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Another design for sundials

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Based on Kepler's laws we calculate the position of the sun as a function of the date, the local time, the latitude and longitude and use the resulting formula to construct a portable sundial which works without being adjusted to the cardinal directions (i.e. there is no need for an additional compass).

1. Introduction

The design of sundials consists of a technical and a creative part. The technical part is to find a formula which tells you the position of the sun (relative to your local coordinate-system) as a function of the date and the time of the day. Such a formula is based upon Kepler's laws and some geometric respectively astronomic constants such as the angle between ecliptic and equatorial plane of the earth. In the first part we give a brief derivation of such a formula (see also Blatter [1]). The second and more creative part is then to find a design for the sundial (based of course on the formula of the first part consisting of

- an object which casts a shadow (the pointer),
- a surface which catches the shadow (the dialplate),
- a scale on the dialplate to read the time as a function of the position (or shape) of the shadow.

There is no 'generic way' to find an 'optimal design', however there is a long way of development between a simple stick in a wall and very sophisticated sundials (as for example, the famous sundial of the german engineer Martin Bernhardt in figure 1) which measure the time with an error of a minute or less; some are even equipped with a nonius to enhance the precision of the reading. Some others show, besides the local standard time, also the local mean sun-time or other astronomic quantities. For more information on old and new sundials see the bibliography.

Most sundials have their fixed (and well-chosen) position at walls or in gardens. Portable sundials usually need to be equipped with a compass in order to adjust

Figure 1. Bernhardt's sundial.

their position according to the cardinal directions. However, we want to give a design of a portable sundial which does not need this additional tool, i.e. a portable sundial which does not require a compass.

2. Kepler's laws

According to Kepler's first law (figure 2) the orbit of the earth in space is an ellipse with the sun in one focus. In polar coordinates an ellipse is given by

$$r(\psi) = \frac{1-\kappa^2}{1+\kappa\cos\psi} \tag{1}$$

For the earth the value of the eccentricity κ is approximately

$$\kappa \approx \frac{1}{60} \tag{2}$$

In the chosen coordinates the area enclosed by the ellipse is

$$A = \pi\sqrt{(1-\kappa^2)} \tag{3}$$

We formulate Kepler's second law in differential form:

$$\mathrm{d}A = \tfrac{1}{2}\sqrt{(1-\kappa^2)}\,\mathrm{d}t \tag{4}$$

Note that by equations (3) and (4) we have now scaled the time t such that 2π corresponds to one earth-year. Since we have, on the other hand,

$$\mathrm{d}A = \tfrac{1}{2}r^2\,\mathrm{d}\psi \tag{5}$$

combination with equation (4) yields the law of motion

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \sqrt{\frac{(1-\kappa^2)}{r^2}} = \frac{(1+\kappa\cos\psi)^2}{(1-\kappa^2)^{3/2}} \tag{6}$$

with intial condition

$$\psi(0) = 0 \tag{7}$$

on 3 January (perihelion-position of the earth). By separating the variables in

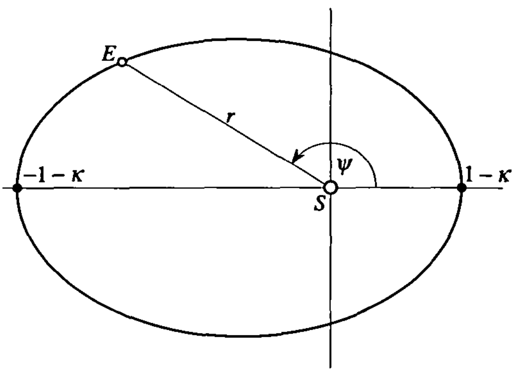

Figure 2. Kepler's first law.

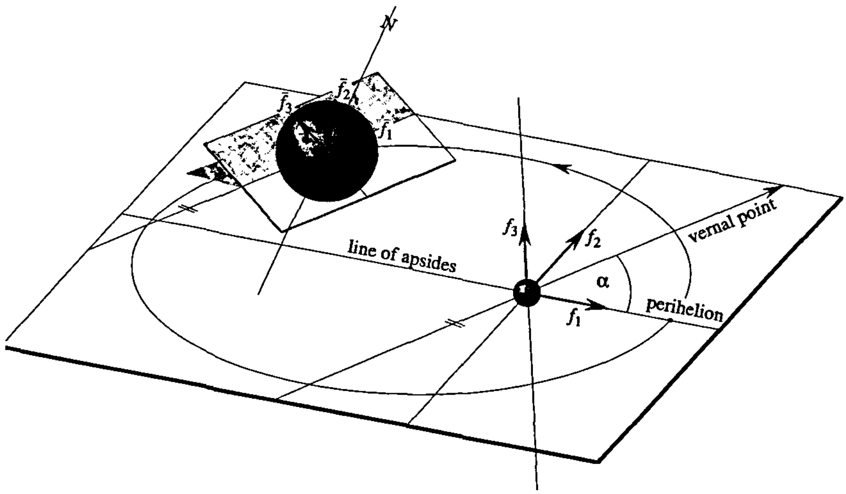

Figure 3. The coordinate systems (f_1, f_2, f_3) and $(\bar{f}_1, \bar{f}_2, \bar{f}_3)$.

equation (6) and then by integration we obtain as solution of equations (6) and (7):

$$t = 2\arctan\left(\sqrt{\frac{1-\kappa}{1+\kappa}}\tan\left(\frac{\psi}{2}\right)\right) - \frac{\kappa\sqrt{(1-\kappa^2)}\sin\psi}{1+\kappa\cos\psi} \tag{8}$$

Of course, we cannot solve equation (8) for ψ but we can easily find $\psi(t)$ numerically, e.g. with Newton's method starting the iteration at $\psi_0 = t$ (observe, that $t(\psi) \approx \psi$).

After the previous physical part, it remains to express the position of the sun (given by equation (8)) in our local coordinate system on earth. This is a purely geometric problem. In order to solve it we introduce several orthonormal coordinate systems: The system (f_1, f_2, f_3) with origin in the centre of the sun and the system $(\bar{f}_1, \bar{f}_2, \bar{f}_3)$ with origin in the centre of the earth are fixed with respect to the fixed-stars. f_1 is pointing to the perihelion, f_2 lies in the ecliptic oriented as indicated in figure 3). $\bar{f}_1$ is pointing to the vernal point (i.e. in the direction of the intersection of the ecliptic and the equatorial plane of the earth) and $\bar{f}_2$ to earth-north (see figure 3). f_3 and $\bar{f}_3$ are chosen such that both systems are positively oriented.

Remark. The angle α between f_1 and $\bar{f}_1$ is approximately $\alpha \approx 78{\cdot}5°$, the angle ε between f_3 and $\bar{f}_2$ approximately $\varepsilon \approx 23{\cdot}45°$.

Now, let us denote the vector from earth to sun by $x := \boldsymbol{ES}$. Thus, in f-coordinates x has the components

$$x_f = \begin{pmatrix} -\cos\psi(t) \\ -\sin\psi(t) \\ 0 \end{pmatrix}$$

It is convenient to introduce the so called *dynamic mean sun y* with coordinates

$$y_f := \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} \tag{10}$$

y would be the position of the sun if the orbit of the earth were a perfect circle. In order to find the coordinates of x (and y) in the $\bar{f}$-system we use the transformation

$$T_{f\bar{f}} = \begin{pmatrix} \cos\alpha & -\sin\alpha\cos\varepsilon & \sin\alpha\sin\varepsilon \\ \sin\alpha & \cos\alpha\cos\varepsilon & -\cos\alpha\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \tag{11}$$

and obtain

$$x_{\bar{f}} = T_{f\bar{f}} x_f = \begin{pmatrix} -\cos(\psi(t)-\alpha) \\ -\sin(\psi(t)-\alpha)\cos\varepsilon \\ \sin(\psi(t)-\alpha)\sin\varepsilon \end{pmatrix}, \qquad y_{\bar{f}} = \begin{pmatrix} -\cos(t-\alpha) \\ -\sin(t-\alpha)\cos\varepsilon \\ \sin(t-\alpha)\sin\varepsilon \end{pmatrix} \tag{12}$$

The so called *mean sun z* rotates with constant angular speed in the equatorial plane of the earth, i.e.

$$z_{\bar{f}} := \begin{pmatrix} -\cos(t-\alpha) \\ -\sin(t-\alpha) \\ 0 \end{pmatrix} \tag{13}$$

Now, we need to introduce one more orthonormal coordinate-system $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ which is fixed with respect to the earth (see figure 4: P is a fixed point on the surface of the earth).

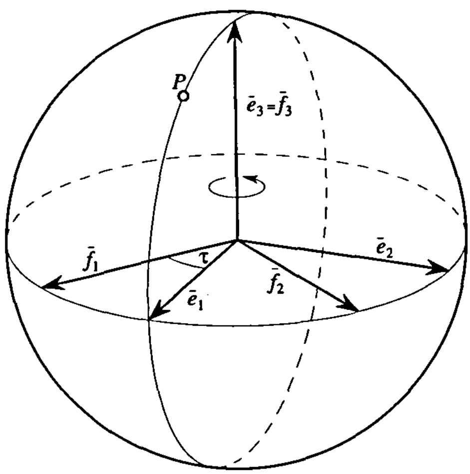

Figure 4. The coordinate system $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$.

The transformation-matrix to switch from the $\bar{e}$ to the $\bar{f}$-system is

$$T_{\bar{f}\bar{e}} = \begin{pmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

τ is called *star time* and one stellar day corresponds to 2π and to one complete revolution of the earth relative to the fixed-stars. We obtain for the mean sun z in $\bar{e}$-coordinates:

$$z_{\bar{e}} = T_{\bar{e}\bar{f}}\, z_{\bar{f}} = \begin{pmatrix} -\cos(\tau - t + \alpha) \\ \sin(\tau - t + \alpha) \\ 0 \end{pmatrix} \tag{15}$$

To define the *local mean time* s on the meridian through the point P let us agree that $s = \pi$ at the moment of culmination of the mean sun (i.e. when the $\bar{e}_1$-component of z is a maximum):

$$s := \tau - t + \alpha \tag{16}$$

Hence, in local mean time the mean sun is given by

$$z_{\bar{e}} = \begin{pmatrix} -\cos s \\ \sin s \\ 0 \end{pmatrix} \tag{17}$$

We now decompose $T_{\bar{f}\bar{e}}$ in the following way

$$T_{\bar{f}\bar{e}} = \begin{pmatrix} \cos(t-\alpha) & -\sin(t-\alpha) & 0 \\ \sin(t-\alpha) & \cos(t-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

and obtain

$$x_{\bar{e}} = T_{\bar{f}\bar{e}} x_{\bar{f}} = \begin{pmatrix} -\cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -\cos\delta(t) + 2\sin^2\left(\frac{\varepsilon}{2}\right)\sin(\psi(t)-\alpha)\sin(t-\alpha) \\ -\sin\delta(t) + 2\sin^2\left(\frac{\varepsilon}{2}\right)\sin(\psi(t)-\alpha)\cos(t-\alpha) \\ \sin(\psi(t)-\alpha)\sin\varepsilon \end{pmatrix}}$$

$$=: w(t) =: \begin{pmatrix} -r(t)\cos\mu(t) \\ r(t)\sin\mu(t) \\ h(t) \end{pmatrix} \tag{19}$$

where $\delta(t) := \psi(t) - t$. Since w is a unit vector we have

$$r(t) = (1 - h^2(t))^{1/2} \tag{20}$$

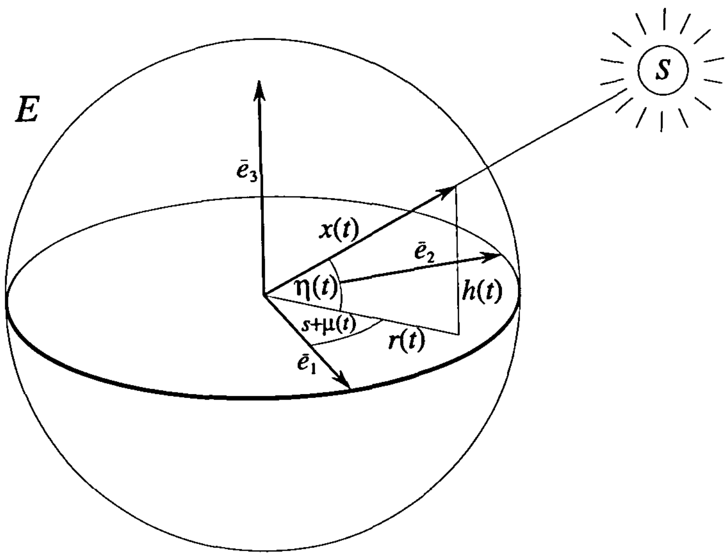

Figure 5. The position of the sun.

with $h(t) = \sin(\psi(t) - \alpha) \sin \varepsilon$. The expression

$$\mu(t) = \arctan \frac{-\sin \delta(t) + \tan^2\left(\frac{\varepsilon}{2}\right) \sin(2(t-\alpha) + \delta(t))}{\cos \delta(t) + \tan^2\left(\frac{\varepsilon}{2}\right) \cos(2(t-\alpha) + \delta(t))} \tag{21}$$

is called the *equation of time*. Rewriting x in the earth-fixed system $\bar{e}$ we obtain

$$x_{\bar{e}} = \begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} w(t) = \begin{pmatrix} -r(t)\cos(s + \mu(t)) \\ r(t)\sin(s + \mu(t)) \\ h(t) \end{pmatrix} \tag{22}$$

Figure 5 shows the situation in space.

So, we may interpret the quantity $\mu(t)$ as deviation of the real sun to the local mean time. Thus, simple sundials which are based on the assumptions that the earth's orbit is a circle (and which hence give the mean sun time) really show this error of about ± 15 minutes in spring and in autumn, respectively (figure 6 shows a plot of $\mu(t)$ in minutes).

Figure 7 is of special interest: it shows the deviation μ as a function of the height η of the sun and it explains the princple of Bernhardt's sundial in figure 1. The special choice of the shape of the pointer exactly compensates this effect.

Finally we need the position of the sun in a coordinate system which is adapted to the position of the observer on earth: see figure 8.

e_1 points to the east, e_2 to the north. To describe the position P we use the geographical longitude φ and the latitude ϑ. Hence the transformation matrix

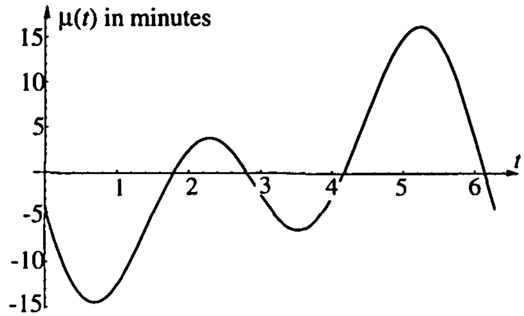

Figure 6. Equation of time ($t = 0$ corresponds to 3 January, a t-period of 2π corresponds to one earth-year).

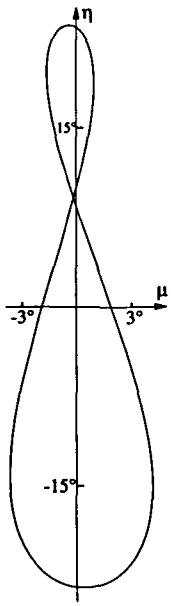

Figure 7. $\mu(t)$ and $\eta(t)$ in degree°.

$T_{\bar{e}e}$ will be

$$T_{\bar{e}e} = \begin{pmatrix} 0 & -\sin\vartheta & \cos\vartheta \\ 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \end{pmatrix} \tag{23}$$

such that we finally obtain

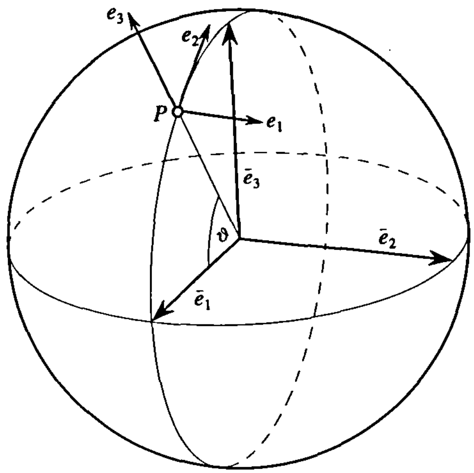

Figure 8. The coordinate system (e_1, e_2, e_3).

$$x_e = T_{e\bar{e}} x_{\bar{e}} = \begin{pmatrix} r(t) \sin(s + \mu(t)) \\ r(t) \cos(s + \mu(t)) \sin \vartheta + h(t) \cos \vartheta \\ -r(t) \cos(s + \mu(t)) \cos \vartheta + h(t) \sin \vartheta \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{24}$$

This is the position of the sun at local mean time s at season t. Now there remains to give the connection of the local mean time s to the standard time S. Let us consider the example of middle Europe. The Middle European Time MET, valid essentially for the whole of Europe (and parts of Africa), is defined as follows:

$$\text{MET} := \text{local mean time on the 15th degree of latitude} =: S \tag{25}$$

Of course, S is measured in hours and $S = 12$ at 'high' noon. Thus we have

$$s = \frac{\pi}{12}\left(S + \frac{\varphi - 15}{15}\right) \tag{26}$$

In the previous calculation we have neglected (among others)

- leap years,
- precession (difference between tropical and sidereal year),
- atmospheric refraction of light,
- daylight-saving time,
- ellipsoidal shape of the earth.

Once we have formulas (24) and (26) the creative part in the construction of a sundial starts.

3. Design of a sundial

Most sundials have a fixed position at walls or in gardens. Portable sundials are usually equipped with a compass in order to adjust them to the north direction. The most simple type of sundials is based on the mean sun – so they differ from the standard time by $\mu(t)$ as we have discussed (see figure 6). More sophisticated

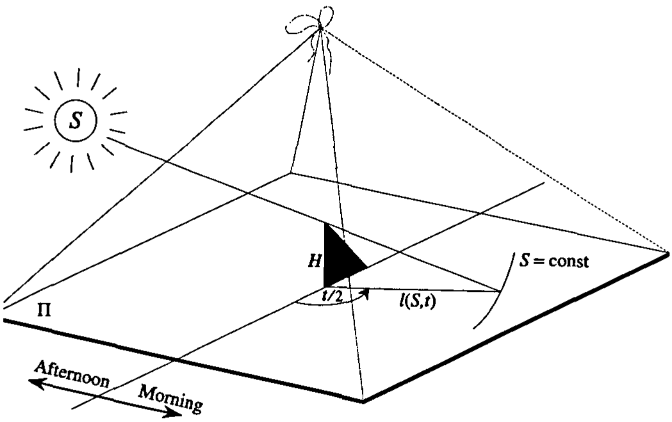

Figure 9. The principle of the sundial.

designs try to correct this effect by using the fact that the height of the culmination point of the sun is also a function of t: either the dial-plate is equipped with curves of the form shown in figure 7 or the object which casts the shadow is formed in a way which effects a correction of the deviation.

Here we use another idea. We wish to have a portable sundial which does not need to be oriented by a compass. Notice that for each day of the year (i.e. for fixed t) the following is true: if we know whether the sun already culminated or not then the time s (and hence S) is determined via equation (24) by the e_3 component of x *alone*. The question is now how to use this fact to design a sundial. We suggest the following solution: consider the horizontal plane Π with a perpendicular pointer of height H in the origin (see figure 9). We will use one half-plane for the morning, one for the afternoon. For each $t \in [0, 2\pi]$ and each S we have for the length $l(S, t)$ of the shadow

$$\frac{l(S, t)}{H} = \left(\frac{1}{x_3^2(S, t)} - 1\right)^{1/2} \tag{27}$$

Then, in each half-plane we draw curves $l(S, t)$ for $S = \text{const}$ (e.g. for the full and the half hours). The result is shown in figure 10 which is produced by an implementation of the derived formulas using a Mathematica-program.

How to measure time with our portable sundial now?

(1) Decide whether it is morning or afternoon (this might be a problem around noon).
(2) Turn the sundial in horizontal position until the shadow lies in the direction of the season (e.g. beginning of July).
(3) Read the time from the curve which is touched by the tip of the shadow.

In (1) and (2) some interpolation process might be needed. If you live in a country where time is changed during the summer months, do not forget to compensate for that.

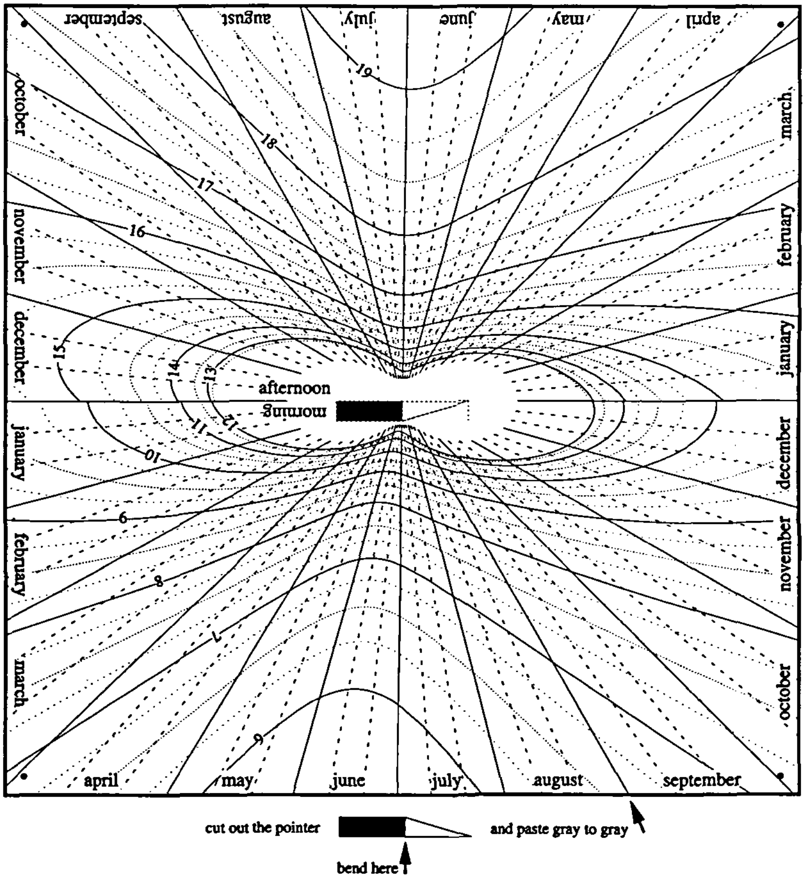

Figure 10. Plate and pointer of the sundial for $\varphi = 8.5°$, $\vartheta = 47.4°$ (Zürich). The dotted lines indicate the half-hours and the quarter-hours, respectively, the dashed lines indicate the 10th and the 20th day in each month.

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