Elemente der Mathematik 77 (2022), 180–186
DOI 10.4171/EM/466
© 2021 Swiss Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Ceva-triangular points of a triangle

Norbert Hungerbühler and Gerhard Wanner

Gerhard Wanner, gebürtiger Tiroler und assimilierter Genfer, wurde 2019 zum Ehrenmitglied der Schweizerischen Mathematischen Gesellschaft ernannt.
Norbert Hungerbühler, gebürtiger Ostschweizer und assimilierter Zürcher, ist Professor für Mathematik und Ausbildung an der ETH Zürich.

"Es ist in der That bewundernswürdig, dass eine so einfache Figur, wie das Dreieck, so unerschöpflich an Eigenschaften ist." (A. L. Crelle [1, p. 176])

It is known since at least a century that the medians of any triangle form the sides of another triangle, i.e., satisfy the triangle inequalities (Eucl. I.20). For a picture, see [3], and for many properties of this triangle and its iterates, see [4]. It was also recognized that most other triangle centers, for example the incenter, do not always possess Cevians with this property (see Figure 3, case $\boxed{2}$, and [2]). We therefore make the following definition.

Definition. We say that a point P is *Ceva-triangular* for a triangle ABC if its Cevians u, v, w (through A, B, C respectively) satisfy the triangle inequalities.

1 Singularities

We can guess from a first plot of the set of these points in Figure 1 (top) that our set has interesting singular points at the vertices A, B, C and also at the vertices U, V, W of what we call the *Gaussian extension* of ABC (Gauss, *Werke*, vol. IV, p. 396).

Die Schwerlinien eines jeden Dreiecks erfüllen die Dreiecksungleichung. Man kann also aus den Schwerlinien selber wieder ein Dreieck konstruieren. Im Gegensatz dazu lässt sich aus den Winkelhalbierenden oder den Höhen eines Dreiecks nicht in jedem Fall ein Dreieck bilden. Bei einem gegebenen Dreieck ABC fragt man sich nun, für welche Punkte P die Ecktransversalen durch P die Dreiecksungleichung erfüllen. Die Menge M dieser Punkte enthält also sicher den Schwerpunkt des Dreiecks. Darüber hinaus hat sie Singularitäten in den Ecken der Gaußschen Erweiterung des Dreiecks ABC. Bei der Untersuchung der Menge M taucht dann ganz überraschend gleich mehrfach der goldene Schnitt auf.

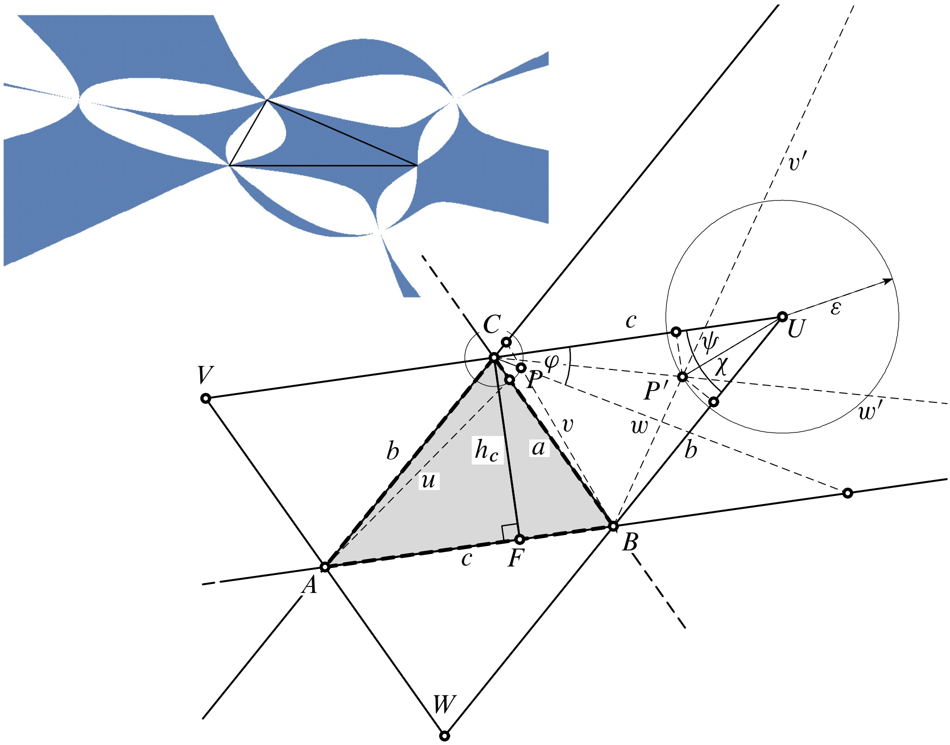


Figure 1. First plot (top), explanation (bottom)

First, we rotate the point P on an (infinitely) small circle around C (Figure 1, bottom). Then the Cevians through A and B behave like $u \approx b$ and $v \approx a$, while the third Cevian is $w = \frac{h_c}{\sin\varphi}$. Thus the triangle inequality is satisfied if

$$|b-a| \le \frac{h_c}{\sin\varphi} \le b+a \quad \text{i.e., if} \quad \arcsin\frac{h_c}{b+a} \le \varphi \le \arcsin\frac{h_c}{|b-a|}. \tag{1}$$

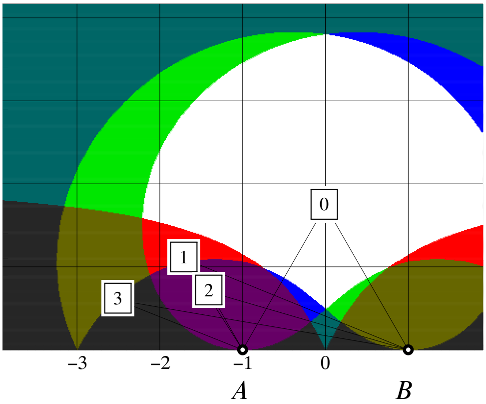
Thus there is always a pair of "forbidden" sectors with $\varphi < \arcsin\frac{h_c}{b+a}$. A second pair of forbidden sectors with $\varphi > \arcsin\frac{h_c}{|b-a|}$ only exists for $h_c < |b-a|$.

Secondly, even if a Cevian w' tends to infinity, the triangle inequality can still be satisfied if, at the same time, a second Cevian, say, v', also tends to infinity with the same speed. This happens in the vicinity of a "Gauss" point like U. Let thus a point P' rotate on another (infinitely) small circle around U. We see from two similar triangles above and below CP' that

$$\frac{CP'}{\varepsilon\sin\psi} = \frac{w'}{h_c} \quad \text{and, for } \varepsilon \to 0, \quad w' \approx \frac{c \cdot h_c}{\varepsilon\sin\psi} \quad \text{and similarly} \quad v' \approx \frac{b \cdot h_b}{\varepsilon\sin\chi}.$$

Since $c \cdot h_c = b \cdot h_b$ (area of the parallelogram $ABUC$), we have

$$w' = v' \quad \text{if} \quad \psi = \chi, \tag{2}$$



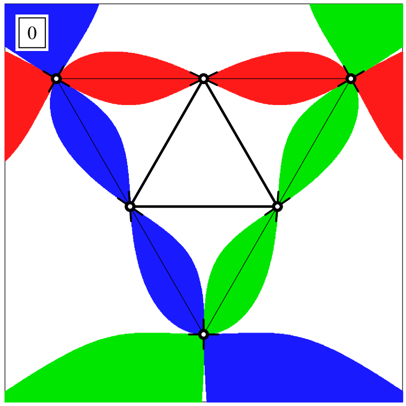


Figure 2. Topology of singular points for various positions of C (left): two pairs of forbidden sectors at C (red), at A (blue), at B (green). The celebrated equilateral triangle (right)

i.e., if P' approaches U along the angle bisector. A similar picture, where w' tends to infinity on the other side of AB, gives a second solution on the outer bisector, i.e., perpendicular to UP'.

Figure 2 (left) indicates the various regions of the position of C for which the inequalities

$$\frac{h_a}{|b-c|} < 1 \text{ (blue)}, \qquad \frac{h_b}{|c-a|} < 1 \text{ (green)}, \qquad \text{and} \qquad \frac{h_c}{|a-b|} < 1 \text{ (red)}$$

are satisfied. Figure 2 (right) and Figure 3 show "forbidden" regions for P at four triangles with continuously increasing complexity.

2 Asymptotes

The triangle with the sides u, v, w collapses to a line if its area, given by Heron's formula, is zero. For $A = (0,0)$, $B = (1,0)$, $C = (c_1, c_2)$, it is elementary, but slightly tedious, to compute the lengths of u, v, w as a function of the coordinates (x, y) of P. Hence, with $s := \frac{1}{2}(u + v + w)$, the boundary of the set of the Ceva-triangular points is given by the equation

$$0 = 16s(s-u)(s-v)(s-w) = (u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4).$$

It turns out that this expression is a polynomial $f(x, y)$ of degree 12. We now want to compute the asymptotes of the curve $f(x, y) = 0$. To this end, we plug in the equation of an asymptote $y = px + q$ in $f(x, y)$, which yields a polynomial $f(x, px + q)$ of degree 12. In this expression, we drop all terms of degree 10 or less. The remaining terms are $A_0(p,q)x^{12} + A_1(p,q)x^{11}$. One finds p, q as solution of the system $A_0(p,q) = 0$, $A_1(p,q) = 0$. The system is also of degree 12. For the vertical asymptote $x = r$, we

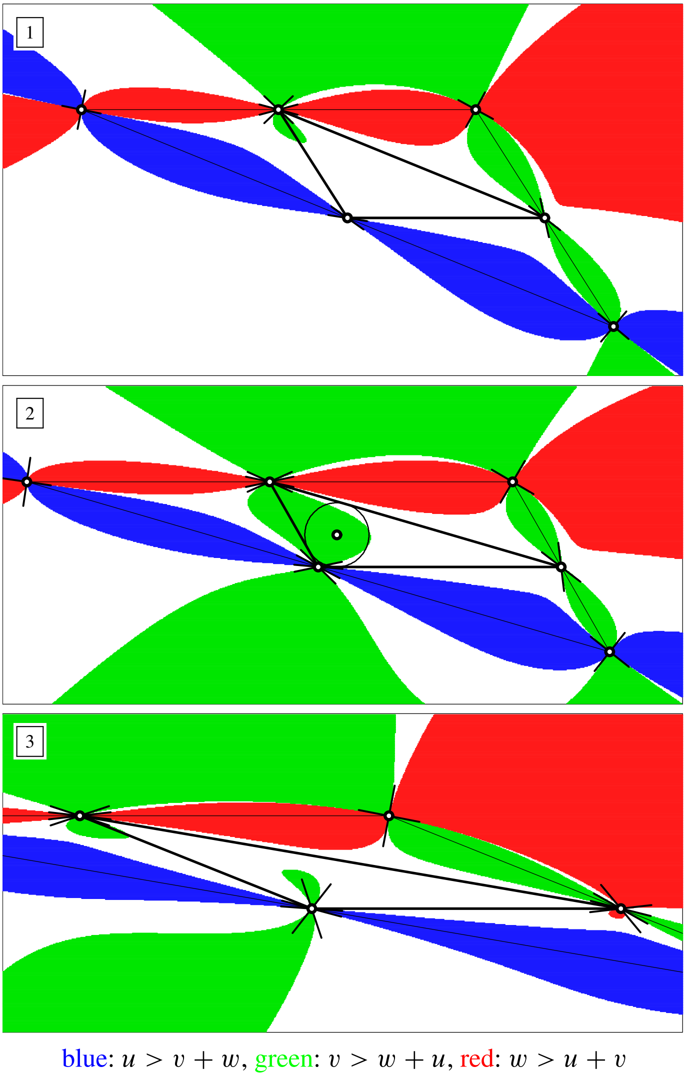


Figure 3. Triangles with 1, 2, and 3 double “forbidden” sectors; little tracts indicate the asymptotic directions (1) and (2). $A = (-1, 0)$, $B = (1, 0)$, 1: $C = (-1.7, 1.1)$, 2: $C = (-1.4, 0.7)$, 3: $C = (-2.5, 0.6)$

proceed analogously: put $f(r, y) = 0$ and solve $A_1(r) = 0$. A computer algebra system yields explicit expressions for p, q and r. It turns out that a vertical asymptote exists if c_1, the endpoint of the altitude in C, divides the side c in the golden ratio (*proportio divina*). The square root of 5 is also present in the case of an oblique asymptote. This suggests that a geometric interpretation of the asymptotes is possible, involving the golden ratio. Indeed, we have the following proposition.

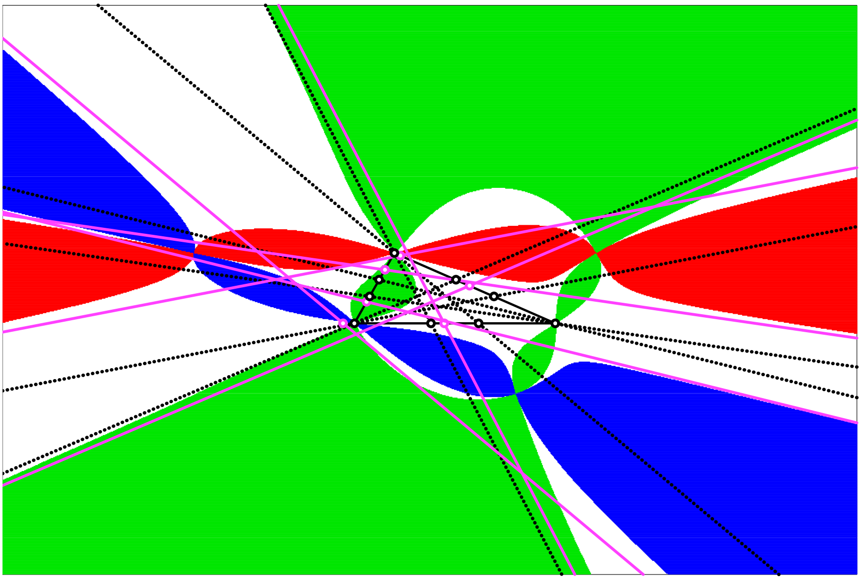


Figure 4. Directions of the asymptotes based on the golden section (Proposition 1, black dotted); corrected asymptotes from Proposition 2 (pink)

Proposition 1. *If the sides of the triangle ABC are divided according to the golden section (from left or right), then the connections of these points with the opposite vertices indicate the* directions *of the six asymptotes for $P \to \infty$ (Figure 4).*

Proof. We normalize AB to 1 (Figure 5 (a)) and choose for G the right "golden" point. Then, using the "golden" number $\Phi = (\sqrt{5}+1)/2$,

$$\text{Thales: } u = \Phi^2 w, \quad \text{Thales: } v = \Phi w \implies v + w = (\Phi + 1)w = \Phi^2 w = u. \qquad (3)$$

For the second golden point, u and v exchange their roles. ■

We remark that the line through the golden point on c which is closer to A and the golden point on a which is closer to B passes through U and is parallel to the line through A and golden point on a which is closer to C. Hence this line is also parallel to an asymptote. The same holds, mutatis mutandis, if we go cyclically around the triangle and if we replace the closer golden point by the more distant golden point.

Proposition 2. *The asymptote parallel to CG is obtained from the line CG (in the case where G is the right golden point on AB) by a parallel move to the left of distance*

$$m = \frac{\Phi \cot(\varphi - \beta) + \frac{1}{\Phi}\cot(\varphi + \alpha)}{\Phi^2 \cot(\varphi - \beta) - \Phi \cot(\varphi + \alpha) - \cot\varphi}, \qquad (4)$$

where φ is $\angle CGA$ (Figure 5 (b)). All six asymptotes are obtained by applying the six permutations of the vertices of ABC (see the pink lines in Figure 4).

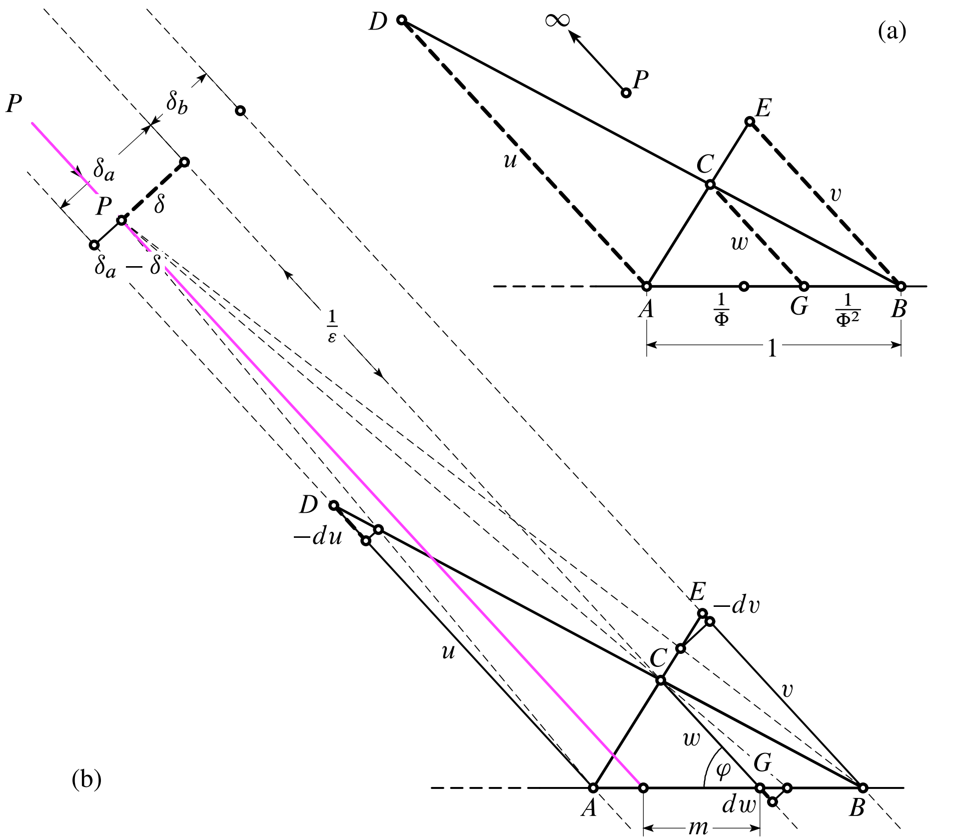


Figure 5. Proof for the correcting shift

Proof. If P is at infinity, we have $u = v + w$ from Proposition 1. We then move P from infinity to a position far outside at distance $\frac{1}{\varepsilon}$ from our triangle and at an unknown distance δ from CG. Then u, v, w will vary by du, dv, dw. In order to preserve $u = v + w$, we require that $du = dv + dw$. Since

$$\angle BDA = \varphi - \beta \quad \text{and} \quad \angle AEB = \angle ACG = \pi - \alpha - \varphi,$$

from narrow similar triangles and using $\cot(\pi - \alpha - \varphi) = -\cot(\varphi + \alpha)$, we have

$$\begin{aligned} du &= -\varepsilon \cdot u \cdot (\delta_a - \delta) \cdot \cot(\varphi - \beta), \\ dv &= \varepsilon \cdot v \cdot (\delta_b + \delta) \cdot \cot(\varphi + \alpha), \\ dw &= \varepsilon \cdot w \cdot \delta \cdot \cot \varphi, \end{aligned}$$

from which (4) follows by solving for δ, using $\delta_a = \frac{1}{\Phi} \cdot \sin\varphi$, $\delta_b = \frac{1}{\Phi^2} \cdot \sin\varphi$, inserting (3) and $m = \frac{\delta}{\sin\varphi}$. ■

References

[1] A. L. Crelle, *Sammlung mathematischer Aufsätze und Bemerkungen*, 1. Band. Maurer, Berlin, 1821.

[2] M. Hajja, P. T. Krasopoulos, H. Martini, The median triangle theorem as an entrance to certain issues in higher-dimensional geometry. *Math. Semesterber.*, to appear.

[3] N. Hungerbühler, Proof without words: The triangle of medians has three-fourths the area of the original triangle. *Math. Mag.* **72** (1999), no. 2, 142.

[4] J.-C. Pont, *La Balade de la Médiane et le Théorème de Pythagoron*. Editions du Tricorne, Genève, 2012.

Norbert Hungerbühler
Department of Mathematics
ETH Zentrum, Rämistrasse 101
8092 Zürich, Switzerland
norbert.hungerbuehler@math.ethz.ch

Gerhard Wanner
37, chemin des Pinsons
1226 Thônex, Switzerland
gerhard.wanner@unige.ch