A MATHEMATICAL MODEL FOR LEVITATION IN A MAGNETIC STIRRER

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We describe and discuss a mathematical model for the magnetic stirrer which explains the magnetic levitation of the stir-bar which has recently been reported in an experimental setting. The model explains the main experimental observations.

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1. Introduction

In [1] Baldwin et al. describe the stunning phenomenon of magnetic levitation in a magnetic stirrer. Recall that according to Earnshaw's theorem [4], a static magnetic or electric field is not capable of maintaining an electrically charged body or a magnetic dipole in a stable equilibrium. However, magnetic levitation has been realised in many ways if the levitated object is dynamically stabilised (see [1] and the references therein). It is surprising that magnetic levitation in magnetic stirrers has only recently been discovered and described in [1], in particular, because this standard laboratory tool is in worldwide use since its invention in 1942*. The authors of [1] describe that the magnetic stir-bar, submerged in a fluid of suitable viscosity and driven at an angular velocity above a critical threshold, jumps up to levitate stably up to several centimetres above the base of the container. Very recently, the effect has also been documented in water and even in air: see [5], and the supplementary video [6] showing the experiment. Other approaches to overcome the limitations of Earnshaw's theorem have recently been dicussed in [9], namely the levitation of a rapidly oscillating magnetic dipole above a metallic sheet, or in [8] where a rotating saddle trap is used. On a much smaller scale, rotating magnetic nanorods in a viscous liquid are very sensitive to an ambient magnetic field: In [7] it is shown that any ambient field stabilizes the synchronous precession of the nanorod, and this allows to control the precession. In this way, rotating magnetic nanorods can detect minute fluctuations of a magnetic field such that they can be used henceforth as sensors of very weak magnetic fields, for microrheology, and generally for magnetic levitation. It is the aim of this paper to develop and discuss a simple mathematical model for

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*Arthur Rosinger of Newark, New Jersey, U.S., obtained the US Patent 2,350,534, for his *Magnetic Stirrer* on 6 June 1944. He filed his application on 5 October 1942.

levitation in a magnetic stirrer which explains the main experimental observations which have been reported in [1] (see also the supplemental material [2]).

2. A mathematical model of the magnetic stirrer

In this section we consider a simplified mathematical model of the magnetic stirrer. The situation is shown in Figure 1. We make certain simplifying assumptions in the modelling, so that a mathematical analysis is still easily possible, but nonetheless all aspects of the experiment are captured.

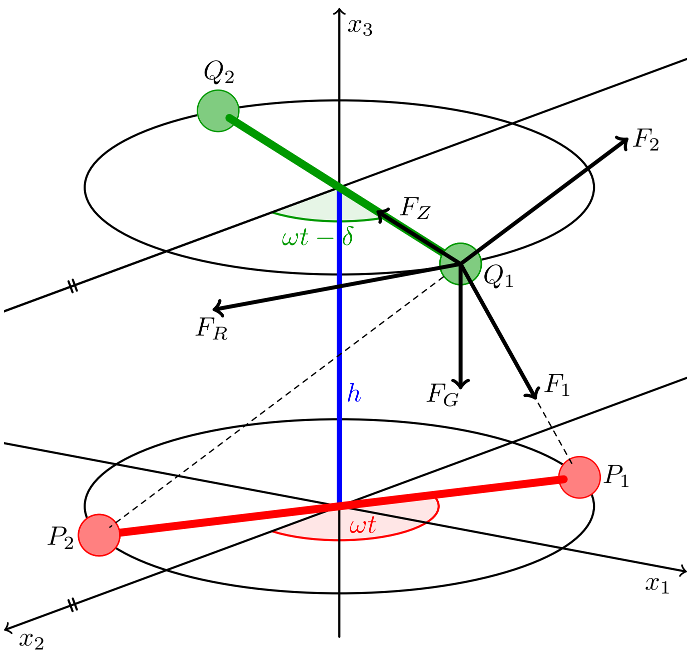

FIGURE 1. The geometrical representation of the magnetic stirrer.

The motor-driven magnet (red) rotates in the (x_1, x_2) plane at an angular velocity of $\omega > 0$. Its one end P_1 has at time t the position

$$P_1 = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix}$$

in Cartesian coordinates (x_1, x_2, x_3). Here, we assume that the length of the red motor-driven magnet is 2. This defines the unit length. The position of the other end of the red magnet is $P_2 = -P_1$. The green dumbbell-shaped stir-bar, the so called flea, rotates at height h above the (x_1, x_2) plane likewise with the angular velocity ω but with a phase shift δ. Its two ends have at time t the positions

$$Q_1 = \begin{pmatrix} \cos(\omega t - \delta) \\ \sin(\omega t - \delta) \\ h \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} -\cos(\omega t - \delta) \\ -\sin(\omega t - \delta) \\ h \end{pmatrix}.$$

Now we calculate the forces acting on the dumbbell head Q_1:

- The gravitational force reduced by the buoyancy force amounts to

$$F_G = -mg \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where m is the mass of the dumbbell head minus the mass of the fluid it displaces.

- We assume that the frictional force exerted by the viscous fluid on the dumbbell is proportional to the velocity vector $\dot{Q}_1$ with proportionality factor $\lambda > 0$. The frictional force is therefore

$$F_R = -\lambda \dot{Q}_1.$$

The constant λ is greater the higher the viscosity of the liquid is.

- The force exerted by the red magnet on the green magnet: As a simplified model, we assume that the force exerted by P_1 on Q_1 is attracting with

$$F_1 = \varepsilon \, \frac{P_1 - Q_1}{|P_1Q_1|^2}$$

(attraction between north pole and south pole). $\varepsilon > 0$ is a proportionality factor that indicates the strength of the magnets. If P_1 and Q_1 were opposite electrical point charges, the denominator would be $|P_1Q_1|^3$ instead of $|P_1Q_1|^2$. However, here we calculate with the square in order to simplify the model and the calculations (see also [3] for a discussion of this approach). Accordingly, the force P_2 exerts on Q_1 is repulsive and equal to

$$F_2 = \varepsilon \, \frac{Q_1 - P_2}{|P_2Q_1|^2}.$$

The distances are

$$|P_1Q_1| = \sqrt{2 - 2\cos\delta + h^2}, \qquad |P_2Q_1| = \sqrt{2 + 2\cos\delta + h^2}.$$

- The force that the green dumbbell bar exerts on the dumbbell head Q_1 is

$$F_Z = -\mu \begin{pmatrix} \cos(\omega t - \delta) \\ \sin(\omega t - \delta) \\ 0 \end{pmatrix},$$

with a constant μ which is to be determined.

The green magnetic stir-bar rotates in an equilibrium if and only if

$$F_G + F_R + F_1 + F_2 + F_Z = 0. \tag{1}$$

In fact, if the sum of the forces in (1) were not 0, the flea would not maintain its height or angular velocity. If we use the above expressions for the forces, we find for (1):

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} \frac{\varepsilon(1-\cos\delta)}{2-2\cos\delta+h^2} + \frac{\varepsilon(\cos\delta+1)}{2+2\cos\delta+h^2} - \lambda\omega\sin\delta - \mu\cos\delta \\ \sin\delta\left(\frac{4\varepsilon\cos\delta}{(h^2+2)^2-4\cos^2(\delta)} + \mu\right) - \lambda\omega\cos\delta \\ -mg - \frac{4h\varepsilon\cos\delta}{(h^2+2)^2-4\cos^2\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

Note that for symmetry reasons it is sufficient to consider the time $t = 0$.

We decompose the resulting force (2) in a radial, tangential and vertical component, i.e., in the direction of the vectors

$$\begin{pmatrix}\cos\delta\\-\sin\delta\\0\end{pmatrix},\ \begin{pmatrix}\sin\delta\\\cos\delta\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix},$$

and get

$$\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} := \frac{1}{c}\begin{pmatrix}2h^2\varepsilon\cos\delta-\mu c\\2(h^2+2)\varepsilon\sin\delta-\lambda\omega c\\4mg\cos^2\delta-4h\varepsilon\cos\delta-mg(h^2+2)^2\end{pmatrix} \stackrel{!}{=} \begin{pmatrix}0\\0\\0\end{pmatrix}, \tag{3}$$

where $c := (h^2+2)^2 - 4\cos^2\delta$. Observe that $c > 0$ for $h > 0$, and also for $h = 0$ provided $\delta \in (0, \pi)$. For a real magnetic stirrer we have $h > 0$, and we will assume this from now on. Observe also that μ occurs only in the first equation in (3), and only linearly. This means that for any solution (h, δ) of the second and third equation in (3) the first equation can also be fulfilled. Hence the flea levitates in an equilibrium at height h with phase shift δ if and only if the two equations

$$2(h^2+2)p\sin\delta + \omega\big(4\cos^2\delta - (h^2+2)^2\big) = 0 \tag{4}$$

$$4\cos^2\delta - 4hq\cos\delta - (h^2+2)^2 = 0 \tag{5}$$

hold with

$$p := \frac{\varepsilon}{\lambda} \quad\text{and}\quad q := \frac{\varepsilon}{mg}.$$

In Section 3 we will discuss the equilibria of the system, i.e., the solutions of (4)–(5). In Section 4 we examine under which conditions the jump of the flea can take place, and in Section 5 we will determine which equilibria are stable or unstable: Only stable solutions can be realised experimentally. Observe that a model where the two magnets are replaced by idealised magnetic dipoles have only unstable solutions. It is therefore essential that the magnets are modelled as extended bars.

3. **Discussion of the equilibrium condition**

Before we discuss the equilibrium conditions (4), (5) mathematically, we make some heuristic physical considerations: If the angular velocity ω and thus the friction force is small, the phase shift δ is also small. Then the distance of the attracting magnetic poles is smaller than the distance of the repelling magnetic poles. Together with the gravitational force we have a resulting force in negative x_3-direction: The flea rotates on the bottom of the container and the normal force compensates for the resulting vertical force. If we increase ω, the friction and thus the phase shift increases. If δ exceeds $\frac{\pi}{2}$, the distance between the attracting poles is greater than the distance of the repulsive poles. As soon as the resulting repulsion is greater than the weight force, the flea rises from the bottom of the container and floats at the height, where the weight force and the repulsion cancel out each other. If ω becomes too large, the friction becomes so large that the coupling force with the driving magnet is no longer sufficient and the driving magnet overtakes the flea.

Let us first determine the threshold value δ_0 so that for all phase shifts $\delta \leq \delta_0$ no stable levitation can occur:

Proposition 3.1. *Let $0 \le \delta_0 \le \pi$ be such that*

$$\cos\delta_0 = -\frac{\sqrt{64 - 144q^2 - 27q^4 + \sqrt{q^2+8}\,(9q^2+8)^{3/2}}}{8\sqrt{2}}. \tag{6}$$

Then, the vertical force component $v_3 < 0$ is strictly negative for all values $\delta \in [0, \delta_0)$. For $\delta = \delta_0$ there is a unique value $h = h_0$ such that $v_3 = 0$, and $v_3 < 0$ for all other values of $h > 0$. This means that for values $0 \le \delta \le \delta_0$ no stable levitation can occur. The function $q \mapsto \delta_0(q)$ is decreasing, with $\lim_{q\to 0} \delta_0(q) = \pi$, and $\lim_{q\to\infty} \delta_0(q) = \frac{\pi}{2}$.

Proof. The function $v_3(h)$ is clearly negative for $0 \le \delta \le \frac{\pi}{2}$, and for $\frac{\pi}{2} < \delta < \pi$ it has exactly one maximum for $h > 0$: In fact, the derivative $\frac{dv_3}{dh}$ has only one positive zero at

$$\bar{h} = \sqrt{\frac{2}{3}}\sqrt{\sqrt{4 - 3\cos^2\delta} - 1}.$$

If this value $h = \bar{h}$ is used in v_3 and the expression thus obtained is set equal to 0, the resulting equation for $\cos\delta$ yields the value specified by (6). □

For all values of δ in the interval $(\delta_0, \pi]$ the equation $v_3 = 0$ has solutions:

Lemma 3.1. *Let $0 < h \le \hat{h}$, where $\hat{h}$ is the positive root of $h(4 + h^2) = 4q$. Then equation (5) is satisfied for the value*

$$\cos\delta = \frac{1}{2}\left(hq - \sqrt{(2+h^2)^2 + h^2q^2}\right), \tag{7}$$

with $\delta_0 < \delta \le \pi$. Levitation at height $h \ge \hat{h}$ cannot occur. The function $q \mapsto \hat{h}(q)$ is increasing, with $\lim_{q\to 0} \hat{h}(q) = 0$, and $\lim_{q\to\infty} \hat{h}(q) = \infty$.

Proof. Equation (5), i.e. $v_3 = 0$, is a quadratic equation for $\cos\delta$. It has two real solutions for $\delta \in (\delta_0, \pi]$, one of them is given by (7). A positive sign in front of the root in (7) gives a value > 1 which is impossible. Hence (7) is the only remaining solution. For $0 < h \le \hat{h}$ the expression on the right hand side of (7) takes values in $[-1, 0)$, hence there is a corresponding solution $\delta \in (\delta_0, \pi]$. For $h > \hat{h}$ the value of the expression is < -1. For $h = \hat{h}$ it follows from $v_3 = 0$ that $\delta = \pi$, but this violates (4). □

It is remarkable that the relation (7) between the phase shift δ and the altitude h only depends on $q = \frac{\varepsilon}{mg}$, but not on the parameter λ for the viscosity.

For the function

$$f : [0, \infty) \to \mathbb{R}, \quad h \mapsto \frac{1}{2}\left(hq - \sqrt{(2+h^2)^2 + h^2q^2}\right)$$

which is given by equation (7) we have:

- f is concave
- $f(0) = -1$, $f'(0) > 0$, $\lim_{h\to\infty} f(h) = -\infty$
- f has a maximum at

$$h_0 = \frac{\sqrt{\sqrt{(8+q^2)(8+9q^2)} - 3q^2 - 8}}{2\sqrt{2}} \tag{8}$$

- $f(h_0) = \cos(\delta_0)$ (see Lemma 3.1).
- $f(\hat{h}) = -1$ (see Lemma 3.1).

This means that there are always two values h in the interval $[0, \hat{h}]$ which yield the same value in (7) for δ. Figure 2 shows the points (δ, h) which satisfy (7), and hence (5), for the numerical value $q = 7$.

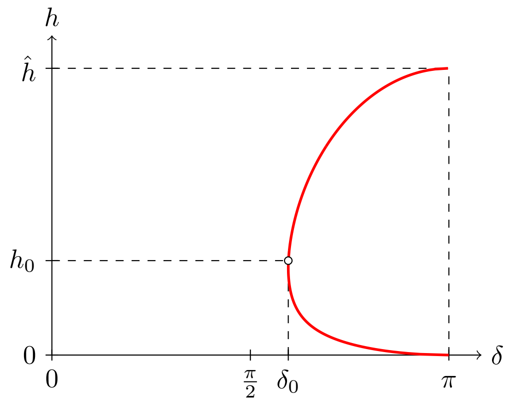

FIGURE 2. The set of solutions of equation (5) for $q = 7$.

Now, we are able to parametrize all solutions of (4)–(5):

Proposition 3.2. *Let $p, q > 0$, $\hat{h}$ be the positive root of $h(4 + h^2) = 4q$, and*

$$\zeta : (0, \hat{h}) \to \mathbb{R}, \quad h \mapsto \sqrt{(2 + h^2)^2 + h^2 q^2}.$$

Suppose that the values $\delta \in (\frac{\pi}{2}, \pi)$ and $\omega > 0$ satisfy (4)–(5) for a given $h \in (0, \hat{h})$. Then we have

$$\delta = \arccos\big(\tfrac{1}{2}\big(hq - \zeta(h)\big)\big) \tag{9}$$

$$\omega = \frac{p\big(hq + \zeta(h)\big)\sqrt{4 - (hq - \zeta(h))^2}}{2qh(2 + h^2)}. \tag{10}$$

Proof. The relation (9) between δ and h has been established in Lemma 3.1. Since (5) is linear in ω it can have only one solution for a given pair (δ, h). It can be checked by hand that the value (10) is this solution of (5) if δ is given by (9). □

The function $h \mapsto \omega(h)$ given by (10) is strictly decreasing. So this is also true for the inverse function $\omega \mapsto h(\omega)$: The levitation height h decreases with increasing ω. This is in accordance with the observations in the real experiment, as reported in [1]. Figure 3 shows the graph of the function $\omega \mapsto h(\omega)$ for the numerical values $p = 1, q = 3$. We will see in Section 5, that all equilibria for a levitation height h below a certain critical threshold h_s are unstable and are hence not observed in a physical experiment. This is in agreement with the results in [1].

In the experiment, the flea first rotates for small angular velocity at the bottom of the container at some fixed height $h_b > 0$. As the angular velocity ω increases, the phase shift δ increases until it reaches a critical value δ_0 where it jumps to the certain height. However, it has been observed in [1], that this phenomenon only occurs if h_b is not too high. So our next aim is to investigate the conditions under which the jump of the flea is actually possible.

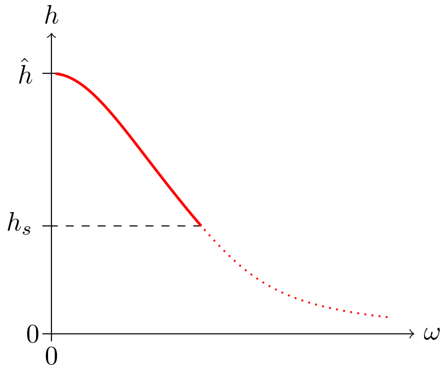

FIGURE 3. The levitation height h in function of the angular velocity ω for $p = 1, q = 3$. Stable levitation is only possible for $h > h_s$ (see Section 5).

4. The jump of the flea

Suppose the flea rotates at the bottom of the container at height $h_b > 0$. Then, the relation between the angular velocity and the phase shift δ is given by (4) and we have:

Lemma 4.1. *Suppose the flea rotates at height $h_b > 0$ at the bottom of the container, i.e., $v_3 \leq 0$. Then the angular velocity is given by*

$$\omega = \frac{2p\left(h_b^2 + 2\right)\sin(\delta)}{(h_b^2+2)^2 - 4\cos^2\delta}. \tag{11}$$

The function $\delta \mapsto \omega(\delta)$ has the following properties:

- $\omega(0) = \omega(\pi) = 0$.
- $\omega(\delta) > 0$ *for* $0 < \delta < \pi$.
- *If* $h_b < \sqrt{2\left(\sqrt{2}-1\right)} =: H = 0.91018\ldots$[†], *then the function has two maxima on the interval* $[0,\pi]$ *at the values* $\sin\delta = \frac{h_b}{2}\sqrt{4+h_b^2}$. *In these points ω assumes the maximum value*

$$\hat{\omega} = \frac{p\left(h_b^2+2\right)}{2h_b\sqrt{h_b^2+4}}. \tag{12}$$

ω has a local minimum at $\delta = \frac{\pi}{2}$ with value

$$\check{\omega} = \frac{2p}{h_b^2+2}.$$

[†]Recall that the unit length is half of the length of the flea.

- *If $h_b \geq H$, then the function $\omega(\delta)$ has only one maximum at $\frac{\pi}{2}$ on the interval $[0, \pi]$ with value*
$$\hat{\omega} = \frac{2p}{h_b^2 + 2}.$$

Proof. Solve (4) with fixed value $h = h_b$ for ω. This yields (11). It is then elementary to determine the extrema. □

Figure 4 shows the graph of the function $\delta \mapsto \omega(\delta)$ given by (11) for two values h_b, one below H, one above H.

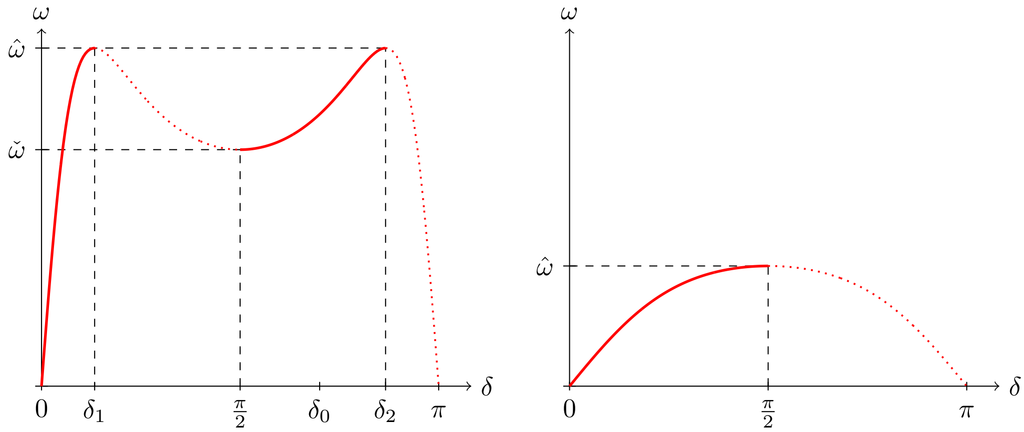

FIGURE 4. Angular velocity ω as a function of the phase shift δ, on the left for $h_b < H$, on the right for $h_b > H$. We will see in Lemma 4.2 that the dotted branches correspond to unstable solutions of (4).

Let us first consider the case $h_b \geq H$ with only one maximum: Here, the phase shift increases continuously together with the angular velocity. At the maximum value $\hat{\omega}$ the phase shift reaches the value $\delta = \frac{\pi}{2}$. If ω is increased further, the coupling between the flea and the driving magnet breaks down: Equation (4) has no solution in this regime. The driving magnet laps the flea which begins to waggle. This effect is reported [1]. Lemma 4.2 shows that solutions on the dotted branch cannot be realized physically, because they correspond to unstable equilibria. A phase shift $\delta \geq \delta_0$ can therefore not be attained and levitation does not occur.

In the case $h_b < H$ the phase shift also increases together with ω. When ω reaches the value $\hat{\omega}$, the phase shift jumps to the second maximum δ_2. If $\delta_2 > \delta_0$ the vertical force v_3 is strictly positive (see Proposition 3.1) and the flea will jump to the height which is given by the solution of (4) and (5) for $\omega = \hat{\omega}$. If, on the other hand $\delta_2 < \delta_0$, we are in the same situation as in the case $h_b > H$: The flea will waggle and no levitation occurs. The critical value h_b^{crit} with the property that a jump to levitation occurs for $h_b < h_b^{\text{crit}}$ solves $\delta_0 = \delta_2$ which yields

$$h_b^{\text{crit}} = \sqrt{2}\sqrt{\sqrt{2 - \arccos^2(\delta_0)} - 1}.$$

Recall that $\arccos(\delta_0)$ is given by (6) and depends only on q, i.e., not on the viscosity. Observe that h_b^{crit} is monotonically increasing in q and has limit H as q tends to ∞

(see Proposition 3.1). Summarizing, we see that the angular velocity $\hat{\omega}(h_b)$ at which the jump occurs is given by (12) for $0 < h_b < h_b^{\text{crit}}$. $\hat{\omega}(h_b)$ is monotonically decreasing: The higher h_b, the smaller the angular velocity $\hat{\omega}$ at which the flea jumps. This phenomenon, and a critical threshold h_b^{crit}, has also been reported in [1]. Figure 5 shows the graph of the function $\hat{\omega}(h_b)$.

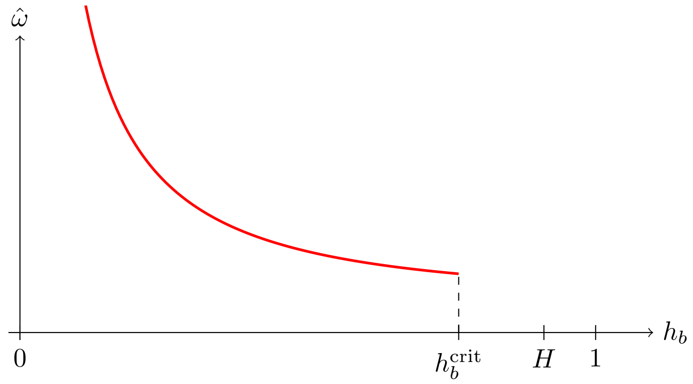

FIGURE 5. Angular velocity $\hat{\omega}$ at which the flea jumps when rotating at the bottom of the container at height h_b.

Lemma 4.2. *Suppose that the flea rotates at the bottom of the container at height $h_b > 0, H > h_b$, with angular velocity ω and phase shift δ. Let $0 < \delta_1 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \delta_2 < \pi$ be the values for which $\sin\delta_1 = \frac{h}{2}\sqrt{4+h^2}$ (these are the values, for which ω assumes its maximum according to Lemma 4.1). Then, the rotation is stable if $0 \le \delta < \delta_1$ or $\frac{\pi}{2} < \delta < \delta_2$. The rotation is unstable if $\delta_1 < \delta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \delta < \pi$. If $h_b > H$ the rotation is stable for $0 \le \delta < \frac{\pi}{2}$ and unstable for $\frac{\pi}{2} < \delta < \pi$.*

Proof. The stability condition is $\frac{dw_2}{d\delta} > 0$, because then the resulting tangential force points towards the equilibrium (see the third component of (3)). We find

$$\frac{dw_2}{d\delta} = \frac{2(2+h^2)\varepsilon\cos\delta(h^4+4h^2-4\sin^2\delta)}{((2+h^2)^2-4\cos^2\delta)^2}.$$

The denominator is obviously positive. The numerator is positive if and only if δ lies in the intervals stated in the lemma. □

5. Stability of the levitation equilibria

It turns out that not all solutions of (3) correspond to stable equilibria. To see this, we consider the vector field in $\mathbb{R}^2$ given by

$$F : \begin{pmatrix} \delta \\ h \end{pmatrix} \mapsto \begin{pmatrix} w_2(\delta, h) \\ w_3(\delta, h) \end{pmatrix}$$

according to (3). If the Jacobian matrix of F in an equilibrium point, i.e., a point where F vanishes, has only eigenvalues with strictly negative real part, then this equilibrium is asymptotically stable. The next lemma shows, that we can replace the vector field F by the vector field

$$G : \begin{pmatrix} \delta \\ h \end{pmatrix} \mapsto \begin{pmatrix} 2(h^2+2)p\sin\delta + \omega\big(4\cos^2\delta - (h^2+2)^2\big) \\ 4\cos^2\delta - 4hq\cos\delta - (h^2+2)^2 \end{pmatrix}$$

given by (4) and (5) for the stability discussion, which is slightly simpler.

Lemma 5.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field with $F(x_0) = 0$ and $c : \mathbb{R}^n \to \mathbb{R}$ a C^1 function with $c(x_0) > 0$. Then the Jacobian matrix $J_F(x_0)$ has only eigenvalues with strictly negative real part if and only if the same is true for the Jacobian matrix $J_G(x_0)$ of the vector field $G = cF$.*

Proof. We have

$$\begin{aligned} \left(\frac{\partial G_i(x)}{\partial x_j}\right)\Big|_{x=x_0} &= \left(\frac{\partial F_i(x)}{\partial x_j}\alpha(x) + F_i(x)\frac{\partial \alpha(x)}{\partial x_j}\right)\Big|_{x=x_0} = \\ &= \alpha(x)\left(\frac{\partial F_i(x)}{\partial x_j}\right)\Big|_{x=x_0} \end{aligned}$$

and the claim follows. □

The eigenvalues of J_G evaluated for δ and ω given by (9) and (10) converge for $h \to 0$ to the values

$$2\left(p + q \pm |p - q|\right) > 0.$$

On the other hand, both eigenvalues of J_G are strictly negative in the point $(\delta, h) = (\pi, \hat{h})$. This can be seen as follows: By observing that $\hat{h}^3 = 4q - 4\hat{h}$ one gets for the eigenvalues of J_G in $\delta = \pi$

$$-2p(\hat{h}^2 + 2) \text{ and } 8\hat{h} - 12q.$$

The first of these two values is obviously strictly negative, but this is also true for the second, since $\hat{h}(4 + \hat{h}^2) = 4q$. This means that equilibria are unstable for small h and stable (and hence physically observable) for h close to $\hat{h}$. So, our next goal is to determine the value h_s such that the equilibria are stable for $h > h_s$ and unstable for $h < h_s$. The stability condition for complex eigenvalues is $\operatorname{trace} J_G < 0$. The next proposition addresses this question.

Proposition 5.1. *Let $p, q > 0$, and $\hat{h}$ be the positive root of $h(4 + h^2) = 4q$. Then the system of the three equations (4), (5) and $\operatorname{trace} J_G = 0$ has exactly one zero $(\delta_s, h_s, \omega_s)$ in the set $(0, \pi) \times (0, \hat{h}) \times (0, \infty)$.*

Proof. In view of the previous remarks, we already know, that for solutions of (4), (5) $\operatorname{trace} J_G$ takes positive and negative values. So, for continuity reasons there must be at least one point with $\operatorname{trace} J_G = 0$. To see that there is at most one zero, we proceed as follows: First, we eliminate ω from equation (4) and $\operatorname{trace} J_G = 0$. This yields an equation in δ and h with parameters p and q. Using (9) such that (5) is automatically satisfied, we can eliminate δ form this equation, resulting in an equation for h alone:

$$\begin{aligned} &h^2\zeta(h)\left(p\left(h^2+2\right)\left(h^2+2q^2+4\right)+4q\left(h^2+q^2+2\right)\right) = \\ &\quad = 2hq\left(p\left(h^2+2\right)\left(h^4+h^2\left(q^2+4\right)+2\right)+q\left(3h^4+2h^2\left(q^2+4\right)+4\right)\right) \end{aligned} \tag{13}$$

The quotient (left hand side of (13) divided by right hand side) is a strictly monotonically increasing function. Indeed, its derivative has the strictly positive denominator

$$2q\zeta(h)\left(p\left(h^2+2\right)\left(h^4+h^2\left(q^2+4\right)+2\right)+q\left(3h^4+2h^2\left(q^2+4\right)+4\right)\right)^2>0$$

and the numerator is an even polynomial with exclusively positive coefficients. This proves the claim. □

The critical value h_s can numerically be computed by solving (13). It turns out that in the point $(\delta_s, h_s, \omega_s)$ $\det J_G > 0$, i.e., the eigenvalues in this point are indeed complex. We end this discussion with stream plots of the forces (w_2, w_3) given by (3) in the (δ, h) plane for a stable and for an unstable equilibrium in Figure 6.

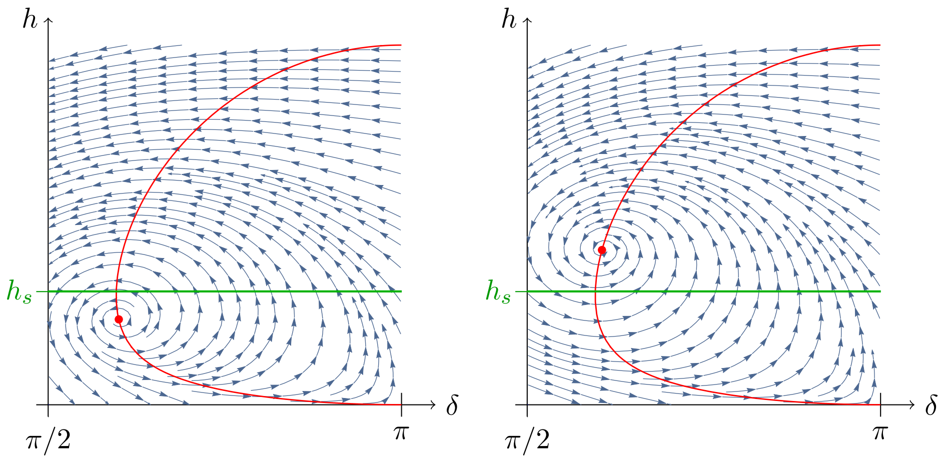

FIGURE 6. Stream plots of the forces (w_2, w_3) in the (δ, h) plane for the parameter values $p = 4, q = 7$. The red line is the locus of all solutions of $w_3 = 0$ as in Figure 2. Solutions of $(w_2, w_3) = (0, 0)$ below the green line $h = h_s$ are unstable (figure on the left), above the green line, they are stable (figure on the right). Observe that $h_s \neq h_0$.

Conclusion. In summary, the proposed model of the magnetic stirrer exhibits the following characteristics which are consistent with the corresponding experimental findings observed in [1]:

- At the beginning of the experiment, the flea rotates at the bottom of the container at height h_b: The phase shift δ increases with the angular velocity ω. If ω reaches a critical value $\hat{\omega}$ the flea jumps if $h_b < h_b^{\text{crit}}$ and it starts to waggle if $h_b > h_b^{\text{crit}}$.
- The angular velocity $\hat{\omega}$ at which the jump occurs is monotonically decreasing as a function of h_b.
- Levitation at height $h \geq \hat{h}$ cannot occur.
- The levitation height h is decreasing as a function of ω and becomes unstable for h below a threshold h_s.

The formulas we developed for the values $\hat{\omega}, \delta_0, \hat{h}, h_b^{\text{crit}}$ show how they depend on the parameters $p = \frac{\varepsilon}{\lambda}$ and $q = \frac{\varepsilon}{mg}$. This opens a door for further research: It would be interesting to check whether the corresponding model predictions can be verified in experiments. E.g., $\hat{\omega}$ is proportional to p. Or: δ_0 is monotonically decreasing, and $\hat{h}$ and h_b^{crit} are monotonically increasing in q.

Furthermore it would be interesting to extend the proposed model to the regime in which the driving magnet and the flea spin asynchronously, since this phenomenon has also been experimentally investigated in [1].

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