldots X_{n-1} \xrightarrow[E_{n-1} \quad E_n]{P_n} X_n \xrightarrow[E_n \quad E_1]{P_n} X_1 \tag{4.4}$$

holds for each point X_1 on $E_1 \setminus g$.

Notice that we do not require the planes E_i to be different.

Proof. We consider again an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes E_i are parallel. Let $X_1 \neq Y_1$ be points in E_1, $X_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(X_1)$ and $Y_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(Y_1)$ on E_n. Observe that the line segments X_1Y_1 and X_nY_n are parallel. Hence the lines X_1X_n and Y_1Y_n intersect in a point P_n. Then, $\varphi := \varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_1} : E_1 \to E_1$ is a translation or a homothety with two different fixed points X_1 and Y_1, and is hence the identity map. □

The next theorem shows another way to generate a porism.

Theorem 11. *Let $E_1, E_2, \ldots, E_n$ be planes in $\mathbb{RP}^3$ which share a common line g, such that $E_i \neq E_{i+1}$ and $E_n \neq E_1$. Let $P_1, P_2, \ldots, P_n$ be points such that $\varphi_{P_i} : E_i \to E_{i+1}$ is well defined. Then there exists a line ℓ with the following property. For each point P_{n+1} on ℓ, P_{n+1} not on the planes $E_1, E_2, \ldots, E_n$, there is a plane E_{n+1} such that*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_3]{P_2} X_3 \xrightarrow[E_3 \quad E_4]{P_3} \ldots X_n \xrightarrow[E_n \quad E_{n+1}]{P_n} X_{n+1} \xrightarrow[E_{n+1} \quad E_1]{P_{n+1}} X_1 \tag{4.5}$$

holds for each point X_1 on $E_1 \setminus g$.

Proof. We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes E_i are perpendicular to the x_3 axis. Let $X_1 \neq Y_1$ be points on E_1, $X_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(X_1)$ and $Y_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(Y_1)$ on E_n. Then, according to Theorem 10, the lines X_1X_n and Y_1Y_n meet in a point P, and $\varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1} = \varphi_P$. Let ℓ be the line PP_n. Then the claim follows from [3, Theorem 14], applied to the front elevation plane and the side elevation plane. □

5. Porisms between straight lines in space

In this section, we consider porisms that arise when the reversion maps act between straight lines. First of all, we note that two corresponding straight lines should not be skewed, otherwise the reversion is only defined for one point. But if two straight lines ℓ_1, ℓ_2 intersect in $\mathbb{R}P^3$, they span a plane, and if P is a point not on ℓ_1 and ℓ_2, then the reversion map $\varphi_P : \ell_1 \to \ell_2$ is well defined. With this setting, we find similar porisms as for reversions between planes.

Proposition 12. *Let $\ell_1, \ell_2, \dots, \ell_n$ be straight lines in $\mathbb{R}P^3$ which intersect in a point O such that $\ell_i \neq \ell_{i+1}$ and $\ell_n \neq \ell_1$. Let $P_1, P_2, \dots, P_{n-1}$ be points such that $\varphi_{P_i} : \ell_i \to \ell_{i+1}$ are well defined. Then, there exists a unique point P_n such that*

$$X_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} X_3 \xrightarrow[\ell_3 \quad \ell_4]{P_3} \dots X_n \xrightarrow[\ell_n \quad \ell_1]{P_n} X_1 \tag{5.1}$$

holds for all X_1 on $\ell_1 \setminus O$.

Notice that we do not require the lines ℓ_i to be different.

Proof. We choose planes $E_1, E_2, \dots, E_n$ which share a common line, such that $E_i \neq E_{i+1}$, $E_n \neq E_1$, and ℓ_i lies in E_i. Then, by Theorem 10 there exists a unique P_n such that (5.1) holds with ℓ_i replaced by E_i for all X_1 on E_1. It follows that (5.1) holds for all X_1 on ℓ_1. We still need to check that there is no other point P_n which works for this restricted case. But this follows easily by considering two different starting points X_1, X_1' on ℓ_1 in (5.1). □

We conclude this discussion with the following theorem, which contains a more general statement.

Theorem 13. *Let $\ell_1, \ell_2, \dots, \ell_n$ be straight lines in $\mathbb{R}P^3$ with $\ell_i \neq \ell_{i+1}, \ell_n \neq \ell_1$, and suppose that ℓ_i and ℓ_{i+1} and ℓ_n and ℓ_1 are concurrent. Let $P_1, P_2, \dots, P_{n-2}$ be given points such that $\varphi_{P_i} : \ell_i \to \ell_{i+1}$ is well defined. Then, there exists a straight line ℓ with the following property. For any point P_{n-1} on ℓ there exists a unique point P_n such that*

$$X_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} X_3 \xrightarrow[\ell_3 \quad \ell_4]{P_3} \dots X_n \xrightarrow[\ell_n \quad \ell_1]{P_n} X_1 \tag{5.2}$$

holds for all points X_1 on ℓ_1.

Proof. We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{R}P^3$ such that the lines ℓ_i are not orthogonal to the ground plane and to the elevation plane. Then, we may apply [3, Theorem 16] in the ground plane and in the elevation plane to find the projections of the line ℓ in these planes. This determines ℓ.

More concretely, following the proof of [3, Theorem 16], we can construct ℓ, P_{n-1} and P_n as follows. Let O_1 be the intersection of ℓ_1 and ℓ_2, and X_1, X_1', X_1'' points on ℓ_1. Consider the points

$$
\begin{aligned}
&X_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} X_{n-1}\\
&X'_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X'_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} X'_{n-1}\\
&X''_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X''_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} X''_{n-1}\\
&O_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} O_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} O_{n-1}.
\end{aligned}
$$

For the cross ratios we have $(O_1, X_1, X'_1, X''_1) = (O_{n-1}, X_{n-1}, X'_{n-1}, X''_{n-1})$. Let X denote the intersection of ℓ_1 and ℓ_n. Then there is a unique point $\tilde{X}$ on ℓ_{n-1} with the property

$$(X, X_1, X'_1, X''_1) = (\tilde{X}, X_{n-1}, X'_{n-1}, X''_{n-1}).$$

Now let ℓ be the line joining X and $\tilde{X}$ and choose P_{n-1} on ℓ, not incident with ℓ_1 and ℓ_{n-1}. In particular, we have $\tilde{X} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X$. Consider the points

$$X_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X_n, \quad X'_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X'_n, \quad X''_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X''_n,$$

$$\text{and } O_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} O_n.$$

The cross ratio of four of the points $X_{n-1}, X'_{n-1}, X''_{n-1}, O_{n-1}, \tilde{X}$ equals the cross ratio of the four corresponding image points X_n, X'_n, X''_n, O_n, X. In particular, the lines $X_1X_n, X'_1X'_n$, $X''_1X''_n, O_1O_n$ are concurrent in a point P_n. □

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KANTONSSCHULE LIMMATTAL
IN DER LUBERZEN 34
8902 URDORF, SWITZERLAND.

Email address: marco.bramato@kslzh.ch

DEPARTMENT OF MATHEMATICS
ETH ZENTRUM, RÄMISTRASSE 101
8092 ZÜRICH, SWITZERLAND.

Email address: norbert.hungerbuehler@math.ethz.ch