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RESEARCH ARTICLE

Closing Theorems for Circle Chains

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ABSTRACT

We consider closed chains of circles $C_1, C_2, \ldots, C_n, C_{n+1} = C_1$ such that two neighbouring circles C_i, C_{i+1} intersect or touch each other with A_i being a common point. We formulate conditions such that a polygon with vertices X_i on C_i, and A_i on the (extended) side X_iX_{i+1}, is closed for every position of the starting point X_1 on C_1. Similar results apply to open chains of circles. It turns out that the intersection of the sides X_iX_{i+1} and X_jX_{j+1} of the polygon lies on a circle C_{ij} through A_i and A_j with the property that C_{ij}, C_{jk} and C_{ki} pass through a common point. The six circles theorem of Miquel and Steiner's quadrilateral Theorem appear as special cases of the general results.

Keywords: *Closing theorems, circle chains, Miquel's six circles theorem, Steiner's quadrilateral theorem*

AMS Subject Classification (2020): *Primary: 51M04; Secondary: 51M15.*

1. Introduction

The treasure trove of geometry contains a wide spectrum of closing theorems. Among the best known are Steiner's closing theorem (Figure 1(a), [3, § 6.5]), which belongs to the Möbius geometry, and Poncelet's porism (Figure 1(b), [4, 6, 8]), one of the deepest results of projective geometry.

Other examples are the classical theorems of Pappus and Desargues, which have been known for a long time and are fundamental to the axiomatics of geometry. These theorems can be formulated in the same style as the theorems of Steiner and Poncelet. They then show their closing character much more clearly than usual. Figure 2(a) illustrates this using the example of the Pappus hexagon theorem. In this formulation, the theorem has very universal generalizations (see [2, 12]). Another well-known closing result is the Butterfly theorem in Figure 2(b). This theorem is also only a special case of a much more general closing result (see [10]). Other examples are the closing theorem of Emch [5], and the zig-zag theorem [1]. In this article we investigate a family of new closing theorems for circle chains.

2. A closing theorem for circle chains

We will use the notation $C_1 \between C_2$ for two circles C_1 and C_2 that intersect either in two points or that touch each other. To describe and prove the closing theorem, we will use the following map.

Definition 1. Let $C_1 \between C_2$ be two intersecting or touching circles and A a common point of C_1 and C_2. Then $\varphi_A : C_1 \to C_2, X \mapsto \varphi_A(X)$, is defined as follows: If $X \neq A$, then the points $X, \varphi_A(X)$, and A are collinear. If

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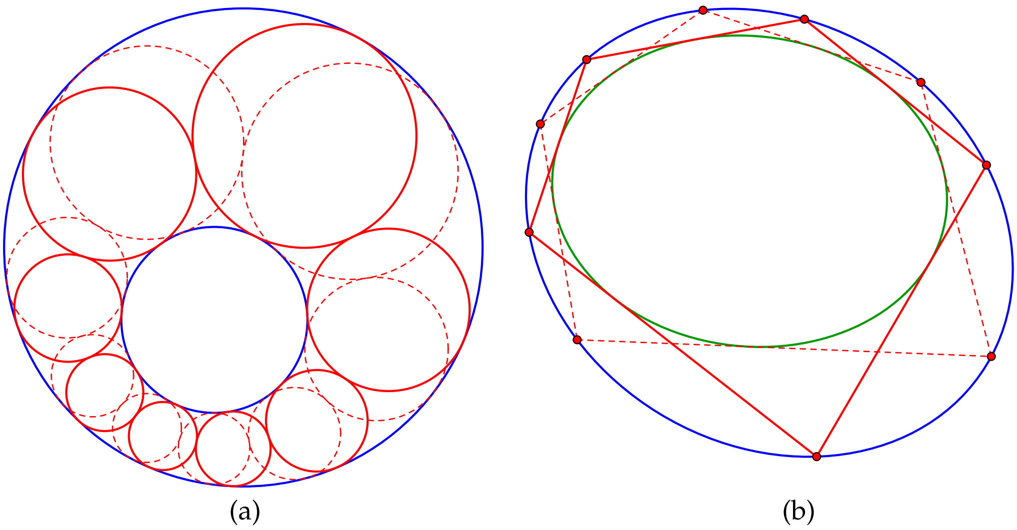


Figure 1. (a) Steiner's closing theorem: Two blue support circles are given, into which a chain of circles is fitted so that neighbouring circles touch each other. If the chain closes in a certain position (red), it closes in every position (dashed). (b) Poncelet's porism: A chain of tangents to a conic (green) with vertices on a second conic (blue) is drawn. If the chain closes in a certain position (red), it closes in every position (dashed).

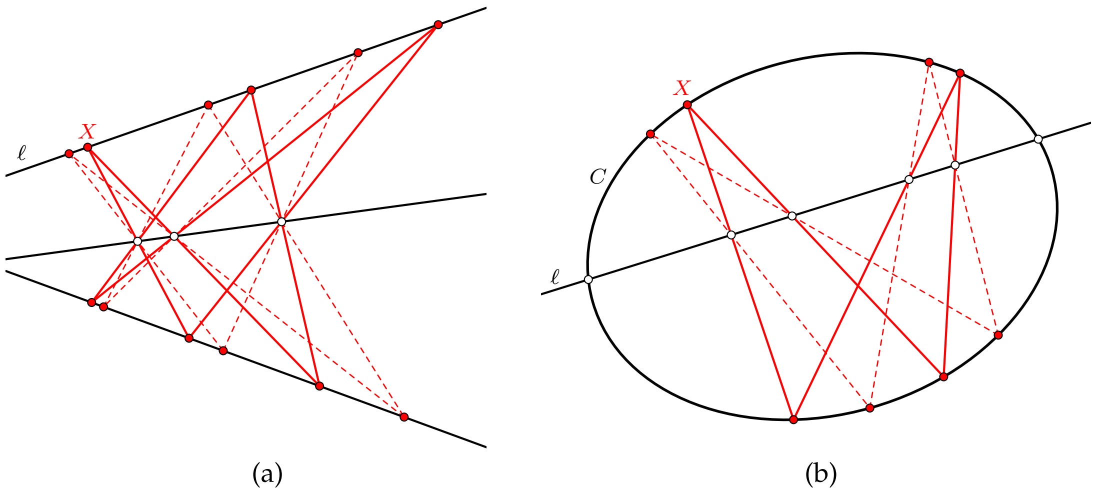


Figure 2. (a) The hexagon theorem of Pappus formulated as closing theorem. If the Pappus hexagon (red) closes for one position of the starting point X on the line ℓ, it closes for any other starting point (dashed). (b) The Butterfly theorem. If the quadrangle (red) with starting point X on a conic C and passing through the four points on a line ℓ is closed, then it closes for any other starting point (dashed).

$X = A$, then the line through the points A and $\varphi_A(X)$ is the tangent to C_1 in A. In particular, if C_1 and C_2 touch at A, then the image of $X = A$ is the point $\varphi_A(X) = A$ (see Figure 3). The point A will be called *pivot* of the *pivot map* φ_A.

Using such pivot maps we can state the first theorem for a closed chain of n circles as follows. Notice that indices will be read cyclically throughout.

Theorem 2. *Let $C_1 \between C_2 \between C_3 \between \cdots \between C_n \between C_1$ be a closed chain of circles. For $i = 1, \ldots, n$ let A_i be a common point of C_i and C_{i+1}. Let $\varphi := \varphi_{A_n} \circ \cdots \circ \varphi_{A_2} \circ \varphi_{A_1}$. Then the following holds: If $\varphi(X) = X$ for one point $X \in C_1$, then φ is the identity map on C_1.*

The statement of Theorem 2 can be formulated in a geometric way visually as follows: Let $X_1 = X \in C_1$ and iteratively $X_{i+1} := \varphi_{A_i}(X_i) \in C_{i+1}$, for $i = 1, 2, \ldots n$. Then we have: If $X = X_{n+1}$, then the corresponding polygon $X_1 X_2 \ldots X_n X_1$ through the pivots $A_1, A_2, \ldots, A_n$ closes for every starting point X_1 on C_1 (see Figure 4).

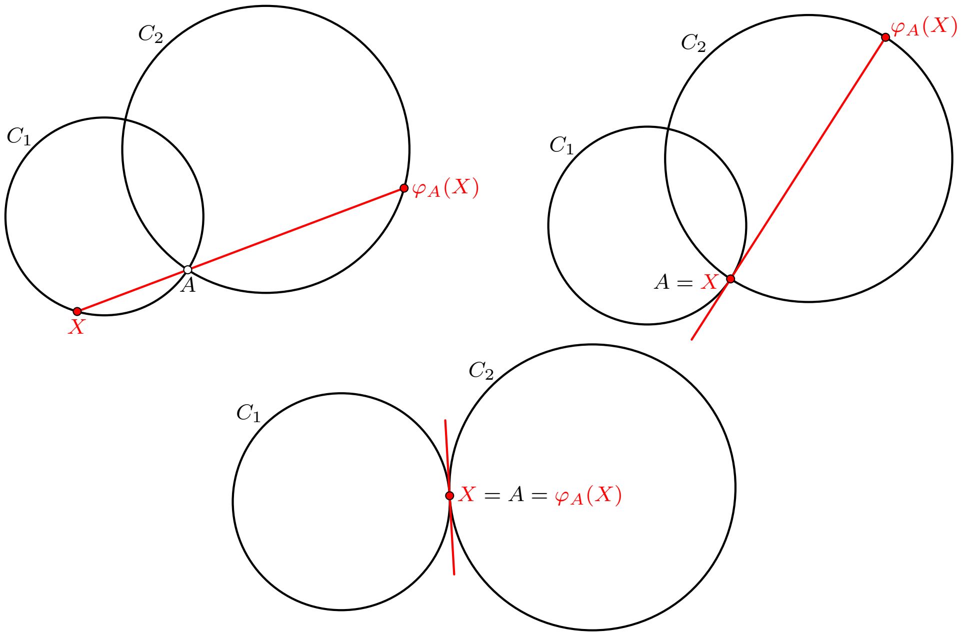


Figure 3. The map $\varphi_A : C_1 \to C_2, X \mapsto \varphi_A(X)$.

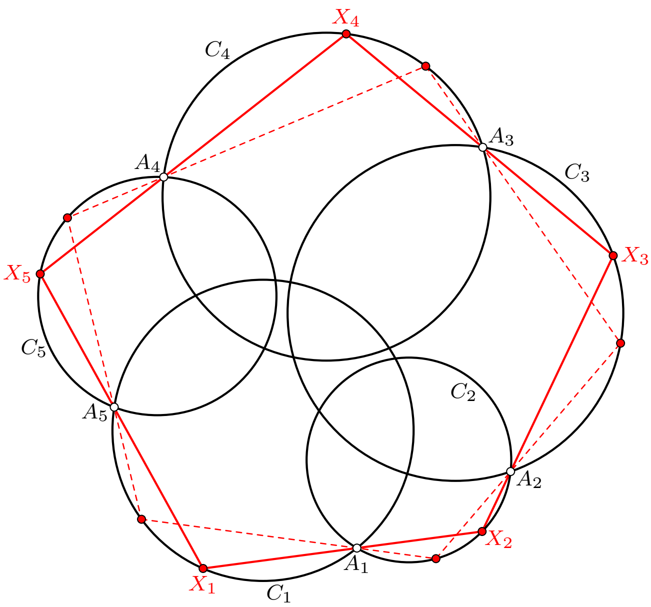


Figure 4. Theorem 2 for five circles. If the polygon with starting point X_1 on C_1 closes, it closes for every starting point on C_1 (dashed).

Proof of Theorem 2. Let M_i denote the center of the circle C_i (see Figure 5). Take two chains $X_{i+1} := \varphi_{A_i}(X_i)$, and $X'_{i+1} := \varphi_{A_i}(X'_i)$ for $i = 1, 2, \ldots n$, for two initial points X_1, X'_1 on C_1. Then we have that for all i the angles $\sphericalangle X_i M_i X'_i$ have the same size, namely $2\sphericalangle X_1 A_1 X'_1 =: 2\varepsilon$. Then, from $\sphericalangle X_{n+1} M_1 X'_{n+1} = 2\varepsilon$ and $X_1 = X_{n+1}$ it follows that $X'_1 = X'_{n+1}$. *q.e.d.*

The proof of Theorem 2 was straightforward. The next result is more surprising.

Theorem 3. *Let $C_1 \between C_2 \between C_3 \between \cdots \between C_n \between C_1$ be a closed chain of circles. Let A_i, B_i be the intersection points of C_i and C_{i+1}, where $A_i = B_i$ if C_i touches C_{i+1}. Let $\varphi_A := \varphi_{A_n} \circ \cdots \circ \varphi_{A_2} \circ \varphi_{A_1}$ and $\varphi_B := \varphi_{B_n} \circ \cdots \circ \varphi_{B_2} \circ \varphi_{B_1}$. Then the following holds: If $\varphi_A(X) = X$ holds for one point $X \in C_1$, then φ_B is the identity map on C_1.*

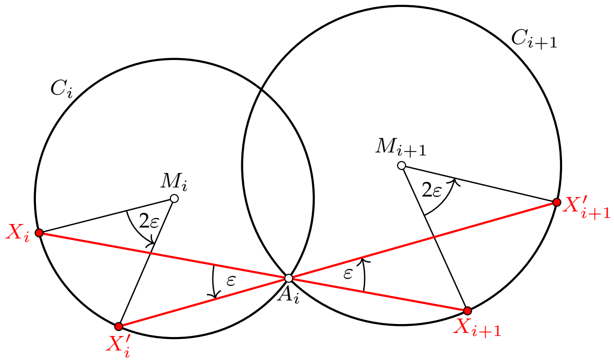


Figure 5. Proof of Theorem 2.

Geometrically speaking, this means, that if the polygon through the pivots $A_1, A_2, \ldots, A_n$ closes for one starting point X on C_1, then the polygon through the pivots $B_1, B_2, \ldots, B_n$ closes for each starting point on C_1 (see Figure 6).

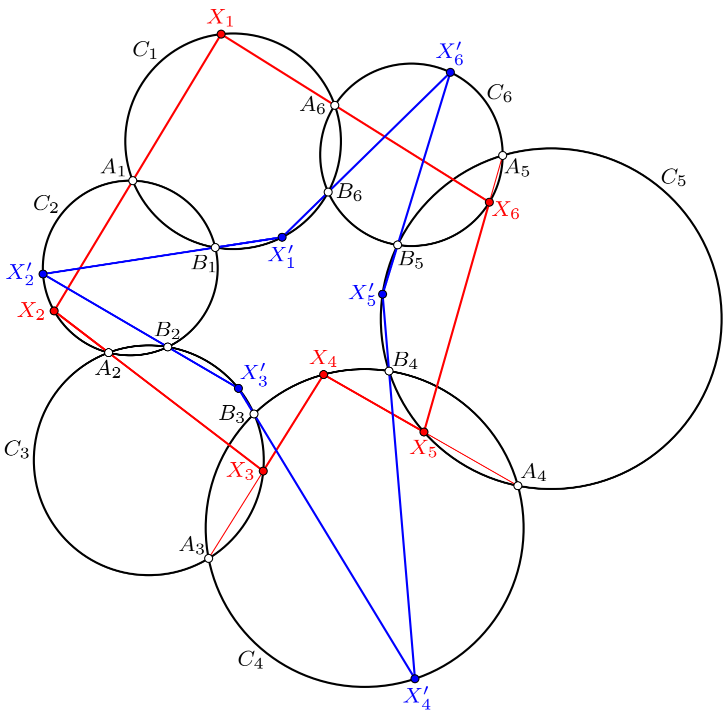


Figure 6. Theorem 3 for six circles. If the red polygon through the pivots A_i with a starting point X_1 on C_1 closes, then the blue polygon through the pivots B_i closes for every starting point X'_1 on C_1.

Before we get to the proof, let us introduce a useful concept which will allow us to formulate an explicit condition on the circle chain for the polygons constructed in this way to close.

Definition 4. Let $C_1 \between C_2$ be two circles with centers M_1 and M_2, and A a common point of C_1 and C_2. Let X be a point on C_1 and $\varphi_A(X)$ on C_2. Let X' be the point on C_2 such that M_1X and M_2X' are parallel in the sense that the vector $\overrightarrow{M_2X'}$ equals $\lambda\overrightarrow{M_1X}$ for a $\lambda > 0$ (see Figure 7). Then $\mu_A := \sphericalangle X'M_2\varphi_A(X)$ is called the *transfer angle* of the pivot map φ_A.

Notice that the transfer angle is well defined as it does not depend on the choice of the point X. It turns out that the transfer angle can be computed in the following way.

Lemma 5. *Let $C_1 \between C_2$ be two circles with centers M_1 and M_2, and A, B the intersection points of C_1 and C_2. Let $\delta_A := \sphericalangle AM_1B, \gamma_A := \sphericalangle BM_2A$. Then the transfer angle μ_A of the map φ_A is given by*

$$\mu_A = \pi - \frac{1}{2}(\delta_A + \gamma_A).$$

Proof. We choose the point X on C_1 in such a way that the point X' in Definition 4 agrees with A (see Figure 7). Then we have the following for the green angles in Figure 8:

$$\begin{aligned}
\sphericalangle \varphi_A(X)AM_2 &= \frac{1}{2}(\pi - \mu_A)\\
\sphericalangle M_2AB &= \frac{1}{2}(\pi - \gamma_A)\\
\sphericalangle BAM_1 &= \frac{1}{2}(\pi - \delta_A)\\
\sphericalangle M_1AX &= \frac{1}{2}(\pi - \mu_A)
\end{aligned}$$

In the last line we have used that M_1X and M_2A are parallel (see Figure 8). Adding these four angles yields

$$\frac{1}{2}(\pi - \mu_A) + \frac{1}{2}(\pi - \gamma_A) + \frac{1}{2}(\pi - \delta_A) + \frac{1}{2}(\pi - \mu_A) = \pi,$$

from which the formula for μ_A follows immediately. *q.e.d.*

The reader is invited to also consider the situation when M_1 and M_2 lie on the same side of the line AB, as well as the case of touching circles with $A = B$.

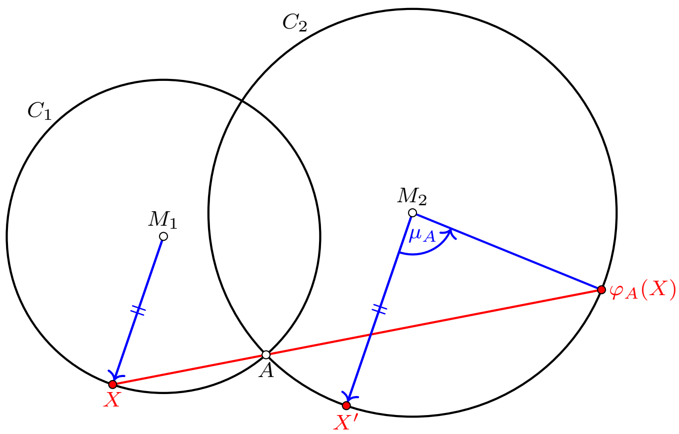


Figure 7. The transfer angle μ_A of the pivot map φ_A.

The transfer angle can also be interpreted geometrically in another way.

Lemma 6. *Let $C_1 \between C_2$ be two circles with centers M_1 and M_2, and A, B the intersection points of C_1 and C_2. Let t_1 and t_2 be the tangents to C_1 and C_2 in A. Then the transfer angle μ_A of the map φ_A is given by $\mu_A = \sphericalangle t_2t_1$ (see Figure 9). Here, t_1 is oriented from A towards the inside of C_2, and t_2 is oriented from A towards the outside of C_1.*

Proof. Recall that the central angle over a chord of a circle is twice the inscribed angle over the same chord, and the inscribed angle over the chord equals the supplementary inscribed angle on the opposite arc, which is also the angle between the chord and the tangent at an endpoint of the chord. In particular, we find the angle $\frac{1}{2}\sphericalangle\delta_A$ between t_1 and the chord AB (see Figure 9), and the angle $\frac{1}{2}\sphericalangle\gamma_A$ between the chord BA and t_2. Hence, according to Lemma 5, the transfer angle μ_A is the angle between t_2 and t_1. *q.e.d.*

Using the notion of the transfer angle we can now formulate the following closing condition.

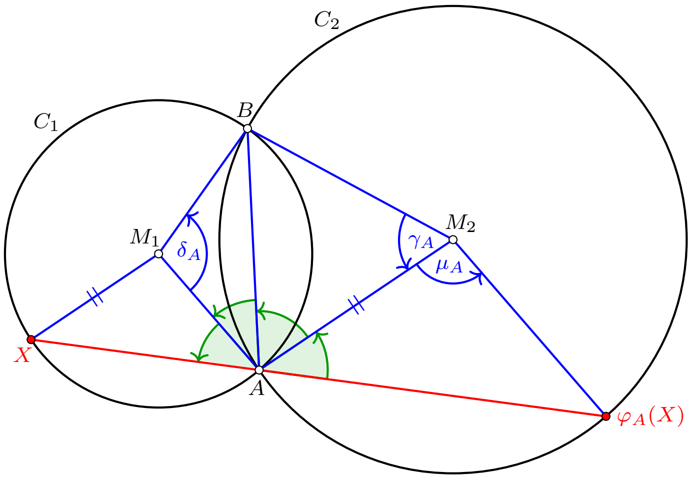


Figure 8. Proof of the formula for the transfer angle μ_A of the pivot map φ_A.

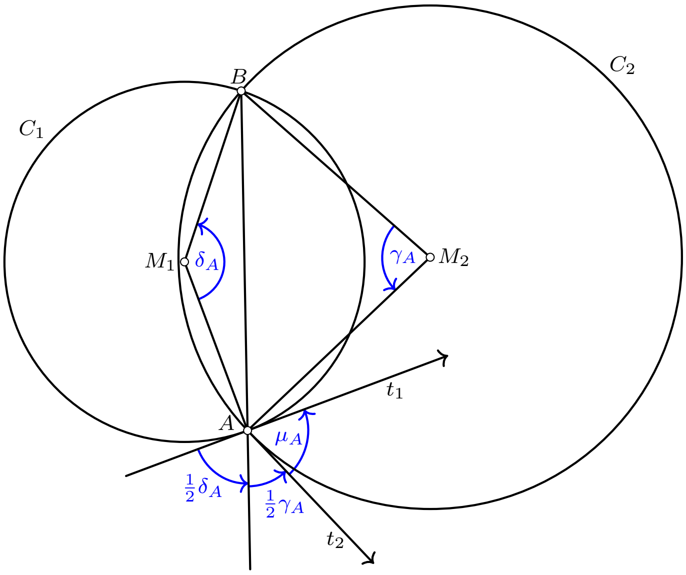


Figure 9. Lemma 6: The transfer angle μ_A of the pivot map φ_A.

Theorem 7. *Let $C_1 \between C_2 \between C_3 \between \cdots \between C_n \between C_1$ be a closed chain of circles with centers $M_1, \ldots, M_n$. Let A_i, B_i be the intersection points of C_i and C_{i+1}, where $A_i = B_i$ if C_i touches C_{i+1}. Let $\delta_{A_i} := \sphericalangle A_i M_i B_i, \gamma_{A_i} := \sphericalangle B_i M_{i+1} A_i$. Then the map $\varphi_A := \varphi_{A_n} \circ \cdots \circ \varphi_{A_2} \circ \varphi_{A_1}$ is the identity map on C_1 if and only if the sum of all transfer angles is a multiple of 2π, i.e.,*

$$n\pi - \frac{1}{2}\sum_{i=1}^{n}(\delta_{A_i} + \gamma_{A_i}) = 2k\pi$$

for an integer k.

Proof. Clearly, the map φ_A is the identity on C_1 if and only if the sum of all transfer angles is a multiple of 2π. Using the formula from Lemma 5 for the transfer angles μ_{A_i} we get the closing condition stated in the theorem. *q.e.d.*

Observe that the situation of two circles with centers M_1 and M_2, intersecting in the points A and B like in Figure 8 is mirror symmetric with respect to the line M_1M_2. Hence for the transfer angles of φ_A and φ_B we have

$$\mu_B = -\mu_A.$$

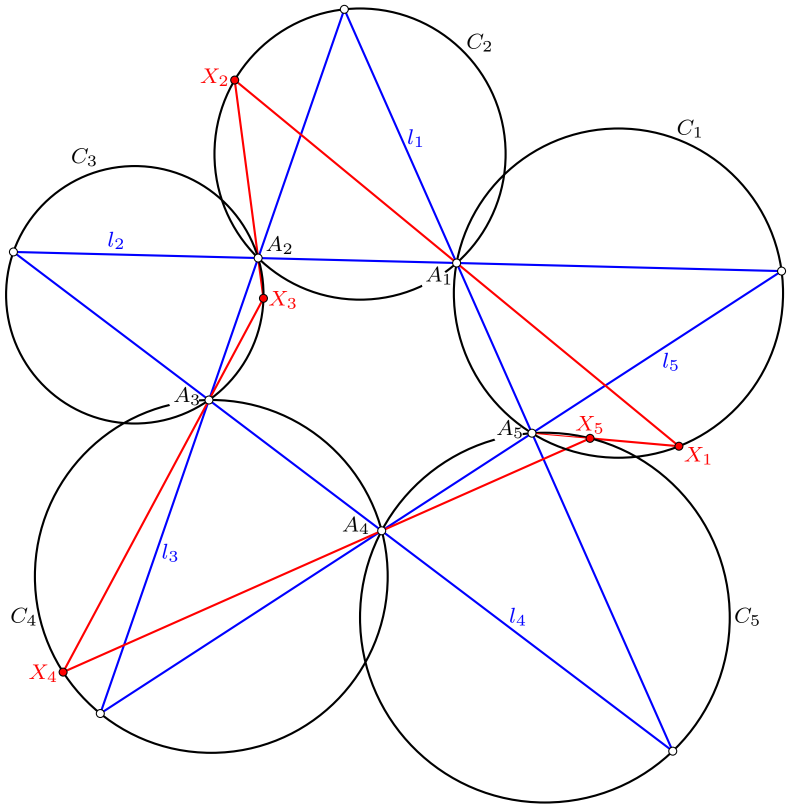


Figure 10. Illustration for Corollary 8 for $n = 5$ lines $l_1, \ldots, l_5$. The red polygon through the pivots $A_1, A_2, \ldots, A_n$ closes for every position of the starting point X_1 on C_1.

Now, if the sum of the transfer angles μ_{A_i} is a multiple of 2π, then the same is true for the sum of the transfer angles $\mu_{B_i} = -\mu_{A_i}$. This proves Theorem 3. Another application of the criterion in Theorem 7 is the following.

Corollary 8. *Let $l_1, l_2, \ldots, l_n, l_{n+1} = l_1, l_{n+2} = l_2$ be a set of lines such that l_{i-1}, l_i, l_{i+1} form a triangle with circumcircle C_i (see Figure 10). Let A_i denote the intersection of l_i and l_{i+1}. Then, $\varphi := \varphi_{A_n} \circ \varphi_{A_{n-1}} \circ \ldots \circ \varphi_{A_1}$ is the identity map on C_1, i.e., the resulting n-gon X_1, $X_{i+1} = \varphi_{A_i}(X_i)$ closes for any initial point X_1 on C_1.*

Note that the situation of Corollary 8 corresponds exactly to that in Morley's Five Circles theorem [9, 17, 18] and of Miquel's Pentagon theorem [13, 16].

Proof. Let $A_{i,i+2}$ denote the intersection of the lines l_i and l_{i+2}, i.e., C_i is the circumcircle of the triangle $A_{i-1}A_iA_{i-1,i+1}$. Consider the point A_i with the transfer angle μ_{A_i} of the map φ_{A_i}. According to Lemma 6 this transfer angle is the angle between the tangents t_{i+1} and t_i (see Figure 11). Let $\varepsilon_i := \sphericalangle A_{i-1}A_{i-1,i+1}A_i$. This angle equals the angle between the line l_i and the tangent t_i (see Figure 11). Similarly, the angle $\varepsilon_{i+1} = \sphericalangle A_iA_{i,i+2}A_{i+1}$ appears also as angle between the tangent t_{i+1} and the line l_{i+1}. The angle ω_i between the lines l_{i+1} and l_i is the exterior angle of the polygon $A_1A_2 \ldots A_n$ in the vertex A_i. We have $\mu_{A_i} = \omega_i + \varepsilon_i + \varepsilon_{i+1}$. Observe that $\varepsilon_i = \pi - (\omega_{i-1} + \omega_i)$, and $\varepsilon_{i+1} = \pi - (\omega_i + \omega_{i+1})$. Hence, we get $\mu_{A_i} = 2\pi - (\omega_{i-1} + \omega_i + \omega_{i+1})$. Taking the sum over all transfer angles results in

$$\sum_{i=1}^{n} \mu_{A_i} = 2n\pi - 3\sum_{i=1}^{n} \omega_i.$$

Since the sum of the exterior angles $\sum_{i=1}^{n} \omega_i$ of a polygon is a multiple of 2π, the closing condition of Theorem 7 is satisfied. *q.e.d.*

We will see in Section 4 that for $n = 4$ Corollary 8 together with Lemma 14 implies Steiner's quadrilateral theorem [19], even in an extended version (see Corollary 17).

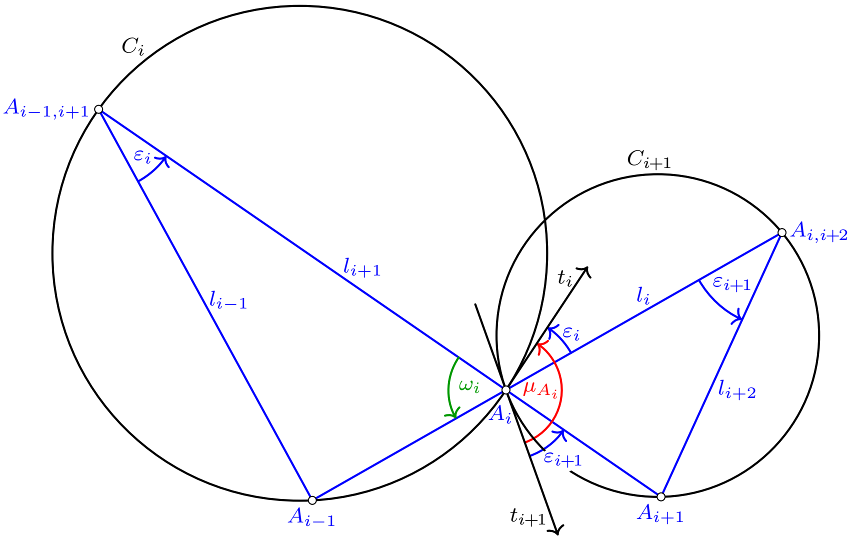


Figure 11. Proof of Corollary 8.

3. Neat special cases

Theorem 9. *Let $C_1 \between C_2 \between C_3 \between \cdots \between C_n \between C_1$ be a closed chain of circles. Let A_i, B_i be the intersection points of C_i and C_{i+1}, where $A_i = B_i$ if C_i touches C_{i+1}. Let $\varphi_A := \varphi_{A_n} \circ \cdots \circ \varphi_{A_2} \circ \varphi_{A_1}$ and $\varphi_B := \varphi_{B_n} \circ \cdots \circ \varphi_{B_2} \circ \varphi_{B_1}$. Then $\varphi_B \circ \varphi_A(X) = X$ holds for all points $X \in C_1$ (see Figure 12).*

Proof. The claim follows immediately from Theorem 7 if we use the fact that the sum of the transfer angles $\mu_{A_i} + \mu_{B_i} = 0$ for all i. Therefore the total sum of the transfer angles along the chain vanishes. *q.e.d.*

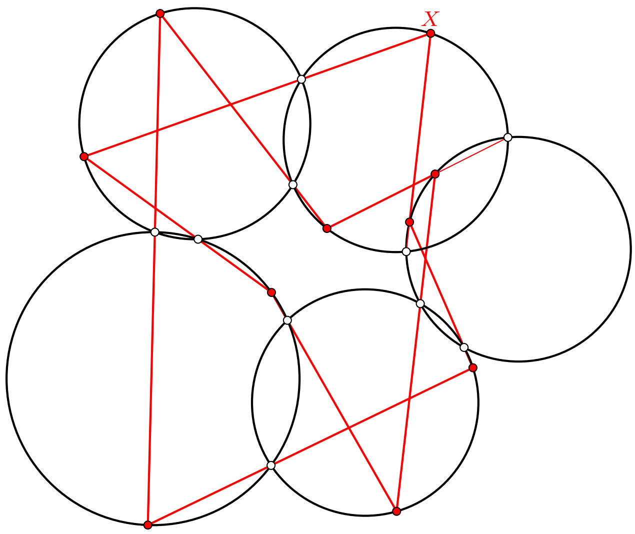
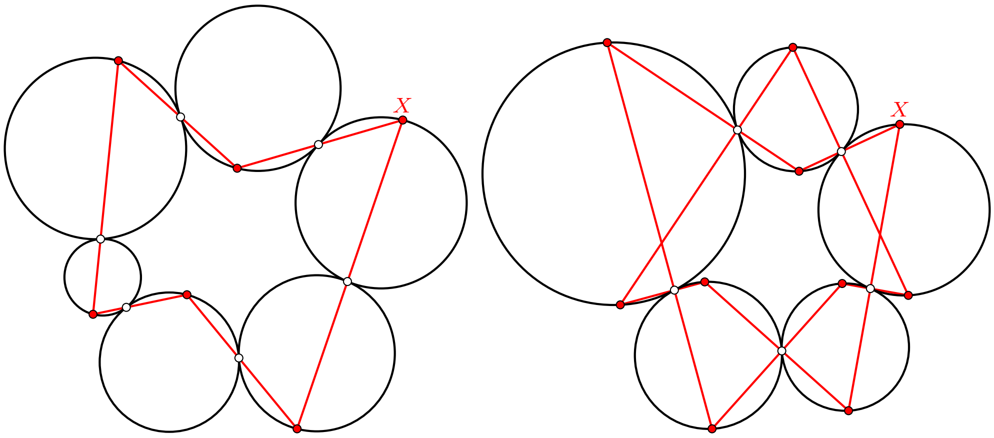


Figure 12. Illustration for Theorem 9. The red polygon closes for every position of the starting point X on the circle.

Let us now consider chains of touching circles. We will use the notation $C_1 \mathbin{)\!(} C_2$ for two touching circles C_1 and C_2. Then we have the following.

Theorem 10. *Let $C_1 \mathbin{)\!(} C_2 \mathbin{)\!(} C_3 \mathbin{)\!(} \cdots \mathbin{)\!(} C_n \mathbin{)\!(} C_1$ be a closed chain of touching circles. Let A_i be the common point of C_i and C_{i+1} and $\varphi = \varphi_{A_n} \circ \cdots \circ \varphi_{A_2} \circ \varphi_{A_1}$. Then the following holds: If n is even, then φ is the identity map on C_1. If n is odd, then $\varphi \circ \varphi$ is the identity map on C_1 (see Figure 13).*

Note that this situation of a chain of touching circle occurs in particular in Steiner's closing theorem.



(a) (b)

Figure 13. Illustration for Theorem 10. (a) For an even number of touching circles the red polygon closes for any position of the starting point X. (b) For an odd number of touching circles the red polygon closes after the second round for any position of the starting point X.

Proof. The claim follows immediately from the fact that we have $\delta_{A_i} = \gamma_{A_i} = 0$ in the case of touching circles. Then, the closing condition in Theorem 7 is trivially satisfied for an even number n. If n is odd then apply the result to the chain that runs through twice

$$C_1)\!(C_2)\!(C_3)\!(\cdots)\!(C_n)\!(C_1)\!(C_2)\!(C_3)\!(\cdots)\!(C_n)\!(C_1$$

and we are done. *q.e.d.*

Miquel's triangle theorem [16, Théorème I, Planche II, Fig. 1] turns out to be a special case of Theorem 3. To see this, consider the case of three circles.

Corollary 11. *Let $C_1 \between C_2 \between C_3 \between C_1$ be a closed chain of three circles which all pass through a point B. Let A_i be the other common point of C_i and C_{i+1}. Then, the map $\varphi_{A_3} \circ \varphi_{A_2} \circ \varphi_{A_1}$ is the identity map on C_1 (see Figure 14(a)).*

Proof. The points $B_i = B$ and A_i are the common points of C_i and C_{i+1}. Obviously the map $\varphi_{B_3} \circ \varphi_{B_2} \circ \varphi_{B_1}$ is the identity map on C_1. Hence, according to Theorem 3 this is also the case for $\varphi_{A_3} \circ \varphi_{A_2} \circ \varphi_{A_1}$. *q.e.d.*

Notice that the statement of Corollary 11 is true for any number $n \geq 3$ of circles which pass through a common point B (see Figure 14(b)).

It follows from Lemma 6 that the transfer angle, and hence the closing property of a chain, is invariant unter Möbius transformations. This means also that we can define the following variant of the pivot map φ_A.

Definition 12. Let $C_1 \between C_2$ be two intersecting or touching circles, A a common point of C_1 and C_2, and I a point not on $C_1 \cup C_2$. Then $\varphi_A^I : C_1 \to C_2, X \mapsto \varphi_A^I(X)$, is defined as follows: If $X \neq A$, then the points $X, \varphi_A^I(X), A$, and I are concyclic. If $X = A$, then the circle through the points $A, \varphi_A^I(X)$, and I is the tangent to C_1 in A. In particular, if C_1 and C_2 touch at A, then the image of $X = A$ is the point $\varphi_A^I(X) = A$ (see Figure 15).

The previous results nicely carry over to the map φ_A^I. As an example we get the six circles theorem of Miquel [15, Théorème I, Planche III, Fig. 1]:

Corollary 13. *Let $C_1 \between C_2 \between C_3 \between C_1$ be a closed chain of three circles which all pass through a point B, and I a point not on any of the three circles. Let A_i be the other common point of C_i and C_{i+1}. Then, the map $\varphi_{A_3}^I \circ \varphi_{A_2}^I \circ \varphi_{A_1}^I$ is the identity map on C_1 (see Figure 16).*

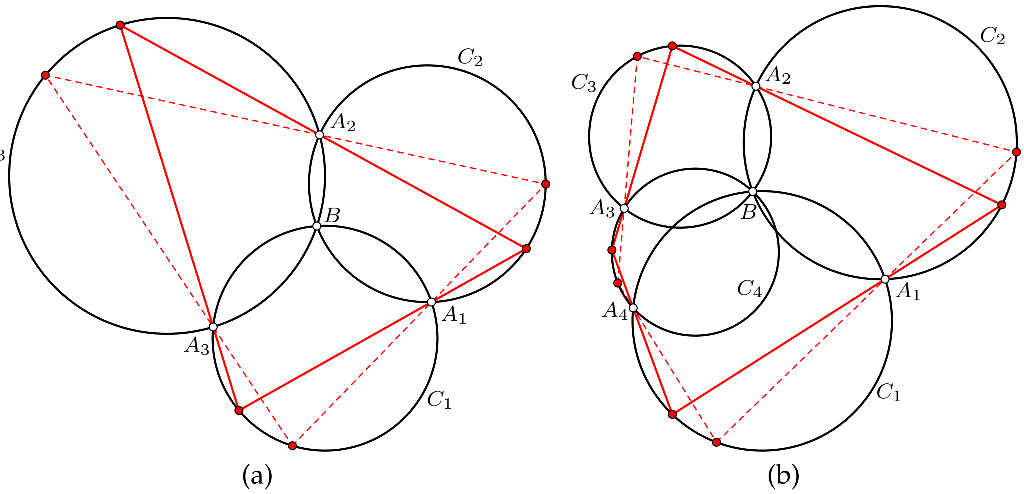


Figure 14. (a) Miquel's triangle theorem. The red triangle through the pivots A_i closes in every position. (b) The closing property for four concurrent circles.

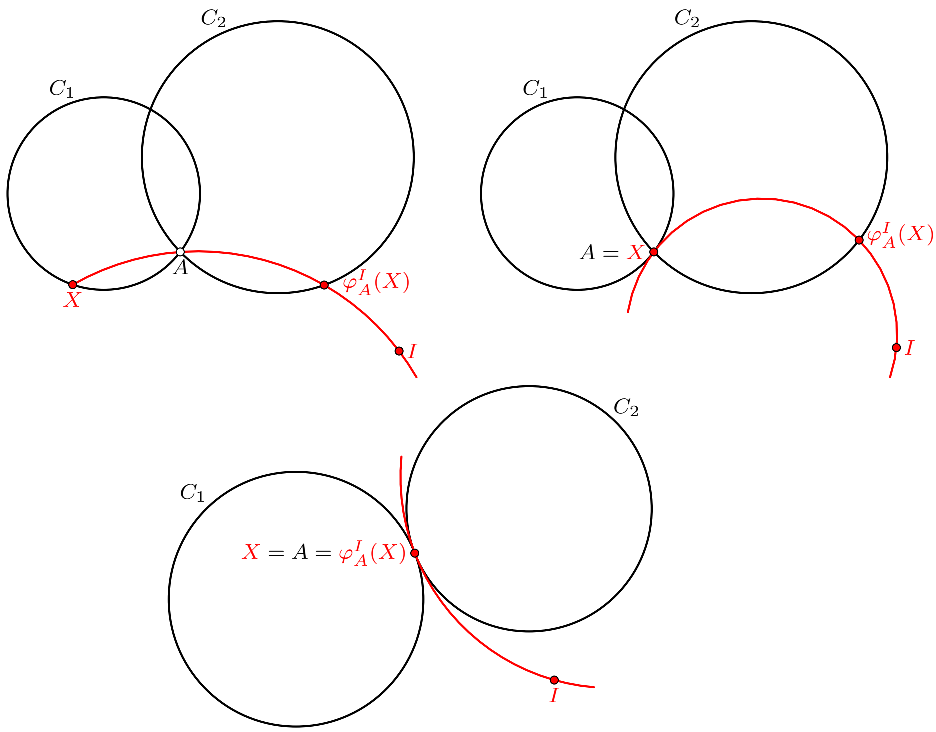


Figure 15. The map $\varphi^I_A : C_1 \to C_2, X \mapsto \varphi^I_A(X)$.

We would also like to point out that the previous closing results can be carried over to circle chains that are *not* closed. Figure 17 shows such a situation for an open chain of four circles. The red polygon closes in every position if it closes in one position. This can be seen as follows. Given an open chain of circles $C_1 \between C_2 \between C_3 \between \ldots \between C_n$ we can applying Theorem 7 to the *closed* chain

$$C_1 \between C_2 \between C_3 \between \ldots C_{n-1} \between C_n \between C_{n-1} \between C_{n-2} \ldots \between C_2 \between C_1.$$

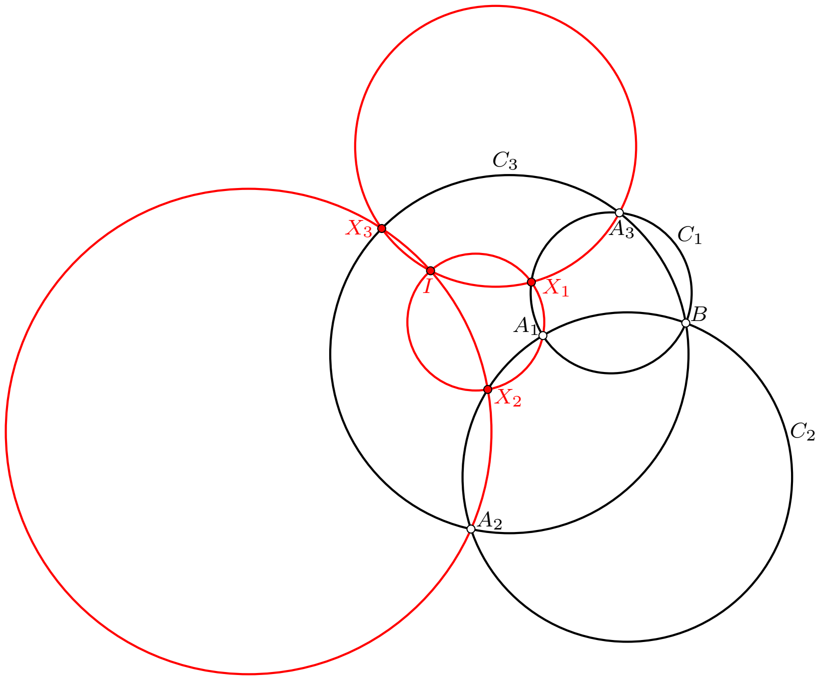


Figure 16. Miquel's six circles theorem follows from Theorem 3 applied to $\varphi^I_{A_3} \circ \varphi^I_{A_2} \circ \varphi^I_{A_1}$.

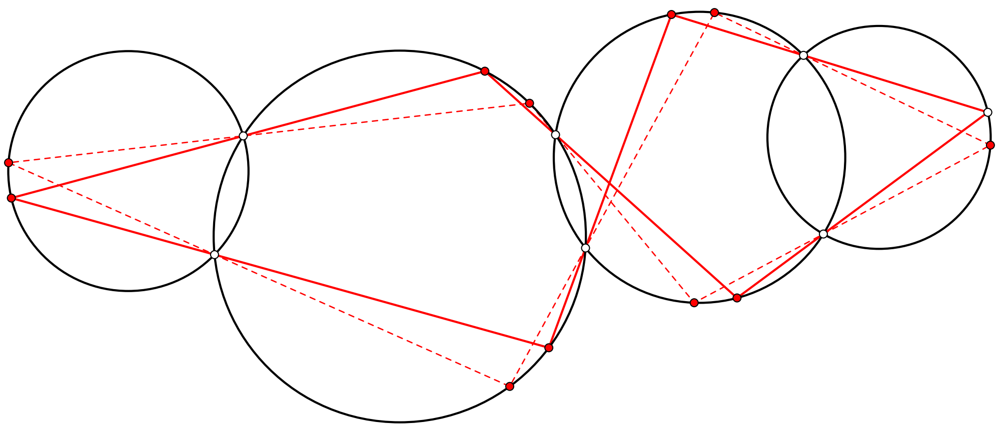


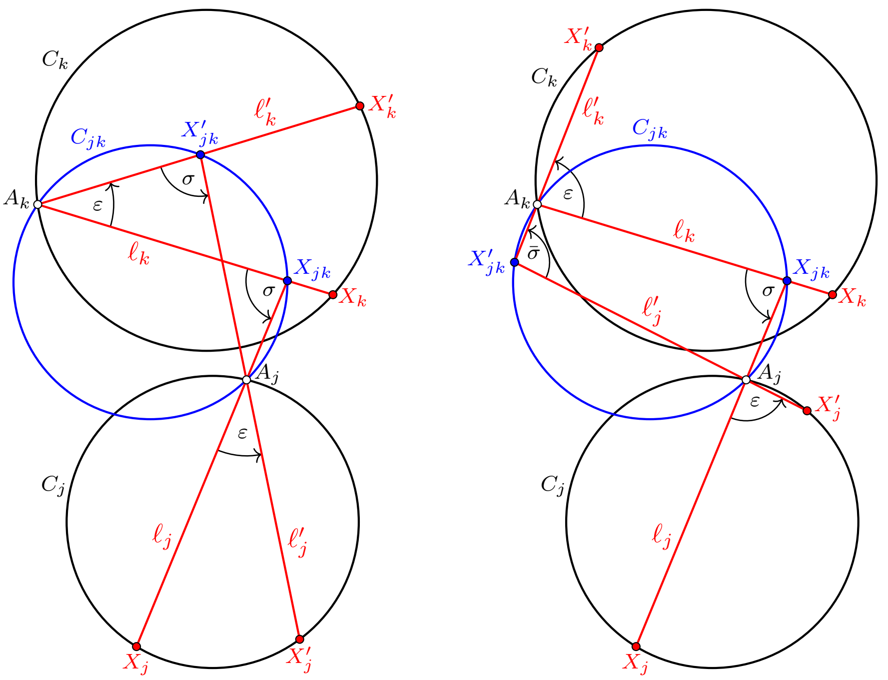
Figure 17. A closing configuration for an open chain of circles. The red polygon closes in every position it it closes in one position.

4. Even more incidences

Let us consider a chain $C_1 \between C_2 \between C_3 \between \cdots$ of intersecting circles, not necessarily closed, with centers M_i, and let A_i be a common point of C_i and C_{i+1}. Take two starting points X_1 and X_1' on C_1, and consider the resulting polygons $X_{i+1} = \varphi_{A_i}(X_i)$, and $X'_{i+1} = \varphi_{A_i}(X'_i)$. By iterating the argument in the proof of Theorem 2 we have that $\sphericalangle X_1 M_1 X_1' = \sphericalangle X_i M_i X_i'$ for all i. An immediate consequence is Lemma 14 below.

Let us first fix some notation that we will use throughout this section. In a situation like the one described above, we set:

- ℓ_j is the line through the points X_j, A_j, X_{j+1}, and ℓ'_j is the line through the points X'_j, A_j, X'_{j+1}.
- X_{ij} is the intersection of the lines ℓ_i and ℓ_j, and X'_{ij} is the intersection of the lines ℓ'_i and ℓ'_j
- For the circle C_i, we will also use the notation $C_{i-1,i}$. Similarly we will write $X_{i-1,i}$ for the point X_i. The reason for this notation will become clear below.



(a) (b)

Figure 18. Proof of Lemma 14.

With these conventions we have the following.

Lemma 14. *The intersection X_{jk} of the lines ℓ_j and ℓ_k lies on a fixed circle C_{jk} through the points A_j and A_k, no matter of the position of the initial point X_1. The circles C_{ij}, C_{jk}, C_{ki} are concurrent in a point P_{ijk} for all triples i, j, k.*

Proof. Let X_1 and X'_1 be two initial points on C_1 and consider the resulting polygons. Consider the points X_{jk} and X'_{jk} in the notation introduced above. We have to treat two cases.

1. case: X_{jk} and X'_{jk} lie on the same side of the line A_jA_k (see Figure 18(a)). Then we have $\sphericalangle A_kX_{jk}A_j = \sphericalangle A_kX'_{jk}A_j$ (these angles are denoted by σ in Figure 18) and hence the points $A_k, X_{jk}, X'_{jk}, A_j$ are concyclic.

2. case: X_{jk} and X_{jk} lie on the opposite sides of the line A_jA_k (see Figure 18(b)). In this case the angles $\sigma = \sphericalangle A_kX_{jk}A_j$ and $\bar{\sigma} = \sphericalangle A_jX'_{jk}A_k$ are supplementary, and hence the points $A_k, X_{jk}, X'_{jk}, A_j$ are again concyclic. This proves the first part of the theorem.

To show that the circles C_{ij}, C_{jk}, C_{ki} are concurrent, consider a situation where the lines ℓ_i and ℓ_j meet in the intersection of C_{ij} and C_{jk} different from the common point A_j. In this case, the line ℓ_k passes also through this point, i.e., $X_{ij} = X_{jk} = X_{ki}$. But his means that the circle C_{ki} also passes through that same point. *q.e.d.*

Figure 19 illustrates Lemma 14 applied to a closed chain $C_1 \between C_2 \between C_3 \between \cdots \between C_6 \between C_1$ with intersection points $A_1, \ldots, A_6$ such that $\varphi_{A_6} \circ \ldots \varphi_{A_1}$ is the identity on C_1. The pivots $A_1, \ldots, A_6$ are represented only as black numbers in small circles. It is now convenient to use the alternative notation $C_{i-1,i}$ for the circle C_i, and $X_{i-1,i}$ for the point X_i, since this is more coherent with the combinatorics of the situation. Observe that this convention is compatible with the general notation X_{ij} for the intersection of the lines ℓ_i and ℓ_j. Also note that $C_{i-1,i}$ is the circumcircle of the triangle A_{i-1}, A_i and $X_{i-1,i}$. In Figure 19 we drop the letter X in X_{ij} and just write red indices ij. The common point P_{ijk} of the circles C_{ij}, C_{jk} and C_{ki} is indicated by light blue indices ijk in the figure. Then we have:

- The polygon $X_{61}X_{12}X_{23}X_{34}X_{45}X_{56}X_{61}$ closes for every position of the starting point X_{61} on the circle C_{61}.
- The intersection X_{ij} of the lines ℓ_i and ℓ_j lies on the circle C_{ij} for every position of the starting point X_{61} on the circle C_{61}.

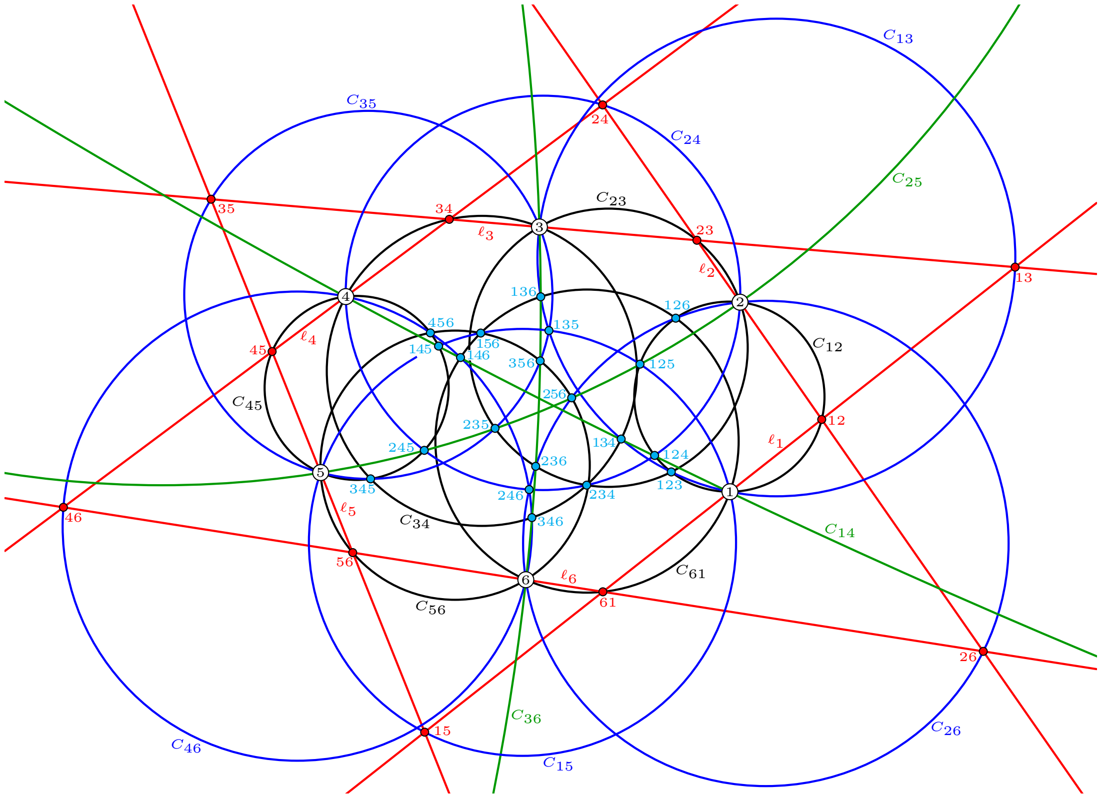


Figure 19. Illustration of Lemma 14.

- For all triples i, j, k the circles C_{ij}, C_{jk}, C_{ki} meet in a common point.

Note that the points X_{14}, X_{25} and X_{36} lie outside the region of the figure on the green circles C_{14}, C_{25} and C_{36}, respectively.

Now let us have a look at the special case of three touching circles.

Corollary 15. *Let $C_1 \asymp C_2 \asymp C_3 \asymp C_1$ be a closed chain of three touching circles, the contact points being $A_1 = A_4, A_2 = A_5, A_3 = A_6$ (see Figure 20). Then the polygon $X_1X_2 \dots X_6X_1$ with vertices $X_{i+1} = \varphi_{A_i}(X_i)$ closes for any staring point X_1 on C_1. Then, $A_1 = X_{14}$, $A_2 = X_{25}$, $A_3 = X_{36}$. The points $X_{135} := X_{13} = X_{35} = X_{51}$ and $X_{246} := X_{24} = X_{46} = X_{62}$ lie on the circumcircle C of the triangle $A_1A_2A_3$. The lines ℓ_i and ℓ_{i+3} are orthogonal, and the midpoint of the segment $X_{135}X_{246}$ is the center of this circle C.*

Proof. Consider, e.g., the circle C_{26} through the points $A_2, A_6 = A_3, X_{26}$. Observe that $X_{26} = A_1$ if $X_1 = A_1$. Hence C_{26} is the circumcircle C of the triangle $A_1A_2A_3$. Similarly $C_{24} = C_{46} = C_{13} = C_{35} = C_{51} = C$. It follows that $X_{13} = X_{35} = X_{51}$ and $X_{24} = X_{46} = X_{62}$ lie on C.

Finally, the sum of the transfer angles of $\varphi_{A_3} \circ \varphi_{A_2} \circ \varphi_{A_1}$ is π, i.e., the segment X_iX_{i+3} is a diameter of the circle C_i. In particular, in the position where the line X_1X_2 passes through the centers of C_1 and C_2 we have that $\sphericalangle A_1A_2X_2$ is a right angle. Hence ℓ_2 and ℓ_5 are orthogonal, independent of the starting point X_1. Similarly, we have that ℓ_1 and ℓ_4, and ℓ_3 and ℓ_6 are orthogonal. In particular, it follows that the center M of C is the midpoint of the segment $X_{135}X_{246}$. *q.e.d.*

The situation of four touching circles is simpler than for three touching circles, since four is an even number (recall Theorem 10).

Corollary 16. *Let $C_1 \asymp C_2 \asymp C_3 \asymp C_4 \asymp C_1$ be a closed chain of four touching circles, the contact points being A_1, A_2, A_3, A_4 (see Figure 21). Then the polygon $X_1X_2 \dots X_4X_1$ with vertices $X_{i+1} = \varphi_{A_i}(X_i)$ closes for any staring point X_1 on C_1. The points X_{13} and X_{24} lie on the circumcircle C of the quadrilateral $A_1A_2A_3A_4$.*

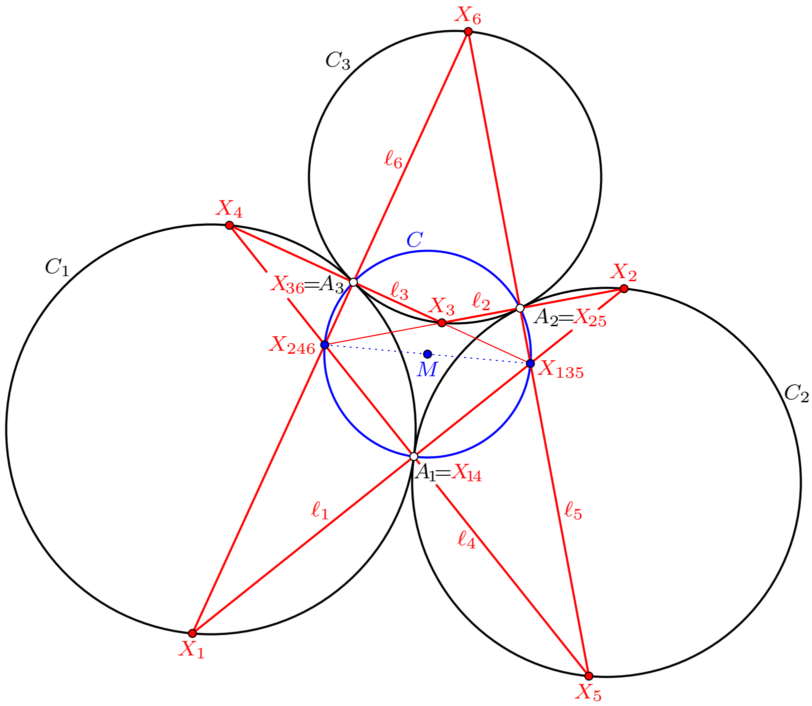


Figure 20. The special case of three touching circles in Corollary 15.

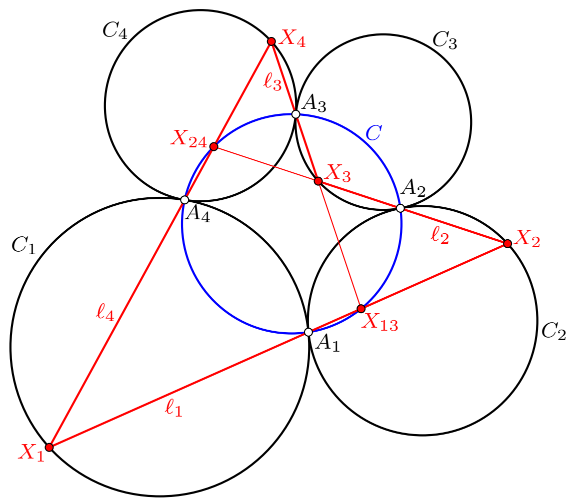


Figure 21. The special case of four touching circles in Corollary 16.

Proof. First recall that the contact points A_1, A_2, A_3, A_4 of four touching circles form always a cyclic quadrilateral. Observe that $X_{13} = A_2$ if $X_2 = X_3 = A_2$. Hence $C_{13} = C$. Similarly $C_{24} = C$, and we are done. *q.e.d.*

Now we can prove an extended version of Steiner's quadrilateral theorem.

Corollary 17. *Consider the complete quadrilateral consisting of four lines l_1, l_2, l_3, l_4 such that l_{i-1}, l_i, l_{i+1} form a triangle with circumcircle C_i for $i = 1, 2, 3, 4$ (see Figure 22). Let A_i denote the intersection of l_i and l_{i+1}, P the intersection of l_2 and l_4, and Q the intersection of l_1 and l_3. Then, $\varphi := \varphi_{A_4} \circ \varphi_{A_3} \circ \varphi_{A_2} \circ \varphi_{A_1}$ is the identity map on C_1, i.e., the resulting quadrilateral X_1, $X_{i+1} = \varphi_{A_i}(X_i)$ closes for any initial point X_1 on C_1. The circles C_1, C_2, C_3, C_4 pass though a common point S, the Steiner point. Moreover we have:*

- *The intersection X_{13} of the lines ℓ_1 and ℓ_3 runs on a circle C_{13} through A_1, A_3 and S.*
- *The intersection X_{24} of the lines ℓ_2 and ℓ_4 runs on a circle C_{24} through A_2, A_4 and S.*
- *The intersection X of the lines X_1X_3 and X_2X_4 runs on a circle C through P, Q and S.*
- *The points X_1, X_2, X_3, X_4 are concyclic on a circle D through S*
- *The points X_1, X_3 and P, as well as X_2, X_4 and Q are collinear.*

Proof. The fact that the polygon $X_1X_2X_3X_4$ closes for any position of the initial point X_1 on C_1 follows directly from Corollary 8. Lemma 14 implies that X_{13} and X_{24} lie on circles C_{13} and C_{24} through A_1, A_3 and A_2, A_4, respectively. The sum of the angles in the quadrilateral $X_3PX_1X_2$ is

$$\sphericalangle X_3PX_1 + \sphericalangle PX_1X_2 + \sphericalangle X_1X_2X_3 + \sphericalangle X_2X_3P = 2\pi.$$

However, by the inscribed angle theorem applied to the circles C_1, C_2, and C_3, the last three of these angles do not depend on the position of X_1. Since $\sphericalangle X_3PX_1 = 0$ for $X_1 = A_1$ it follows that X_1, P, X_3 are always collinear (see Figure 23). Similarly, we have that X_2, Q, X_4 are collinear. Then, again by the inscribed angle theorem, it follows that the quadrilateral $X_1X_3X_2X_4$ has fixed vertex and diagonal angles. Therefore, all these quadrilaterals are similar when X_1 moves along C_1 (see [11], or [14]). If $X_1 = A_1$ this is a cyclic quadrilateral (with circumcircle C_2, see Figure 23), so this is always the case. Let D denote the circumcircle of the cyclic quadrilateral $X_1X_3X_2X_4$ (this is the green, dashed circle in Figure 22). The circles C_1 and C_3 meet in a point S. Therefore, if $X_1 = S$ then $X_1 = X_3$ and hence the quadrilateral $X_1X_3X_2X_4$ with the diagonal points X_{13} and X_{24} degenerates to a point. Therefore, also C_2, C_4, C_{13}, and C_{24} meet in S. If we consider the circles C_1, C_3, C_2, C_4 and the maps $\varphi_P, \varphi_{A_2}, \varphi_Q, \varphi_{A_4}$ we see that the polygon $X_1X_3X_2X_4$ is closed for every position of X_1 on C_1. Therefore the intersection X of X_1X_3 and X_2X_4 lies on a circle C through P and Q. Also the point X collapses together with the other points X_i when $X_1 = S$. Hence C passes also through S. It remains to show that the circumcircle D of the points X_1, X_2, X_3, X_4 passes through S independent of the position of X_1 on C_1. This follows from the fact that the angles $\sphericalangle A_4X_1S$ and $\sphericalangle SX_4A_4$ are independent of the position of X_1. Therefore this is also the case for $\sphericalangle X_1SX_4$. Hence $X_1X_3X_4S$ is always a cyclic quadrilateral, since this is the case for the position $X_1 = A_1$ (see Figure 23). *q.e.d.*

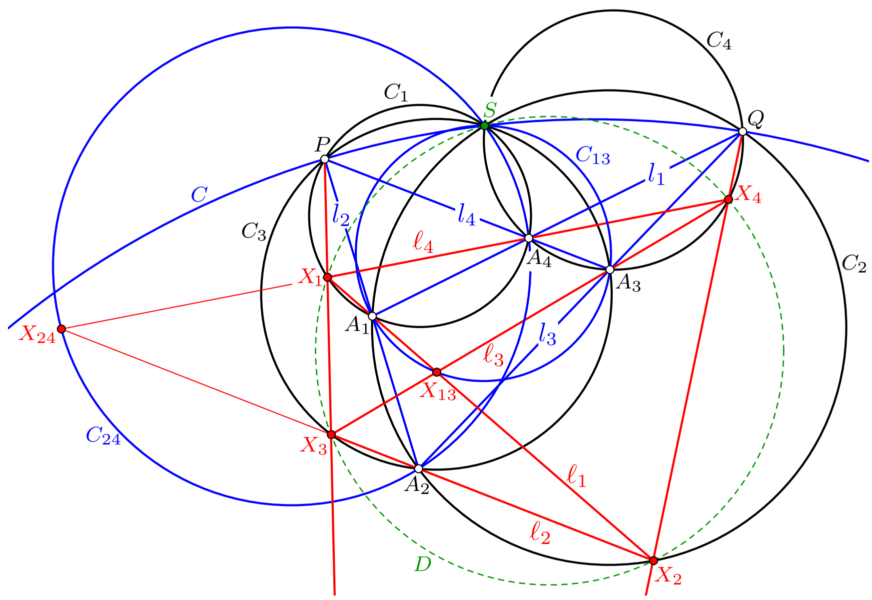


Figure 22. Steiner's quadrilateral theorem follows from Corollary 8 and Lemma 14 when we take $n = 4$.

Figure 24 illustrates Lemma 14 applied to a closed chain of three intersection circles C_1, C_2, C_3 with intersection points A_i, B_i. It follows from Theorem 9 that $\varphi_{B_3} \circ \varphi_{B_2} \circ \varphi_{B_1} \circ \varphi_{A_3} \circ \varphi_{A_2} \circ \varphi_{A_1}$ is the identity map on C_1, i.e., the corresponding polygon $X_1X_2 \dots X_6X_1$ closes for every position of the starting point X_1 on C_1. The intersections X_{ij} of the lines ℓ_i and ℓ_j lie on the blue circles C_{ij}, where $C_{24} = C_{46} = C_{62} =: C_{246}$ and $C_{13} = C_{35} = C_{51} =: C_{135}$. Moreover C_{25}, C_{36} and C_{14} are touching C_{246} and C_{135}.

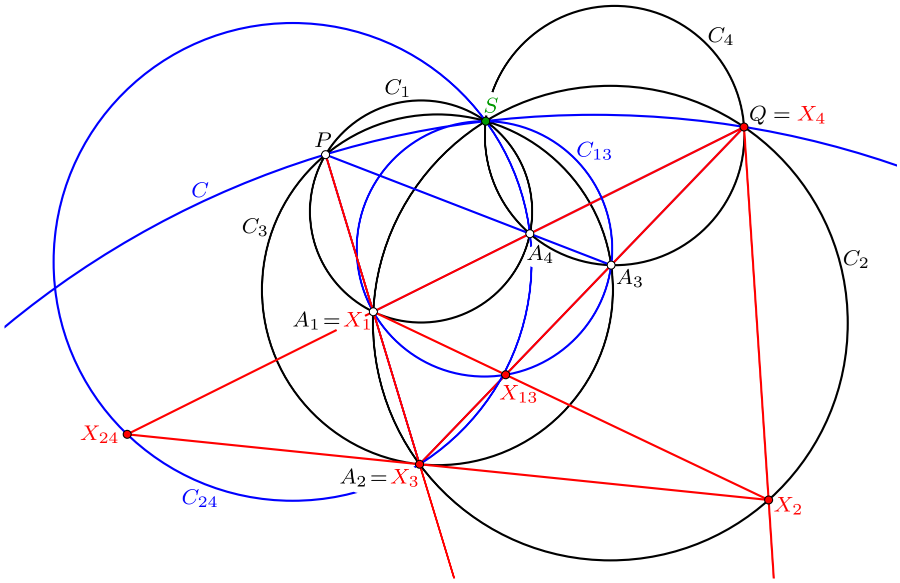


Figure 23. Proof of Corollary 17. The special case $X_1 = A_1$.

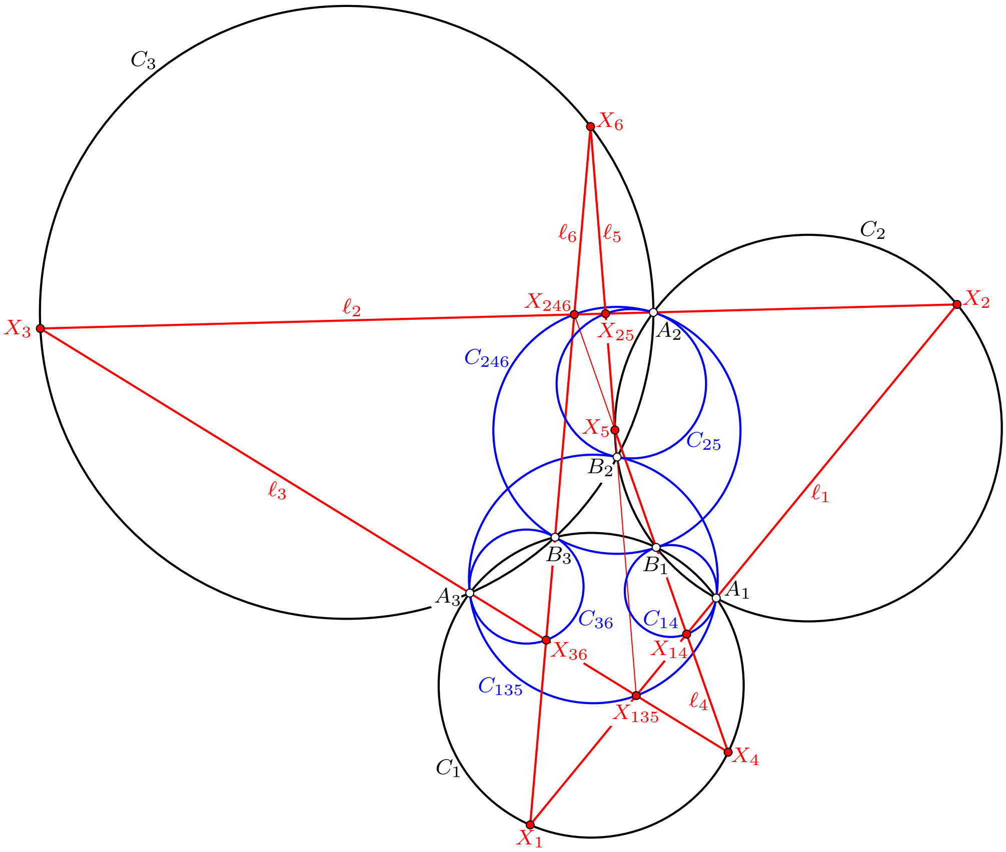


Figure 24. Lemma 14 applied to a closed chain of three intersecting circles.

As a final remark we mention the following.

Corollary 18. *Let $C_1 \between C_2 \between C_3 \between \cdots \between C_n \between C_1$ be a closed chain of circles. Let A_i, B_i be the intersection points of C_i and C_{i+1}, where $A_i = B_i$ is allowed. Let $\delta_{A_i} := \sphericalangle A_i M_i B_i, \gamma_{A_i} := \sphericalangle B_i M_{i+1} A_i$ and $\varphi := \varphi_{A_n} \circ \cdots \circ \varphi_{A_2} \circ \varphi_{A_1}$. If*

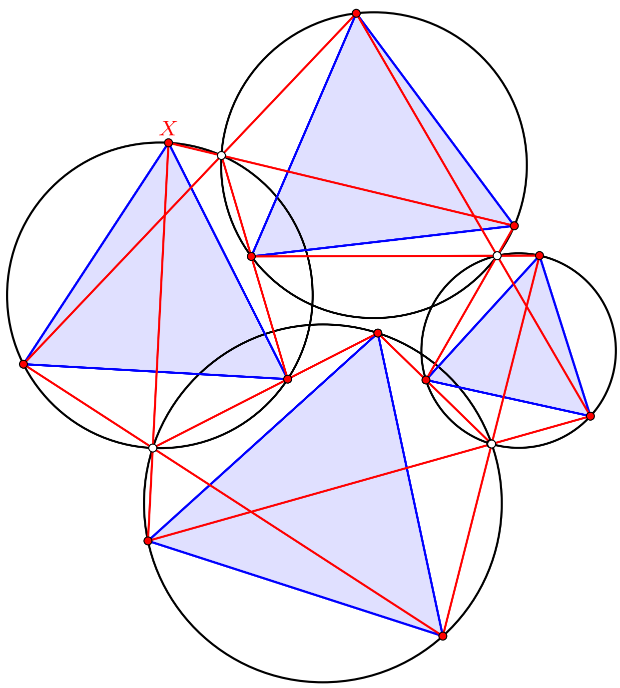


Figure 25. Illustration of Corollary 18 with four circles such that the polygon closes after three runs. Then the vertices on each circle form a regular triangle. The triangles "dance" synchronized pirouettes when X runs on the circle. If three of the circles are given, it is an easy exercise to construct a matching fourth circle.

$\sum_{i=1}^{n} \delta_{A_i} + \gamma_{A_i}$ is a rational multiple of π, then there is a natural number k such that the map φ^k is the identity on C_1. That is, starting with any point X on C_1, the resulting polygon will eventually close on X after a finite number of steps. The points $X_i X_{i+n} X_{i+2n} X_{i+kn}$ form a regular polygon in the circle C_i (see Figure 25).

Observe that Corollary 18 can be interpreted as a stacked version of the Lighthouse theorem [7]: Each pair of points A_i, A_{i+1} can be considered as lighthouses while the lines ℓ_k passing through them correspond to the light beams.

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Author's contributions

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