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An integral that counts the zeros of a function

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Abstract: Given a real function f on an interval $[a, b]$ satisfying mild regularity conditions, we determine the number of zeros of f by evaluating a certain integral. The integrand depends on f, f' and f''. In particular, by approximating the integral with the trapezoidal rule on a fine enough grid, we can compute the number of zeros of f by evaluating finitely many values of f, f' and f''. A variant of the integral even allows to determine the number of the zeros broken down by their multiplicity.

Keywords: number of zeros on an interval, multiplicity of zeros

MSC: 30C15

1 Introduction

Counting the zeros of a given function f in a certain region belongs to the basic tasks in analysis. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, the Argument Principle and Rouché's Theorem are tools which allow to find the number of zeros of f, counted with multiplicity, in a bounded domain of $\mathbb{C}$ with sufficiently regular boundary (see, e.g. [4] for an overview of methods used for analytic functions). Descartes' Sign Rule is a method of determining the maximum number of positive and negative real roots (counted with multiplicity) of a polynomial. The Fourier-Budan Theorem yields the maximum number of roots (counted with multiplicity) of a polynomial in an interval. Sturm's Theorem, a refinement of Descartes' Sign Rule and the Fourier-Budan Theorem, allows to count the exact number of distinct roots of a polynomial on a real interval (see, e.g., [5], [2], [8]). The mentioned methods are restricted to holomorphic functions and polynomials, respectively. On the other end of the regularity spectrum, for a merely continuous function f, the Theorem of Bolzano yields the information that at least one zero exists on an interval $[a, b]$ if f has opposite signs at its endpoints, though, it does not count the zeros. Here, we want to construct a method which gives the number of zeros of a real function under only mild regularity assumptions. More precisely, we want to express the number of zeros of a function f by a certain integral (and boundary terms). The integrand depends on f, f' and f''. If f is sufficiently regular, the integral (and hence the number of zeros of f) can be expressed by evaluating the integrand on a sufficiently fine partition of $[a, b]$. Modifications of the integral even allow to determine the number of the zeros broken down by their multiplicity.

To explain the basic idea, we consider the following elementary connection between the number of zeros of a periodic function and the winding number of the related kinematic curve in the state space with respect to the origin:

Lemma 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic C^2 function with only simple zeros, i.e. points x with $f(x) = 0 \neq f'(x)$. Then, the number n of zeros of f in $[0, 2\pi)$ equals twice the winding number of the curve $\gamma : [0, 2\pi) \to \mathbb{R}^2$, $x \mapsto$*

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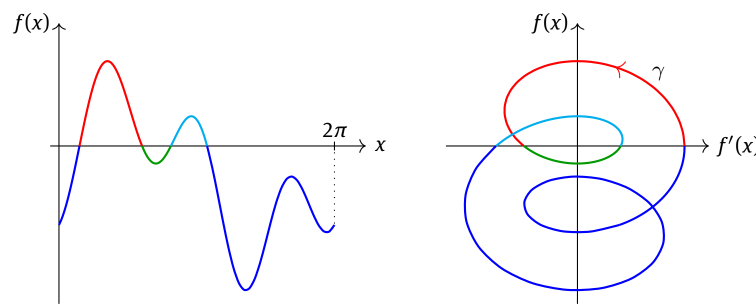
$(f'(x), f(x))$ with respect to the origin. Hence

$$n = \frac{1}{\pi} \int_0^{2\pi} \frac{f'(x)^2 - f(x)f''(x)}{f(x)^2 + f'(x)^2}\, \mathrm{d}x.$$

Figure 1 illustrates a heuristic proof without words: Each colored arc between two zeros of f adds $\frac{1}{2}$ to the winding number of γ. In the sequel, we will rigorously prove much more general versions and variants of this result. We will develop integrals that count the number of zeros with and without multiplicity, and we will even be able to determine the number of zeros of a given multiplicity. As a byproduct, a coherent definition of a fractional multiplicity of zeros will be possible. To start with, it is necessary to analyze the nature of zeros of a function.

Figure 1: Number of zeros of f vs. winding number of (f', f).



2 Zeros of Functions

A function $f : (a, b) \to \mathbb{R}$ may, in general, show a quite pathological behavior in the neighborhood of one of its zeros (see, e.g., Examples 2.2.3 and 2.9 below). To exclude such exotic cases but still be sufficiently general to cover most of the relevant cases, we use the following definition.

Definition 2.1. A zero $x_0 \in (a, b)$ of a function $f \in C^0(a, b) \cap C^1((a, b) \setminus \{x_0\})$ will be called *admissible* provided

$$\lim_{x \nearrow x_0} \frac{f'(x)}{f(x)} = -\infty \text{ and } \lim_{x \searrow x_0} \frac{f'(x)}{f(x)} = \infty. \tag{2.1}$$

If f extends continuously to a (or b) and $f(a) = 0$ (or $f(b) = 0$), we will say that f has an *admissible zero* in a (or b) if

$$\lim_{x \searrow a} \frac{f'(x)}{f(x)} = \infty \left(\text{or } \lim_{x \nearrow b} \frac{f'(x)}{f(x)} = -\infty \right).$$

Remarks.

1. An admissible zero is necessarily an isolated zero. In fact, if the zero x_0 is an accumulation point of zeros of f then, by Rolle's Theorem, it is also an accumulation point of zeros of f' and the limits in Definition 2.1 cannot be plus or minus infinity.
2. The condition on the limits given in (2.1) is in fact equivalent to

$$\lim_{x \to x_0} \left| \frac{\mathrm{d}}{\mathrm{d}x} \ln |f(x)| \right| = \infty. \tag{2.2}$$

Indeed, if (2.2) holds true, it follows that x_0 is an isolated zero of f, hence f does not change its sign on $(x_0, x_0+\varepsilon)$ and on $(x_0-\varepsilon, x_0)$ for $\varepsilon > 0$ small enough. Moreover $0 < |f(x)| < |f'(x)|$ on a punctured neighborhood of x_0. Hence, f' cannot change sign and the claim follows by distinction of cases. The condition (2.2) is slightly more compact than (2.1), however, (2.1) is easier to handle in the calculations below.

3. A simple zero $x_0 \in (a,b)$ of $f \in C^1(a,b)$, i.e. $f(x_0) = 0$ and $f'(x_0) \neq 0$ is admissible. It suffices to consider $x_0 = 0$:
$$\lim_{x\searrow 0}\frac{f'(x)}{f(x)} = \lim_{x\searrow 0}\frac{f'(0)+o(1)}{f(0)+xf'(0)+o(x)} = \lim_{x\searrow 0}\frac{1}{x}\cdot\frac{f'(0)+o(1)}{f'(0)+o(1)} = \infty.$$
The limit $x \nearrow 0$ is analogous.
4. If $f(x_0) = f'(x_0) = 0$ and f' is monotone on $(x_0, x_0+\epsilon)$ and on $(x_0-\epsilon, x_0)$ for some $\epsilon > 0$, then x_0 is an admissible zero: Indeed, for $x_0 < x < x_0+\epsilon$ and f' non-decreasing (if f' is non-increasing consider $-f$) on $(x_0, x_0+\epsilon)$, we have $f(x) = \int_{x_0}^{x} f'(t)\,\mathrm{d}t \le (x-x_0)f'(x)$ and thus $\frac{f'(x)}{f(x)} \ge \frac{1}{x-x_0} \to \infty$ for $x \searrow x_0$. The argument for the limit $x \nearrow x_0$ is analogous.
5. If $f \in C^k(a,b)$ and $x_0 \in (a,b)$ is a zero of multiplicity $k > 1$, i.e. $f^{(\ell)}(x_0) = 0$ for all $\ell = 0,\dots,k-1$ and $f^{(k)}(x_0) \neq 0$, then x_0 is admissible. This follows easily by an iterated application of L'Hôpital's rule. Hence the zeros of real-analytic functions and a fortiori zeros of polynomials are admissible.
6. If $f(x) = |x-x_0|^\alpha g(x)$ for a C^1-function g with $g(x_0) \neq 0$ and $0 < \alpha \in \mathbb{R}$, then x_0 is an admissible zero of f.
7. Every $f \in C^1([a,b])$ can be extended to $\tilde{f} \in C^1(I)$, where $I \supset [a,b]$ is an open interval and the limits
$$\lim_{x\nearrow a}\frac{f'(x)}{f(x)} \text{ and } \lim_{x\searrow b}\frac{f'(x)}{f(x)} \tag{2.3}$$
can be defined via $\tilde{f}$, provided $f(a), f(b) \neq 0$. If f has an admissible zero in a (or b), f can be extended antisymmetrically with respect to a (or b) to an extension $\tilde{f}$ for which a (or b) is an admissible zero. We will henceforth use this particular extension when computing limits like in (2.3).

Example 2.2. 1. The function $f_1 \in C^0(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$, $x \mapsto \sqrt{|x|}$ has an admissible zero in $x = 0$ (see Remark 6 above).
2. The C^∞-function
$$f_2(x) := \begin{cases} \exp\left(-\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0, \end{cases}$$
has an admissible zero of infinite multiplicity at $x = 0$ (see Remark 4 above).
3. An example of an isolated zero which is *not* admissible is given by the C^∞-function
$$f_3(x) := f_2(x)\left(\sin\left(\frac{1}{x^3}\right) + 2\right),$$
which vanishes (together with all derivatives) in 0 but the corresponding limits (2.1) do not exist.

Definition 2.3. A function $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{A}^k([a,b])$, $k \in \mathbb{N}$, if the following holds:

1. $f \in C^0([a,b])$.
2. f has only admissible (and therefore finitely many) zeros $x_1 < \dots < x_n$ and $f|_{(x_i,x_{i+1})}$ $(i = 1,\dots,n-1)$, $f|_{(a,x_1)}$ and $f|_{(x_n,b)}$ are of class C^{k+1}.
3. There exists a partition $a = y_1 < y_2 < \dots < y_m = b$ such that $f|_{(y_i,y_{i+1})}$ is of class C^{k+2} for all $i = 1,\dots,m-1$.

If $f \in \mathcal{A}^0([a,b])$, f will be called *admissible*.

Remarks.

1. Observe that $\mathcal{A}^{k+1}([a,b]) \subset \mathcal{A}^k([a,b])$ for all $k \in \mathbb{N}$ by construction.
2. Every analytic function is in $\mathcal{A}^\infty([a,b])$.

3. $f : [-1, 1] \to \mathbb{R}, x \mapsto \sqrt{|x|}$ is in $\mathcal{A}^\infty([a, b])$.
4. If f is admissible, then $x \mapsto (f'(x), f(x))$ is not necessarily a continuous curve.

As a building block of the intended results we need the following: For $\sigma \in [-\infty, \infty]$, let

$$\mathrm{H}(x) = \int_\sigma^x h(t)\,\mathrm{d}t, \tag{2.4}$$

where $h : \mathbb{R} \to \mathbb{R}$ is any piecewise continuous function such that the improper integral $\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = 1$. Then we have the following theorem (recall (2.3) in order to make sense of the limits that appear).

Theorem 2.4. *Let $f \in \mathcal{A}^0([a, b])$. The number of zeros $n(f)$ of f in $[a, b]$ is given by*

$$n(f) = \int_a^b h\left(\frac{f'(x)}{f(x)}\right)\frac{f'(x)^2 - f(x)f''(x)}{f(x)^2}\,\mathrm{d}x + \lim_{x\searrow b}\mathrm{H}\left(\frac{f'(x)}{f(x)}\right) - \lim_{x\nearrow a}\mathrm{H}\left(\frac{f'(x)}{f(x)}\right)$$

and the number of zeros $\mathring{n}(f)$ of f in (a, b) by

$$\mathring{n}(f) = \int_a^b h\left(\frac{f'(x)}{f(x)}\right)\frac{f'(x)^2 - f(x)f''(x)}{f(x)^2}\,\mathrm{d}x + \lim_{x\nearrow b}\mathrm{H}\left(\frac{f'(x)}{f(x)}\right) - \lim_{x\searrow a}\mathrm{H}\left(\frac{f'(x)}{f(x)}\right).$$

Proof. Consider first the case, where $f(a), f(b) \neq 0$. Then the zeros of f are given by $a < x_1 < x_2 < \ldots < x_{n(f)} < b$. The integrand of

$$\int_a^b h\left(\frac{f'(x)}{f(x)}\right)\frac{f'(x)^2 - f(x)f''(x)}{f(x)^2}\,\mathrm{d}x =: \int_a^b \mathrm{I}(x)\,\mathrm{d}x$$

is a priori undefined whenever f vanishes or whenever f'' is undefined. We decompose the integral and compute the resulting improper integrals using unilateral limits. Since f is admissible, we have

$$\int_{x_j}^{x_{j+1}} \mathrm{I}(x)\,\mathrm{d}x = \lim_{x\searrow x_j}\mathrm{H}_x - \lim_{x\nearrow x_{j+1}}\mathrm{H}_x = 1$$

for all $j = 1, \ldots, n(f) - 1$, where $\mathrm{H}_x := \mathrm{H}(f'(x)/f(x))$. Integrating over a neighborhood of a point y where f'' is undefined does not introduce further boundary terms since $\lim_{x\searrow y}\mathrm{H}_x - \lim_{x\nearrow y}\mathrm{H}_x = 0$. Hence

$$\begin{aligned}\int_a^b \mathrm{I}(x)\,\mathrm{d}x &= \int_a^{x_1} \mathrm{I}(x)\,\mathrm{d}x + \sum_{j=1}^{n(f)-1}\int_{x_j}^{x_{j+1}} \mathrm{I}(x)\,\mathrm{d}x + \int_{x_{n(f)}}^b \mathrm{I}(x)\,\mathrm{d}x = \\ &= \mathrm{H}_a - \lim_{x\nearrow x_1}\mathrm{H}_x + (n(f) - 1) + \lim_{x\searrow x_{n(f)}}\mathrm{H}_x - \mathrm{H}_b\end{aligned} \tag{2.5}$$

and therefore

$$n(f) = \int_a^b \mathrm{I}(x)\,\mathrm{d}x + \mathrm{H}_b - \mathrm{H}_a. \tag{2.6}$$

The computation above suggests that $n(f) > 1$ but one can check that formula (2.6) holds true for $n(f) = 1$ and $n(f) = 0$ as well.

If f has zeros in a and b and therefore $x_1 = a$, $x_{n(f)} = b$, computation (2.5) gives

$$n(f) = \int_a^b \mathrm{I}(x)\,\mathrm{d}x + 1. \tag{2.7}$$

According to (2.3), $\lim_{x\searrow b} \mathrm{H}_x - \lim_{x\nearrow a} \mathrm{H}_x = 1$ and (2.7) becomes

$$n(f) = \int_a^b \mathrm{I}(x)\,\mathrm{d}x + \lim_{x\searrow b} \mathrm{H}_x - \lim_{x\nearrow a} \mathrm{H}_x \tag{2.8}$$

and hence (2.8) counts the zeros of f in $[a, b]$ since it reduces to (2.6) if $f(a), f(b) \neq 0$ and one can check that the remaining cases $f(a) = 0 \neq f(b)$ and $f(a) \neq 0 = f(b)$ are also covered. Let now

$$\mathring{n}(f) = \int_a^b \mathrm{I}(x)\,\mathrm{d}x + \lim_{x\nearrow b} \mathrm{H}_x - \lim_{x\searrow a} \mathrm{H}_x.$$

Since

$$\begin{aligned} n(f) - \mathring{n}(f) &= \lim_{x\searrow b} \mathrm{H}_x - \lim_{x\nearrow a} \mathrm{H}_x - \left(\lim_{x\nearrow b} \mathrm{H}_x - \lim_{x\searrow a} \mathrm{H}_x\right) = \\ &= \begin{cases} 0, & \text{if } f(a), f(b) \neq 0 \\ 1, & \text{if either } f(a) = 0 \text{ or } f(b) = 0 \\ 2, & \text{if } f(a) = f(b) = 0 \end{cases} \end{aligned}$$

we conclude that $\mathring{n}(f)$ counts the zeros of f in (a, b). □

Remarks.

1. Putting $g(x) := f'(x)/f(x)$, the integrand in Theorem 2.4 reads $-(h \circ g)(x)g'(x)$. With respect to the signed Borel-Lebesgue-Stieltjes-Measure $\mathrm{d}g(x) := g'(x)\,\mathrm{d}x$ (see [9]), the integral can be written more compactly as

$$-\int_a^b h(g)\,\mathrm{d}g.$$

2. If $h(x) := 1/(\pi(1 + x^2))$, i.e. h equals the *Cauchy Density* and f is an admissible 2π-periodic function, then the number n of zeros of f in $[0, 2\pi)$ equals

$$\begin{aligned} n &= \frac{1}{\pi}\left[\int_0^{2\pi} \frac{f'(x)^2 - f(x)f''(x)}{f(x)^2 + f'(x)^2}\,\mathrm{d}x + \lim_{x\nearrow 2\pi} \arctan\left(\frac{f'(x)}{f(x)}\right) - \lim_{x\nearrow 0} \arctan\left(\frac{f'(x)}{f(x)}\right)\right] = \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{f'(x)^2 - f(x)f''(x)}{f(x)^2 + f'(x)^2}\,\mathrm{d}x, \end{aligned} \tag{2.9}$$

since the integral-free terms cancel out in this case. In this way we obtain Lemma 1.1 as a corollary of Theorem 2.4. Observe that a 2π-periodic C^2 function with an odd number of zeros on $[0, 2\pi)$ gives rise to a curve $x \mapsto (f'(x), f(x))$ having a half-integer valued winding number. This idea, further developed, leads to a generalized version of the Residue Theorem (see [3]).

Observe, that for a C^2 function f with only zeros of multiplicity one, the integrand in (2.9) is continuous provided h is continuous. This remains true for zeros of higher multiplicity in the following way:

Proposition 2.5. *Let $h : \mathbb{R} \to \mathbb{R}$ be continuous and $h(x) \sim \frac{C}{x^2}$ for $|x| \to \infty$. Then, the integrand in Theorem 2.4*

$$\mathrm{I} := h\left(\frac{f'}{f}\right)\frac{f'^2 - ff''}{f^2}$$

is continuous if $f \in C^n([a, b])$, $n \geq 2$, has only zeros of multiplicity $\leq n$.

Proof. It suffices to show that I is continuous in 0 if $x = 0$ is a zero of f of multiplicity n. Then, by Taylor expansion, we have

$$f(x) = \left(\frac{f^{(n)}(0)}{n!} + r_0(x)\right) x^n$$
$$f'(x) = \left(\frac{f^{(n)}(0)}{(n-1)!} + r_1(x)\right) x^{n-1}$$
$$f''(x) = \left(\frac{f^{(n)}(0)}{(n-2)!} + r_2(x)\right) x^{n-2}$$

where r_i are continuous functions with $\lim_{x\to 0} r_i(x) = 0$. Using these expressions in I, we get

$$\mathrm{I}(x) = h\left(\frac{s_1(x)}{x}\right)\frac{s_2(x)}{x^2}$$

for continuous functions s_i with $\lim_{x\to 0} s_i(x) = n$. Thus

$$\mathrm{I}(x) \sim \frac{Cx^2}{s_1^2(x)}\frac{s_2(x)}{x^2} \to \frac{C}{n}$$

for $x \to 0$. □

If we only assume that $h(x) = O(1/x^2)$ for $|x| \to \infty$ in the previous proposition, the proof shows that then I is at least bounded.

As a corollary of Proposition 2.5 we obtain that if h is continuous and $h(x) \sim \frac{C}{x^2}$, then I is in C^0 provided f is analytic. Nontheless, the function f may behave in the neighborhood of a zero in such a pathological way, that I becomes unbounded (see Example 2.7.3). This is why, in general, the integrals in Theorem 2.4 have to be interpreted as improper integrals. This means that the concrete computation requires the zeros of f to be known a priori in order to evaluate the improper integrals. It is therefore of practical importance to formulate conditions (see Propositions 2.8 and 2.10) with additional assumptions which guarantee that I is in L^1: To this end we will slightly sharpen the admissibility condition for a function and impose some conditions on the behaviour of the zeros of f'' in neighborhoods of the zeros of f. Furthermore we will require h to have at least quadratic decay at infinity.

The proof of Proposition 2.5 for the case $C = 1$ indicates, how we can generalize the notion of multiplicity of zeros in a natural manner:

Definition 2.6. The multiplicity $\mu_f(x_0)$ of a zero x_0 of $f \in \mathcal{A}^0$ is defined to be

$$\mu_f(x_0) = \lim_{x\to x_0}\frac{f'(x)^2}{f'(x)^2 - f(x)f''(x)}.$$

Since the zeros of functions in $\mathcal{A}^0$ are admissible, it follows that $\mu_f(x_0) \geq 0$ whenever it exists, however, it can take values in $[0, \infty]$ (see Example 2.7.3 and 2.7.4 below). This definition of the multiplicity of a zero will be useful for a variant of Theorem 2.4 that takes the multiplicities of the zeros into account.

Example 2.7.
1. A function $f \in C^n$, $n \geq 2$ with $0 = f(x_0) = f'(x_0) = \ldots = f^{(n-1)}(x_0) \neq f^{(n)}(x_0)$ has a zero of multiplicity n in x_0: the Definition 2.6 is compatible with the usual notion of multiplicity.
2. The function $f(x) = |x|^r,\ r > 0$ has a zero of multiplicity r in $x = 0$.
3. The function
$$f(x) = \begin{cases}\dfrac{1}{\ln|x|}, & x \neq 0\\ 0, & x = 0\end{cases}$$
has a zero of multiplicity 0 in $x = 0$.
4. The function f_2 in Example 2.2.2 has a zero in $x = 0$ with $\mu_{f_2}(0) = \infty$.

Proposition 2.8. *Let $h : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function such that $h(x) = O(1/x^2)$ for $|x| \to \infty$ and let $f \in \mathcal{A}^0([a,b]) \cap W^{2,1}(a,b)$ have only zeros of positive multiplicity in the sense of Definition 2.6. Furthermore we assume that for each zero x_0 we have a neighborhood U such that either $f''(x) \equiv 0$ on $U \setminus \{x_0\}$ or*

$$\sum_{k=1}^{\infty} |z_k - x_0| < \infty,$$

where $z_1, z_2, \dots$ denote the countably many zeros of f'' in U. Then

$$\mathrm{I} := h\left(\frac{f'}{f}\right)\frac{f'^2 - ff''}{f^2} \in L^1(a,b).$$

Proof. Choose neighborhoods $U_1, \dots, U_n$ of the n zeros of f, which do not (with the possible exception of the respective zero itself) contain singular points of f'' or zeros of f' and let

$$\mathrm{U} = \bigcup_{i=1}^{n} U_i.$$

Since $|f| \geq \eta$ for some $\eta > 0$ on the complement U^c and $W^{2,1}(a,b) \hookrightarrow C^1([a,b])$ we can estimate

$$\int_{\mathrm{U}^c} |\mathrm{I}(x)|\, \mathrm{d}x = \eta^{-2}\|h\|_{L^\infty(\mathbb{R})}\left(\|f'^2\|_{C^0([a,b])}|b-a| + \|f\|_{C^0([a,b])}\|f''\|_{L^1(a,b)}\right) < \infty.$$

Consider now wlog the neighborhood U_i of the zero $x_i = 0$ and assume $U_i = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. We need to show that $\mathrm{I}|_{(-\varepsilon,\varepsilon)} \in L^1$. Since $h(x) = O(1/x^2)$ for $|x| \to \infty$, there exists a constant $C > 0$ such that

$$|\mathrm{I}(x)| \leq C\left(1 + \left|\frac{f(x)f''(x)}{f'(x)^2}\right|\right). \tag{2.10}$$

Note that $ff''/f'^2 \in L^1(-\varepsilon, \varepsilon)$ if and only if $\mathrm{N} \in \mathrm{BV}(-\varepsilon, \varepsilon)$, where $\mathrm{N}(x) = x - f(x)/f'(x)$ denotes the *Newton-Operator* of f and $\mathrm{BV}(-\varepsilon, \varepsilon)$ denotes the space of functions $g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ of bounded variation. It follows from the admissibility of the zero that $\mathrm{N} : (-\varepsilon, \varepsilon) \setminus \{0\} \to \mathbb{R}$ can be continuously extended to $\mathrm{N}(0) = 0$ and it holds that

$$\mathrm{N}'(x) = \frac{f(x)f''(x)}{f'(x)^2},$$

for $x \neq 0$. Let $\mu > 0$ denote the multiplicity of the zero according to Definition 2.6. It holds that

$$\lim_{x \to 0} \mathrm{N}'(x) = \begin{cases} \dfrac{\mu - 1}{\mu}, & \mu < \infty \\ 1, & \mu = \infty. \end{cases}$$

According to the mean value theorem we have $\mathrm{N}(x)/x = \mathrm{N}'(\xi)$ for some ξ between 0 and x and deduce that $\mathrm{N} \in C^1(-\varepsilon, \varepsilon)$. The Taylor expansion of N around $x = 0$ is given by

$$\mathrm{N}(x) = \begin{cases} \dfrac{\mu - 1}{\mu} x + o(x), & \mu < \infty \\ x + o(x), & \mu = \infty. \end{cases}$$

In any case there exists a constant $K > 0$ such that

$$|\mathrm{N}(x)| \leq K|x|, \quad |x| < \varepsilon. \tag{2.11}$$

We will now show that $\mathrm{N} \in \mathrm{BV}([0, \varepsilon))$, the argument on $(-\varepsilon, 0]$ being similar. We start by noticing that N is absolutely continuous on $[\delta, \varepsilon)$ for every $0 < \delta < \varepsilon$ since x, $f(x)$ and $f'(x)$ are absolutely continuous and $f'(x) \neq 0$ on $[\delta, \varepsilon)$. In particular, $\mathrm{N} \in \mathrm{BV}([\delta, \varepsilon))$ for every $0 < \delta < \varepsilon$.

We will now distinguish two cases: If $f'' \equiv 0$ on $(0, \varepsilon)$, then $\mathrm{N} \equiv 0$ and we are done. In the remaining case we first consider the case when the set of zeros of f'' in $(0, \varepsilon)$ is empty: Then N is monotone on $[0, \varepsilon)$ and hence $\mathrm{N} \in \mathrm{BV}([0, \varepsilon))$. Otherwise the zeros of f'' in $[0, \varepsilon)$ are given by $z_1 > z_2 > \dots$ and we may set $\delta := z_1$. According to (2.11) and since the zeros of f'' are precisely the zeros of N' we can estimate the total variation of N on (z_{k+1}, z_k) by

$$\int_{z_{k+1}}^{z_k} |\mathrm{N}'(x)|\,\mathrm{d}x \leq 2Kz_k.$$

The total variation of N on $[0, \varepsilon)$ is bounded by

$$\sum_{k=1}^{\infty} \int_{z_{k+1}}^{z_k} |\mathrm{N}'(x)|\,\mathrm{d}x + \int_{\delta}^{\varepsilon} |\mathrm{N}'(x)|\,\mathrm{d}x \leq 2K \sum_{k=1}^{\infty} z_k + \int_{\delta}^{\varepsilon} |\mathrm{N}'(x)|\,\mathrm{d}x,$$

where the series converges by assumption and the integral is finite since $\mathrm{N} \in \mathrm{BV}([\delta, \varepsilon))$. We conclude that $\mathrm{N} \in \mathrm{BV}([0, \varepsilon))$, which finishes the proof. □

Remark. The key estimate (2.11) in the proof above follows from the admissibility and the positive multiplicity of the zeros. We will however formulate a variant of Proposition 2.8 below (Proposition 2.10), which covers admissible functions that have zeros of ill-defined multiplicity for which (2.11) still holds true: Take e.g. the C^1 function $f : x \mapsto x^3 \left(\sin(1/x) + 2\right) + x$ which has an admissible zero in $x = 0$, but for which $\mu_f(0)$ does not exist, however, (2.11) holds true since $f(x)/(xf'(x))$ is bounded near 0 – in fact

$$\lim_{x \to 0} \frac{f(x)}{xf'(x)} = 1.$$

Example 2.7.3 shows an admissible function for which (2.11) does not hold true. In the mentioned example, the first derivative is unbounded. But even functions with higher regularity may behave in such a pathological way near an admissible zero, that (2.11) does not hold true, as the following example shows:

Example 2.9. Let

$$k(x) = \begin{cases} x^3 + \left(\sqrt{|x|^7} - x^3\right) \cos\left(\pi \log_2 |x|\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then $f(x) = \int_0^x k(t)\,\mathrm{d}t$ is of class C^3 and has an admissible zero in $x = 0$ but $f(x)/(xf'(x))$ is unbounded near 0.

Proposition 2.10. *Let $h : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function such that $h(x) = O(1/x^2)$ for $|x| \to \infty$ and let $f \in \mathcal{A}^0([a, b]) \cap W^{2,1}(a, b)$ be such that that for every zero x_0 of f there exists a relatively open neighborhood $U \subset [a, b]$ such that*

$$0 < \left| \frac{f(x)}{(x - x_0)f'(x)} \right| < \widetilde{K} \tag{2.12}$$

on $U \setminus \{x_0\}$ and such that either $f'' \equiv 0$ on $U \setminus \{x_0\}$, or

$$\sum_{k=1}^{\infty} |z_k - x_0| < \infty,$$

where $z_1, z_2, \dots$ denote the countably many zeros of f'' in $U \setminus \{x_0\}$. Then

$$\mathrm{I} := h\left(\frac{f'}{f}\right) \frac{f'^2 - ff''}{f^2} \in L^1(a, b).$$

Proof. Choose neighborhoods $U_1, \dots, U_n$ of the n zeros of f, which do not (with the possible exception of the respective zero itself) contain singular points of f'' or zeros of f' such that (2.12) holds on each punctured neighborhood. As in the proof of Proposition 2.8 we obtain $\|\mathrm{I}\|_{L^1(\mathrm{U}^c)} < \infty$, where $\mathrm{U} = U_1 \cup \dots \cup U_n$ and the estimate (2.10). Let wlog 0 be a zero of f and let $(-\varepsilon, \varepsilon)$ be its respective neighborhood for some $\varepsilon > 0$. As in the proof of Proposition 2.8, we are done if we show that $\mathrm{N} \in \mathrm{BV}([0,\varepsilon))$. The condition $0 < |f(x)/(xf'(x))| < \widetilde{K}$ on $(-\varepsilon,\varepsilon)\setminus\{0\}$ implies that

$$0 < \left|\frac{f(x)}{f'(x)}\right| < \widetilde{K}|x|, \tag{2.13}$$

from which we conclude that N extends continuously to $[0,\varepsilon)$ (where $\mathrm{N}(0) = 0$) and

$$|\mathrm{N}(x)| \le (\widetilde{K}+1)\,x, \quad x \in [0,\varepsilon). \tag{2.14}$$

This is just estimate (2.11) with $K = \widetilde{K} + 1$. The rest of the proof is exactly the same as the one of Proposition 2.8. □

3 Counting Zeros with Multiplicities

Let again $h : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function such that $\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = 1$ and define H as before in (2.4). Moreover, let

$$\begin{aligned}
\mathrm{I}_g(x) &= h\left(\frac{f'(x)}{f(x)}\right) g(x)\frac{f'(x)^2 - f(x)f''(x)}{f(x)^2} - \mathrm{H}\left(\frac{f'(x)}{f(x)}\right) g'(x),\\
g_1(x) &= \frac{f'(x)^2}{f'(x)^2 - f(x)f''(x) + cf(x)^2},\\
g_2(x) &= \exp\left(\frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2 + f(x)^2}\right),
\end{aligned}$$

where $c \in \mathbb{R}$. Note that if x_0 is a zero of multiplicity $\mu_f(x_0)$, then $g_1(x) \to \mu_f(x_0)$ as $x \to x_0$ for every value c in the definition of g_1 and if $\mu_f(x_0) > 0$, then $g_2(x) \to \exp\left(\frac{1}{\mu_f(x_0)}\right)$ as $x \to x_0$.

Lemma 3.1. *Let all the zeros of $f \in \mathcal{A}^0([a,b]) \cap C^2([a,b])$ have well-defined multiplicities. Then there exists $c \in \mathbb{R}$ such that g_1 has no poles.*

Proof. If x_0 is a zero of f, we have that $g_1(x) \to \mu_f(x_0)$ as $x \to x_0$. In other words g_1 extends continuously to the zeros of f. Hence there are open neighborhoods of the zeros of f, where g_1 has no poles. On the complement of these neighborhoods, there exists a number $\delta > 0$ such that $|f(x)| \ge \delta$. Hence $f'(x)^2 + cf(x)^2 \ge f'(x)^2 + c\delta^2$. If we choose $c > \delta^{-2}\|ff''\|_{C^0([a,b])}$, then g_1 has no poles. In particular, if f is analytic, this choice of c ensures that g_1 is analytic as well. □

We have the following theorem for analytic functions $f : [a,b] \to \mathbb{R}$:

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be an analytic function and choose c in the definition of g_1 such that g_1 is analytic. If $h(x) = O(1/x^2)$ for $|x| \to \infty$, then $\mathrm{I}_{g_1}, \mathrm{I}_{g_2} \in L^\infty(a,b)$ and if f has n_ℓ zeros of multiplicity ℓ in $[a,b]$ and $\mathring{n}_\ell$ zeros of mutliplicity ℓ in (a,b), then*

$$\int_a^b \mathrm{I}_{g_1}(x)\,\mathrm{d}x + \lim_{x\searrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] - \lim_{x\nearrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] = \sum_{\ell=1}^{\infty} n_\ell \ell,$$

$$\int_a^b \mathrm{I}_{g_1}(x)\,\mathrm{d}x + \lim_{x\nearrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] - \lim_{x\searrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] = \sum_{\ell=1}^{\infty} \mathring{n}_\ell \ell,$$

$$\int_a^b \mathrm{I}_{g_2}(x)\,\mathrm{d}x + \lim_{x\searrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right] - \lim_{x\nearrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right] = \sum_{\ell=1}^{\infty} n_\ell \exp\left(\tfrac{1}{\ell}\right),$$

$$\int_a^b \mathrm{I}_{g_2}(x)\,\mathrm{d}x + \lim_{x\nearrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right] - \lim_{x\searrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right] = \sum_{\ell=1}^{\infty} \mathring{n}_\ell \exp\left(\tfrac{1}{\ell}\right).$$

Proof. We first prove the L^∞-bounds: It suffices to show that I_{g_1} and I_{g_2} are bounded near the zeros of f. Let x_0 be a zero of multiplicity k and write (locally) $f(x) = (x - x_0)^k j(x)$, where j is analytic and $j(x_0) \neq 0$. Since

$$\lim_{x \to x_0} g_1'(x) = \frac{2j'(x_0)}{j(x_0)}$$

we find the limits

$$\lim_{x \searrow x_0} \mathrm{H}\left(\frac{f'(x)}{f(x)}\right) g_1'(x) = \frac{2j'(x_0)}{j(x_0)}$$
$$\lim_{x \nearrow x_0} \mathrm{H}\left(\frac{f'(x)}{f(x)}\right) g_1'(x) = 0.$$

If

$$h\left(\frac{f'(x)}{f(x)}\right) g(x) \frac{f'(x)^2 - f(x)f''(x)}{f(x)^2}$$

is bounded near x_0, the claim follows. Since $\left|h\left(f'(x)/f(x)\right)\right| \leq C \cdot f(x)^2/f'(x)^2$ and

$$\lim_{x \to x_0} C|g_1(x)| \frac{f'(x)^2 + |f(x)f''(x)|}{f'(x)^2} = C(2k-1),$$

we obtain $\mathrm{I}_{g_1} \in L^\infty(a, b)$. For I_{g_2}, observe that

$$\lim_{x \to x_0} g_2'(x) = -\frac{2\exp\left(\frac{1}{k}\right) j'(x_0)}{k^2 j(x_0)}$$

and therefore

$$\lim_{x \searrow x_0} \mathrm{H}\left(\frac{f'(x)}{f(x)}\right) g_2'(x) = -\frac{2\exp\left(\frac{1}{k}\right) j'(x_0)}{k^2 j(x_0)}$$
$$\lim_{x \nearrow x_0} \mathrm{H}\left(\frac{f'(x)}{f(x)}\right) g_2'(x) = 0.$$

Proceeding as for g_1 we find

$$\lim_{x \to x_0} C|g_2(x)| \frac{f'(x)^2 + |f(x)f''(x)|}{f'(x)^2} = C\exp\left(\frac{1}{k}\right) \frac{2k-1}{k}$$

and hence $\mathrm{I}_{g_2} \in L^\infty(a, b)$. The computation of the integrals is done as in the proof of Theorem 2.4. □

Remark. If $f \in \mathcal{A}^1([a, b]) \cap C^2([a, b])$ only has zeros of well-defined multiplicities and if the set of zeros of f in (a, b) is given by $\mathring{N}$ and the set of zeros of f in $[a, b]$ by N, then

$$\int_a^b \mathrm{I}_{g_1}(x)\,\mathrm{d}x + \lim_{x \searrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] - \lim_{x \nearrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] = \sum_{x \in N} \mu_f(x),$$
$$\int_a^b \mathrm{I}_{g_1}(x)\,\mathrm{d}x + \lim_{x \nearrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] - \lim_{x \searrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_1(x)\right] = \sum_{x \in \mathring{N}} \mu_f(x).$$

Lemma 3.3. *Let $\mathcal{N}$ be the set of sequences with natural entries of which only finitely many are non-zero. Then the map $\mathcal{F} : \mathcal{N} \to \mathbb{R}$ defined by $\mathcal{F}(k_1, \ldots) = \sum_{\ell=1}^\infty k_\ell \exp\left(\frac{1}{\ell}\right)$ is injective.*

Proof. The difference $\mathcal{F}(k_1, \ldots) - \mathcal{F}(k_1', \ldots)$ is equal to the finite sum

$$\sum_{\ell=1}^\infty (k_\ell - k_\ell') \exp\left(\tfrac{1}{\ell}\right).$$

If this sum vanishes, $k_\ell = k_\ell'$ for all ℓ by the von Lindemann-Weierstrass theorem (see [7, §3]). □

Corollary 3.4. *Let $f : [a, b] \to \mathbb{R}$ be analytic. If f has n_ℓ zeros of multiplicity ℓ in $[a, b]$ and $\mathring{n}_\ell$ zeros of mutliplicity ℓ in (a, b), then*

$$(n_1, \dots) = \mathcal{F}^{-1}\left(\int_a^b \mathrm{I}_{g_2}(x)\,\mathrm{d}x + \lim_{x\searrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right] - \lim_{x\nearrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right]\right)$$

$$(\mathring{n}_1, \dots) = \mathcal{F}^{-1}\left(\int_a^b \mathrm{I}_{g_2}(x)\,\mathrm{d}x + \lim_{x\nearrow b}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right] - \lim_{x\searrow a}\left[\mathrm{H}\left(\tfrac{f'(x)}{f(x)}\right) g_2(x)\right]\right).$$

Example 3.5. Let $f(x) = \cos(2x) + x^2 \sin(2x) - \frac{1}{2}\sqrt{\mathrm{e}^x} + \frac{x-2}{4}$. Using Theorem 2.4 and 3.2 on $[0, 2\pi]$ we obtain

$$\sum_{\ell=1}^{\infty} n_\ell = 3, \quad \sum_{\ell=1}^{\infty} \mathring{n}_\ell = 2, \quad \sum_{\ell=1}^{\infty} n_\ell \ell = 4, \quad \sum_{\ell=1}^{\infty} \mathring{n}_\ell \ell = 2.$$

and we conclude that f has two zeros in $(0, 2\pi)$ and a double zero on the boundary of $[0, 2\pi]$.

Example 3.6. Let $f(x) = x^7 - 2x^6 + x^5 - x^3 + 2x^2 - x$ have n_ℓ zeros of multiplicity ℓ on $\mathbb{R}$. By Theorem 2.4 and 3.2 on $\mathbb{R}$ (observe that the boundary terms of the integrals cancel out in this case) we find that

$$\sum_{\ell=1}^{\infty} n_\ell = 3 \text{ and } \sum_{\ell=1}^{\infty} n_\ell \ell = 5.$$

Hence $(n_1, \dots)$ either equals $(1, 2, 0, \dots)$ or $(2, 0, 1, \dots)$. In particular $n_\ell = 0$, for $\ell \geq 4$. Using again Theorem 3.2 we get

$$\sum_{i=1}^{3} n_i \exp\left(\tfrac{1}{i}\right) \approx 6.8322.$$

Since $1 \cdot \mathrm{e} + 2 \cdot \sqrt{\mathrm{e}} \approx 6.0157$ and $2 \cdot \mathrm{e} + 1 \cdot \sqrt[3]{\mathrm{e}} \approx 6.8322$ we conclude that f has two simple zeros and one of multiplicity 3.

4 Numerical Aspects

The number of zeros of a function f in a given interval $[a, b]$ is of course an integer. Therefore is suffices to compute the integral in Theorem 2.4 with an error $\varepsilon < \frac{1}{2}$. In particular, for the trapezoidal rule

$$T_N(\mathrm{I}) := \frac{b-a}{N}\left(\frac{\mathrm{I}(a) + \mathrm{I}(b)}{2} + \sum_{k=1}^{N-1} \mathrm{I}\left(a + k\tfrac{b-a}{N}\right)\right)$$

with $N + 1$ equidistant grid points, the error $\varepsilon(N)$ is estimated by

$$\varepsilon(N) = \left|\int_a^b \mathrm{I}(x)\,\mathrm{d}x - T_N(\mathrm{I})\right| \leq \frac{(b-a)^3}{12N^2}\|\mathrm{I}''\|_{L^\infty}$$

(see, e.g., [6] or [1]). Thus we have

Theorem 4.1. *Let f satisfy the assumptions in Theorem 2.4. If*

$$N > \sqrt{\frac{(b-a)^3}{6}\|\mathrm{I}''\|_{L^\infty}},$$

then one can replace the integral in Theorem 2.4 by the finite sum $T_N(\mathrm{I})$ and round the result to the closest integer to get the values $n(f)$ and $\mathring{n}(f)$, respectively.

This theorem is quite remarkable: It allows to compute the number of zeros of a function f on $[a, b]$ by evaluating finitely many values of f, f' and f''.

Example 4.2. Let $f : \mathbb{R} \to \mathbb{R},\ x \mapsto J_0(x)$, be the zeroth Bessel function of the first kind. If h is the Cauchy density, one can verify that $\|\mathrm{I}''\|_{L^\infty} < \frac{1}{\pi}$. We want to compute the number of zeros of J_0 on $[0, 2\pi]$ by Theorem 4.1. It suffices to employ the trapezoidal rule with only

$$N = \left\lceil \frac{2\pi}{\sqrt{3}} \right\rceil = 4$$

equidistant intervals. We find

$$T_4(\mathrm{I}) = \frac{\pi}{2}\left(\frac{\mathrm{I}(0) + \mathrm{I}(2\pi)}{2} + \sum_{k=1}^{3} \mathrm{I}\big(k\tfrac{\pi}{2}\big)\right) \approx 1.76479$$

and thus

$$T_4(\mathrm{I}) - \frac{1}{\pi}\arctan\left(\frac{J_1(2\pi)}{J_0(2\pi)}\right) \approx 1.76479 + 0.24419 = 2.00898$$

and hence, J_0 has two zeros on $[0, 2\pi]$.

If we compute the number of zeros of J_0 on $[0, 100\pi]$, we have to choose

$$N = \left\lceil \frac{500\sqrt{6}}{3} \cdot \pi \right\rceil = 1283.$$

(Actually, a finer analysis shows that a much smaller N suffices). In this case, we get

$$T_{1283}(\mathrm{I}) = \frac{100\pi}{1283}\left(\frac{\mathrm{I}(0) + \mathrm{I}(100\pi)}{2} + \sum_{k=1}^{1282} \mathrm{I}\big(k\tfrac{100\pi}{1283}\big)\right) \approx 99.75013$$

and

$$T_{1283}(\mathrm{I}) - \frac{1}{\pi}\arctan\left(\frac{J_1(100\pi)}{J_0(100\pi)}\right) \approx 99.75013 + 0.24987 = 100,$$

hence we conclude that J_0 has $n = 100$ zeros on $[0, 100\pi]$, in accordance with the well known distribution of zeros of J_0. Surprisingly, the routine CountRoots of Mathematica™ is giving up on this simple problem after giving it some thought.

From a practical point of view, it is desirable to keep $\|\mathrm{I}''\|_{L^\infty}$ (and hence N) as small as possible. This can be achieved in several ways: First of all, we have the freedom to choose the function h. Below there is a small table of possible choices of h and the resulting function H in Theorem 2.4 (in each case, the integrand I turns out rather nicely).

Moreover, with smooth functions γ and κ that satisfy $\operatorname{sign}\gamma(x) = \operatorname{sign}\kappa(x) = \operatorname{sign} x$ for all $x \neq 0$ and $\gamma(x) \sim C_1|x|^\alpha \operatorname{sgn} x$ and $\kappa(x) \sim C_2|x|^\beta \operatorname{sgn} x$ as $x \to 0$, where $0 < \alpha \le \beta$, one can modify the integrand I as follows and the proof of Theorem 2.4 still goes through:

$$\mathrm{I}(x) = h\left(\frac{\gamma(f'(x))}{\kappa(f(x))}\right)\left(\frac{\gamma(f'(x))f'(x)\kappa'(f(x)) - \gamma'(f'(x))f''(x)\kappa(f(x))}{\kappa(f(x))^2}\right).$$

In this case the boundary terms in a and b have to be taken with the function

$$\mathrm{H}\left(\frac{\gamma(f'(x))}{\kappa(f(x))}\right).$$

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$h(x)$	$\mathrm{H}(x)$
$\frac{1}{\pi(1+x^2)}$	$\frac{\arctan x}{\pi}$
$\frac{1}{2\left(x^2+1\right)^{3/2}}$	$\frac{x}{2\sqrt{x^2+1}}$
$\frac{\exp\left(-x^2\right)}{\sqrt{\pi}}$	$\frac{1}{2}\operatorname{erf}(x)$
$\frac{1}{4x^2}-\frac{1}{4x^2\sqrt{4x^2+1}}$	$\frac{\sqrt{4x^2+1}-1}{4x}$
$\operatorname{sech}(2x)^2$	$\frac{1}{2}\tanh(2x)$
$\frac{e^x}{(1+e^x)^2}$	$-\frac{1}{1+e^x}$
$\operatorname{UnitBox}(x)$	$\begin{cases} 0 & \text{if } 2x<-1 \\ x+\frac{1}{2} & \text{if } -\frac{1}{2}<x\le\frac{1}{2} \\ 1 & \text{if } 2x>1 \end{cases}$
$\operatorname{UnitTriangle}(x)$	$\begin{cases} 0 & \text{if } x\le-1 \\ \frac{1}{2}(1+x)^2 & \text{if } -1<x\le 0 \\ -\frac{1}{2}(1-x)^2+1 & \text{if } 0<x\le 1 \\ 1 & \text{if } 1<x \end{cases}$

References

[1] Gautschi W., *Numerical analysis*. An introduction, 1997, Birkhäuser Boston, Inc., Boston.

[2] Henrici P., *Applied and computational complex analysis*. Volume 1: Power series—integration—conformal mapping—location of zeros, Pure and Applied Mathematics, 1974, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney.

[3] Hungerbühler N., Wasem M., A generalized version of the residue theorem, *ArXiv e-prints* 1808.00997, August 2018.

[4] Kravanja P., Van Barel M., *Computing the zeros of analytic functions*, volume 1727 of *Lecture Notes in Mathematics*, 2000, Springer-Verlag, Berlin.

[5] Obreschkoff N., *Verteilung und Berechnung der Nullstellen reeller Polynome*, 1963, VEB Deutscher Verlag der Wissenschaften, Berlin.

[6] Schwarz H.R., *Numerical analysis*. John Wiley & Sons, Ltd., Chichester, 1989, A comprehensive introduction, with a contribution by Jörg Waldvogel.

[7] Weierstraß, K., Zu Lindemann's Abhandlung: „Über die Ludolph'sche Zahl". *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 1885, 5, 1067–1085.

[8] Sturmfels B., *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, 2002, Washington, DC; by the American Mathematical Society, Providence, RI.

[9] Dshalalow J.H., Real Analysis: An Introduction to the Theory of Real Functions and Integration, *Studies in Advanced Mathematics*, 2000. ISBN=9781584880738, Taylor & Francis.