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A selfdual generalization of the Theorems of Pascal and Brianchon

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ABSTRACT. We introduce the technique of polarization in the context of projective incidence geometry. It is discussed how to obtain new incidence results by polarizing known theorems. The approach leads to a selfdual theorem which contains as special cases both Pascal's and Brianchon's theorem. As corollaries, we find generalizations of both theorems. A similar technique, regularization, is used to find a generalization of de La Hire's fundamental theorem of polarity.

1. INTRODUCTION

A tangent p to a conic E with point of tangency P is at the same time the polar line of P with respect to E. Stated the other way round, a polar line p with respect to a conic E of a point P is a generalization of a tangent with point of tangency P. Following this line of thought we investigate in this paper the following scheme: Suppose we find in a geometric configuration a tangent p to a conic E with point of tangency P. Then we replace the pair *tangent/point of tangency* by a pair *polar line/pole* and investigate the geometric properties of the situation thus modified.

Definition 1.1. *Formally, when we substitute the pair of words* tangent/point of tangency *by the pair* polar line/pole *in an incidence statement, we call this manipulation* polarizing the statement.

In general, the polarized form of a true incidence statement is not automatically true. Interestingly, however, there are such cases. We begin below by showing that the polarization of the Nobbs-Gergonne theorem yields the theorem of Chasles. In the subsequent sections we will present examples where polarizing known theorems yields new results.

So, let us first illustrate the idea of polarization by starting with a projective version of the Nobbs-Gergonne theorem (see [9]):

Theorem 1.1. *Let $\triangle = ABC$ be a triangle with sides a, b, c, and E a conic inscribed in $\triangle$. The points of tangency of E on the sides a, b, c are denoted by A', B', C' (see Figure 1). Then the lines AA', BB' and CC' are concurrent and meet in the Gergonne point G. Let A'' be the intersection of a with $B'C'$, B'' the intersection of b with $C'A'$, and C'' be the intersection of c with $A'B'$. Then the Nobbs points A'', B'' and C'' are collinear and lie on the Gergonne line g which is the polar line of the Gergonne point G with respect to E and at the same time the trilinear polar of G.*

Polarizing Theorem 1.1, i.e., replacing *tangent/point of tangency* by *polar line/pole* in the statement, we actually get the well known Theorem of Chasles (see [2, Theorem 5·61]).

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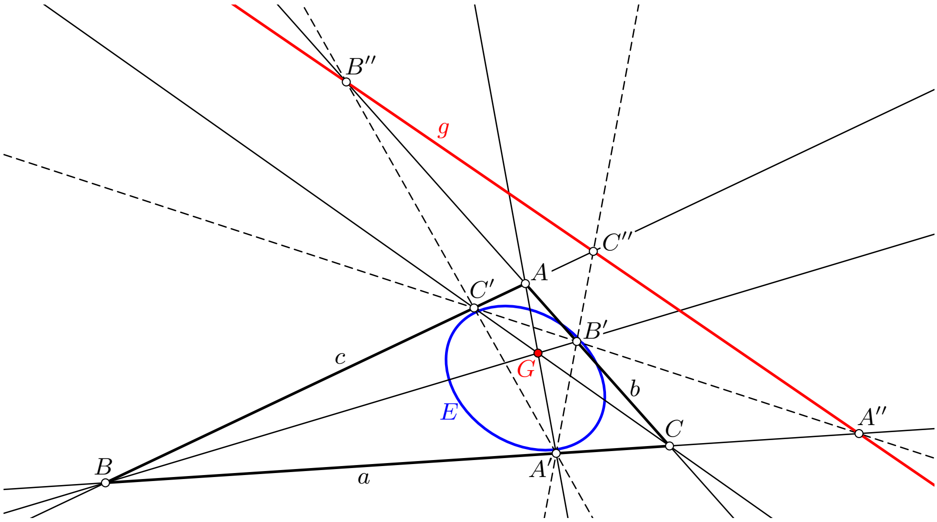

FIGURE 1. Projective version of the Nobbs-Gergonne theorem

Theorem 1.2. *A triangle $\triangle = ABC$ with sides a, b, c and its polar triangle $\triangle' = A'B'C'$ with sides a', b', c' with respect to a conic E are Desargues triangles: The lines AA', BB', CC' meet in a point G, and the intersections A'' of a and a', B'' of b and b', and C'' of c and c' lie on a line g (see Figure 2). The line g is the polar line of G with respect to E.*

For completeness we note that Chasles's theorem has a converse, namely von Staudt's theorem which says that any pair of Desargues triangles are polar triangles with respect to a conic (see [12, p. 135, §241]).

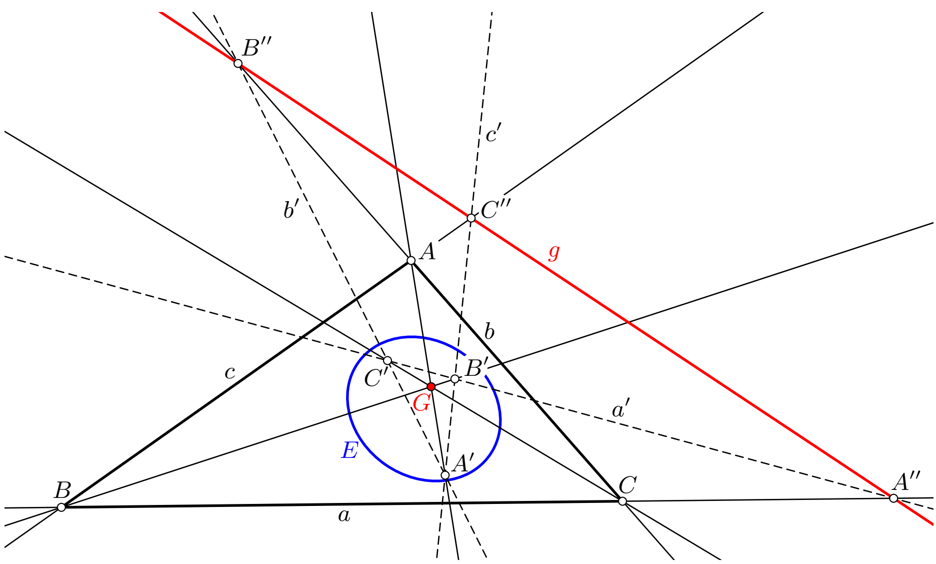

FIGURE 2. Chasles's theorem

Another example where polarization was used to get new incidence results was discussed in [7], where a generalization of the Steinbart theorem was proved.

2. Generalizing Pascal's theorem and Brianchon's theorem by polarization

We will now illustrate that new results can be obtained from known theorems by formally applying the polarization procedure. To put this idea into action, we first need to set up a few tools, namely the Theorems 2.3 and 2.4. First, we recall de La Hire's fundamental theorem of polar theory (see [8]).

Theorem 2.3 (de La Hire)**.** *Let E be a conic. Then the polar lines p of the points P on a line c with respect to E pass through the pole C of the line c with respect to E. Stated differently: A point P' lies on the polar of point P iff P lies on the polar of P'.*

In analogy to a triangle and its polar triangle we shall say that a hexagon H_2 is the polar hexagon of another hexagon H_1 with respect to a conic E if the vertices of H_2 are the poles of H_1 with respect to E. By Theorem 2.3, H_2 is the polar hexagon of H_1 iff H_1 is the polar hexagon of H_2, so we can just say that H_1 and H_2 are polar hexagons with respect to E.

The following theorem contains the theorems of Pascal (see [2, Theorem 7·21] and Theorem 2.7 below) and Brianchon (see [2, Theorem 7·22] and Theorem 2.5 below) and their respective converse. It has the additional nice feature that it is selfdual.

Theorem 2.4. *Let H_1 and H_2 be polar hexagons with respect to a conic E. Then the diagonals of H_1 are concurrent iff H_1 is circumscribed around a conic C iff the intersections of opposite sides of H_2 are collinear iff H_2 is inscribed in a conic D.*

We will see in Section 3 that the conics C and D are conjugate with respect to the conic E.

Proof of Theorem 2.4. Let $P_1, P_2, \ldots, P_6$ denote the vertices of H_1. Then the sides of H_2 are the polar lines $p_1, p_2, \ldots, p_6$ of these vertices with respect to E. Let Q_i be the intersection of the opposite sides p_i and p_{i+3} of H_2 for $i = 1, 2, 3$. Then the diagonal P_iP_{i+3} of H_1 is the polar line of Q_i with respect to E. It follows from Theorem 2.3 that Q_1, Q_2, Q_3 are collinear on a line h iff the corresponding polar lines P_1P_4, P_2P_5, P_3P_6 are concurrent in the pole H of h. The remaining equivalences follow directly from Pascal's theorem, its converse (the Braikenridge-Maclaurin theorem, see, e.g., [4, p. 76]) and Brianchon's theorem and its converse (see, e.g., [4, p. 78]). □

Generalizing the first part of the above proof we find as a byproduct:

Proposition 2.1. *Let H_1 and H_2 be polar polygons with respect to a conic E with an even number of vertices. Then the diagonals of H_1, connecting opposite vertices, are concurrent iff the intersections of opposite sides of H_2 are collinear.*

Let uns now turn our attention to Brianchon's theorem (see [2, Theorem 7·22]).

Theorem 2.5 (Brianchon)**.** *Let $p_1, p_2, \ldots, p_6$ be tangent lines of a conic E with points of tangency $P_1, P_2, \ldots, P_6$. Then the extended diagonals of the hexagon formed by the tangent lines, each connecting opposite vertices, intersect at the Brianchon point B (see Figure 3).*

The Nobbs-Gergonne Theorem 1.1 can be viewed as a degenerate case of Brianchon's Theorem 2.5, namely when each two adjacent contact points of the hexagon with the conic coincide. On the other hand, Chasles's Theorem 1.2 was the polarized form of the Nobbs-Gergonne Theorem 1.1. Now, what is the polarized form of Brianchon's theorem? If we formally replace *tangent/point of tangency* by *polar line/pole* in Theorem 2.5 we get the following theorem.

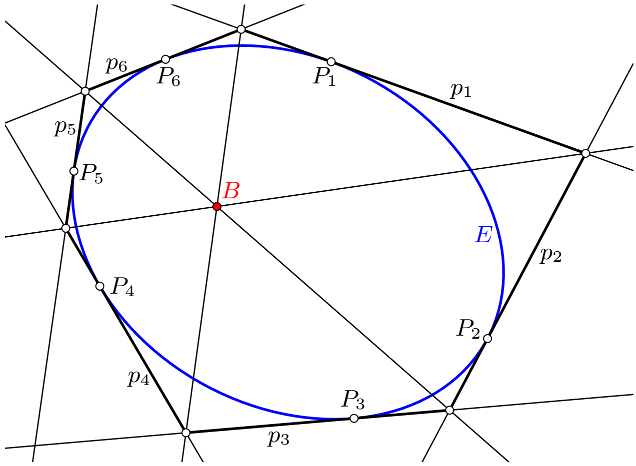

FIGURE 3. Brianchon's theorem

Theorem 2.6 (Polarized Brianchon)**.** *Let $P_1, P_2, \ldots, P_6$ be points on a conic C and $p_1, p_2, \ldots, p_6$ be the polar lines of these points with respect to a conic E. Then the extended diagonals of the hexagon formed by these polar lines, each connecting opposite vertices, are concurrent (see Figure 4).*

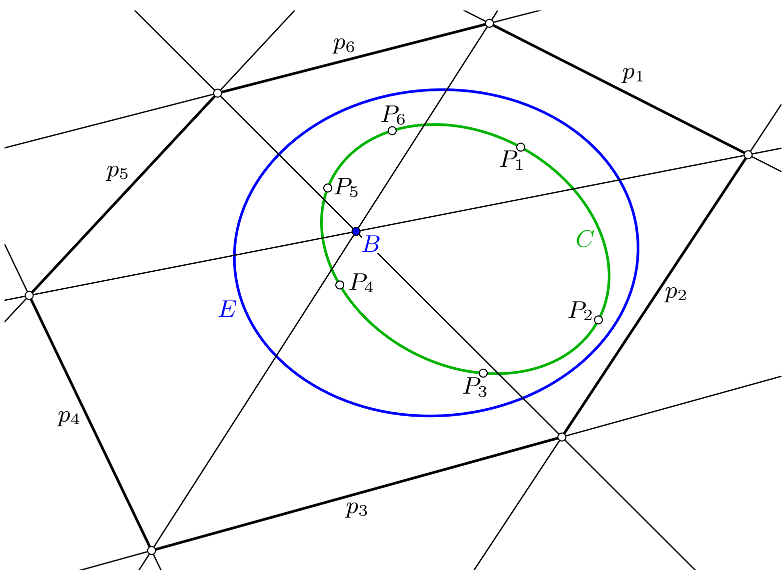

FIGURE 4. Polarized Brianchon theorem

Proof. The points $P_1, P_2, \ldots, P_6$ form a hexagon inscribed in the conic C. Hence, by Pascal's theorem, the intersections of opposite sides are collinear. By de La Hire's Theorem 2.3 the intersection of p_i and p_{i+1} is the pole of the line through P_i and P_{i+1} with respect to the conic E. Hence the polygons $P_1, \ldots, P_6$ and $p_1, \ldots, p_6$ are polar to each other with respect to E. Thus the claim follows directly by Theorem 2.4. $\square$

Note that Theorem 2.6 contains Brianchon's Theorem 2.5, namely if the conics E and C coincide.

Also note that by Theorem 2.3 the polar line of the intersection of opposite sides p_i and p_{i+3} with respect to E is the line through P_i and P_{i+3}. Similarly, the polar line of the intersection of the line through P_i and P_{i+1} and through P_{i+3} and P_{i+4} with respect to E passes through the intersection of p_i and p_{i+1} and through the intersection of p_{i+3} and p_{i+4}.

Pascal's hexagon theorem (see [2, Theorem 7·21]) is the dual of Brianchon's theorem.

Theorem 2.7 (Pascal)**.** *Let $P_1, P_2, \ldots, P_6$ be points of a conic E. Then the intersections of opposite sides of the hexagon $P_1P_2P_3P_4P_5P_6$ are collinear on the Pascal line p (see Figure 5).*

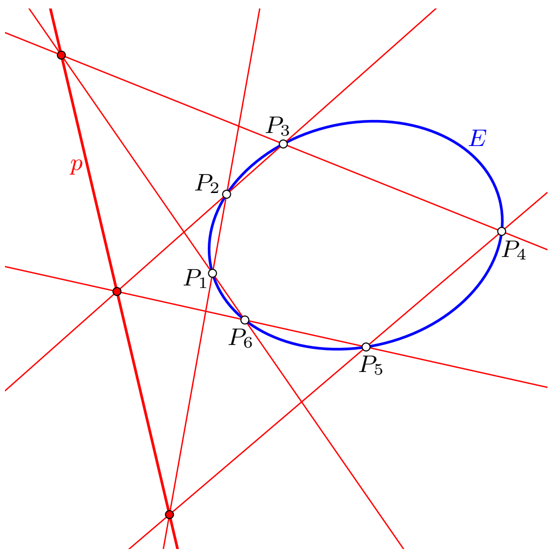

FIGURE 5. Pascal's hexagon theorem

The dual of the polarized Brianchon Theorem 2.6 is the polarized Pascal theorem:

Theorem 2.8 (Polarized Pascal)**.** *Let $p_1, p_2, \ldots, p_6$ be tangent lines of a conic C and $P_1, P_2, \ldots,$ P_6 be the poles of these lines with respect to a conic E. Then the intersections of opposite sides of this hexagon are collinear and lie on a line p (see Figure 6).*

Proof. The claim follows from Theorem 2.6 by dualizing the statement. Alternatively we can again argue with Theorem 2.4: The hexagon formed by the lines $p_1, p_2, \ldots, p_6$ are circumscribed around the conic C. Hence, by Brianchon's theorem, the diagonals connecting opposite vertices are concurrent. Then the claim follows indeed directly from Theorem 2.4. □

Theorem 2.8 contains Pascals hexagon theorem, namely if C and E coincide. Also note that, again by Theorem 2.3, the polar line of the intersection of p_i and p_{i+1} with respect to E passes through P_i and P_{i+1}. Similarly, the pole of the line joining the intersection of p_i and p_{i+1} and the intersection of p_{i+3} and p_{i+4} with respect to E is the intersection of the lines through P_i and P_{i+1} and through P_{i+3} and P_{i+4}.

We note the following additional property.

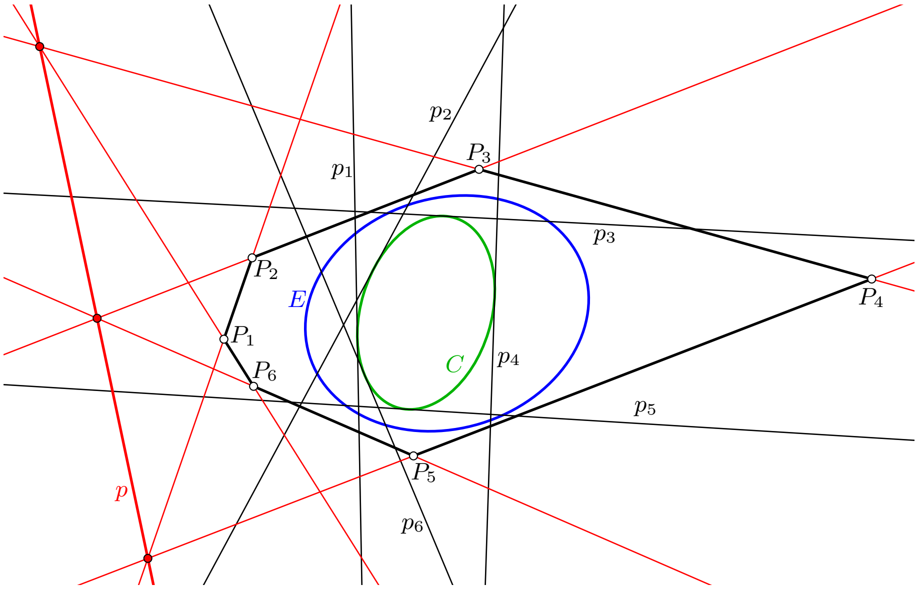

FIGURE 6. Polarized Pascal theorem

Proposition 2.2. *Let H_1 and H_2 be polar hexagons with respect to a conic E. If the diagonals of H_1 meet in a point H and hence the opposite sides of H_2 meet on a line h, then h ist the polar line of H with respect to E.*

Proof. We can just repeat the first part of the proof of Theorem 2.4. □

Summarizing, we see that Theorem 2.4 can be viewed as an amalgamated version of Theorem 2.6 and Theorem 2.8 to one single, selfdual theorem which contains the classical theorems of Brianchon and Pascal.

3. A GENERALIZED VERSION OF DE LA HIRE'S THEOREM

Formal manipulations of statements such as dualization have a long tradition in mathematics. Michael Atiyah expressed it this way: *Duality in mathematics is not a theorem, but a "principle"* (see [1]). Especially in projective geometry dualization is a powerful tool (see [3]). Let us now look at another formal manipulation of statements that can lead to new results in projective geometry. Consider Pappus's theorem (see [2, Theorem 4·31]):

Theorem 3.9 (Pappus). *Let g, h be two straight lines in the projective plane and $P_1, \ldots, P_6$ be a hexagon where the vertices lie alternately on g and h. Then the intersections of opposite sides of the hexagon are collinear.*

Pascal was only 16 years old when noticed in his *Essay pour les coniques*, 1639, that this theorem remains true if he replaced the degenerate conic consisting of the lines g and h by a regular conic. This led him to what became known as Pascal's hexagon theorem (see [10, p. 243–260] and Theorem 2.7 above). We could call this process of replacing in a statement a degenerate conic by a regular one, *regularization* of the statement. Let us see, if other theorems can be regularized in this way. E.g., by regularizing the Scissors theorem we get the Butterfly theorem (see [6]). This idea also works perfectly well if we consider de La Hire's Theorem 2.3. Namely, if we replace in Theorem 2.3 the line c (considered as a degenerate conic) by a regular conic C', we get:

Theorem 3.10. *Let C' and E be conics. Then there is a unique conic C, called the conjugate conic of C' with respect to E, which has the following property: The polar line p' of any point $P' \in C'$ with respect to E is tangent to C. Vice versa, if $P \in C$ is the point of tangency on p', then the polar line p of P with respect to E is tangent to C' in point P'. In particular, this relation is symmetric, i.e., C is the conjugate of C' with respect to E iff C' is the conjugate conic of C with respect to E (see Figure 7).*

De La Hire's Theorem 2.3 can indeed be viewed as a degenerate case of Theorem 3.10, namely if the conic C degenerates to a point and the conjugate conic C' degenerates to a line (see the Remark after Definition 1.6 in [5]).

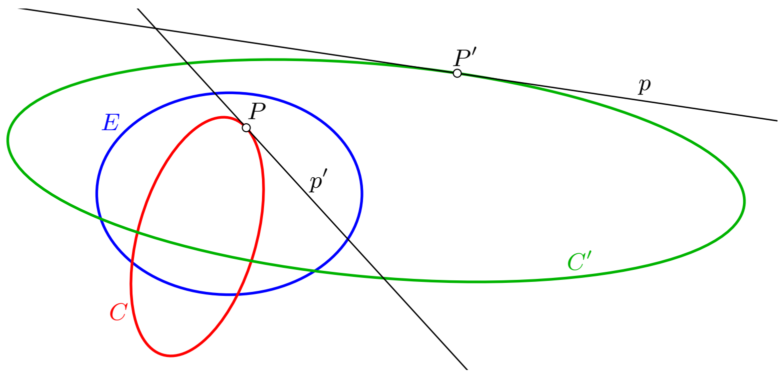

FIGURE 7. Illustration of Theorem 3.10: When the point P' moves along the conic C', its polar line p' with respect to E is tangent to C in a point P. Vice versa, the polar line p of P with respect to E is tangent to C' in point P'.

Proof of Theorem 3.10. We first give a geometric proof based on Theorem 2.4 and then add an algebraic proof, which gives some additional insight.

Choose five fixed points $C'_1, C'_2, C'_3, C'_4, C'_5$ on C'. Let $c'_1, c'_2, c'_3, c'_4, c'_5$ be the polar lines of these five points with respect to E. These five polar lines define a unique conic section C which they all touch simultaneously. Now let P' be a further point on C' and p' the corresponding polar line. The points $C'_1, C'_2, C'_3, C'_4, C'_5$ and P' all lie on C', so it follows from Theorem 2.4 that the polar lines $c'_1, c'_2, c'_3, c'_4, c'_5$ and p' are tangents to a conic section. But this must be the conic section C, consequently p' is a tangent to C. Let P be the point where p' touches the conic section C. We consider the polar line p of this point with respect to E. Since P lies on p', p passes through P'. We have to show that p is the tangent to C' at the point P'. Let us assume that this is not the case. Then there is a second intersection point R' of p and C'. Since R' lies on C', the polar r' of R' is a tangent to C (use the same argument as above for the polar line p'). At the same time r' passes through P, since R' lies on p. r' is therefore a tangent to C and passes through the point P on C from which it follows that r' must be the tangent to C at the point P. Because of the uniqueness of the tangent, and since p' is tangent to C in P, we conclude $r' = p'$, which in turn means $R' = P'$. This contradicts the assumption. Consequently, p is indeed the tangent to C' at the point P'.

An algebraic proof uses projective coordinates in the projective plane $\mathbb{R}P^2$. In this framework, a conic is given by an equation $\langle P, CP \rangle = 0$, where C is a regular symmetric 3×3 matrix with mixed signature (see [5]). In the following we identify a conic with the corresponding matrix. So let P' be a point on the conic C'. The polar of P' with respect to E is $p' = EP'$. We claim that the conjugate conic C of C' with respect to E is given by the matrix $C = EC'^{-1}E$. Indeed, p' is tangent to C since the pole $C^{-1}p'$ of p' with respect to E is incident with p':

$$\langle C^{-1}p', p'\rangle = \langle (E^{-1}C'E^{-1})p', p'\rangle = \langle E^{-1}C'P', EP'\rangle = \langle C'P', P'\rangle = 0.$$

Hence the point of tangency is $P' = C^{-1}p' = E^{-1}C'P'$. Obviously, $C = EC'^{-1}E$ is equivalent to $C' = EC^{-1}E$. Note also, that C is symmetric, regular and has, by Sylvester's law of inertia (see [11]), mixed signature. □

4. Conclusions

In self-dual projective planes, a true incidence statement is transformed through dualization into another true incidence statement. Dualization is done by formally exchanging the terms "point" and "straight line" (along with any necessary grammatical adjustments). Here we propose a similar formal manipulation of incidence statements: polarization. The terms "tangent" and "point of contact" are replaced by the terms "polar" and "pole". This transformation does not necessarily result in true incidence statements again. However we show in several examples that polarization can lead to interesting new results, and old results appear in a new light. In a similar technique, regularization, a double line or a pair of lines is regarded as a degenerate conic section and formally replaced by a non-degenerate conic section in an incidence statement. We use examples to show that in this process, similar to polarization, new incidence results can arise from known incidence results. It is to be expected that the proposed techniques, polarization and regularization, will also provide new results in other cases.

References

[1] Atiyah, M. F. *Duality in mathematics and physics*. Lecture notes of the Facultat de Matemàtiques i Estadística of the Universitat Politècnica de Catalunya, 2007. https://api.semanticscholar.org/CorpusID:229550466
[2] Coxeter, H. S. M. *The real projective plane*. Springer, New York, third edition, 1993.
[3] Richter-Gebert, J. *Perspectives on projective geometry*. Springer, Heidelberg, 2011.
[4] Coxeter, H. S. M.; Greitzer, S. L. *Geometry revisited*, volume 19 of *New Mathematical Library*. Random House, Inc., New York, 1967.
[5] Halbeisen, L.; Hungerbühler, N. Conjugate conics and closed chains of Poncelet polygons. *Mitt. Math. Ges. Hamburg*, 36:5–28, 2016.
[6] Halbeisen, L.; Hungerbühler, N.; Schiltknecht, M. Reversion porisms in conics. *Int. Electron. J. Geom.*, 14(2):371–382, 2021.
[7] Hungerbühler, N. The Steinbart theorem, polar triangles and the reversion map. *Glob. J. Adv. Res. Class. Mod. Geom.*, 13(1):34–44, 2024.
[8] de La Hire, P. *Sectiones conicae*. P. Michalet, Parisiis, 1685.
[9] Oldknow, A. The Euler-Gergonne-Soddy triangle of a triangle. *Amer. Math. Monthly*, 103(4):319–329, 1996.
[10] Pascal, B. *Œuvres de Blaise Pascal, Tome I*. Hachette, Paris, 2e éd., 1923.
[11] Sylvester, J. J. A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 4(23):138–142, 1852.
[12] von Staudt, K. G. C. *Geometrie der Lage*. Bauer und Raspe, 1847.

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