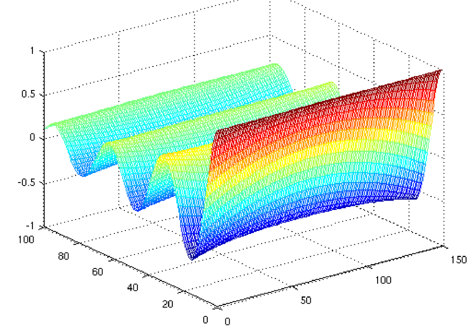
Error estimates for optimized domain decomposition methods applied to the one-dimensional heat equation



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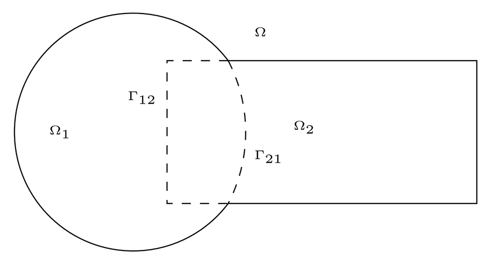
Abstract

DOMAIN Decomposition Methods (DDM) were first introduced by Schwarz in 1869 for proving the Dirichlet Principle. After a century of latency, Lions revived Schwarz's theory with very innovative convergence proofs, and Dryja and Widlund studied Schwarz methods in a discrete setting for parallel computing purposes. Later, optimized Schwarz methods were introduced, based on an optimization of the communication between the subdomains. These methods are of interest because of their fast convergence rates.

This poster provides a short introduction to DDM, followed by a discussion of Optimized DDMs with Robin transmission conditions applied to the one-dimensional heat equation. The efficiency of the Robin transmission conditions will be emphasized. We present existing results for these algorithms, and a very recent performance analysis for short time intervals.

1. Overlapping Schwarz methods

TO introduce Overlapping Schwarz Methods we take the example introduced by Schwarz himself in 1869.



The domain Ω is divided into two subdomains Ω_1 and Ω_2 as shown in the figure. Then the Schwarz method is defined for the heat equation,

$$\partial_t u = \Delta u \text{ on } \Omega \times (0,T),$$

as an iterative system:

$$\begin{cases} \partial_t u_1^n(x,t) = \Delta u_1^n(x,t) & \text{on } \Omega_1 \times (0,T), \\ \mathcal{B}_1 u_1^n(x,t) = \mathcal{B}_1 u_2^{n-1}(x,t) & x \in \Gamma_{12}, t \in (0,T), \\ u_1^n(x,0) = g(x) & x \in \partial\Omega \cap \Omega_1, \end{cases}$$

$$\begin{cases} \partial_t u_2^n(x,t) = \Delta u_2^n(x,t) & \text{on } \Omega_2 \times (0,T), \\ \mathcal{B}_2 u_2^n(x,t) = \mathcal{B}_2 u_1^{n-1}(x,t) & x \in \Gamma_{21}, t \in (0,T), \\ u_2^n(x,0) = g(x) & x \in \partial\Omega \cap \Omega_2, \end{cases}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are pseudo differential operators. We will be interested in the analysis of the following transmission conditions:

- **Dirichlet:** $\mathcal{B}_1$ and $\mathcal{B}_2$ are the identity.
- **Robin:** $\mathcal{B}_1 = \partial_n + p$ and $\mathcal{B}_2 = \partial_n + p$ with p real positive and ∂_n the unit outward normal derivative.

Remark 1. *The derivative adds essential information that is communicated to the neighboring subdomains. Fewer iterations are then needed to obtain the same precision as illustrated in the next figure.*

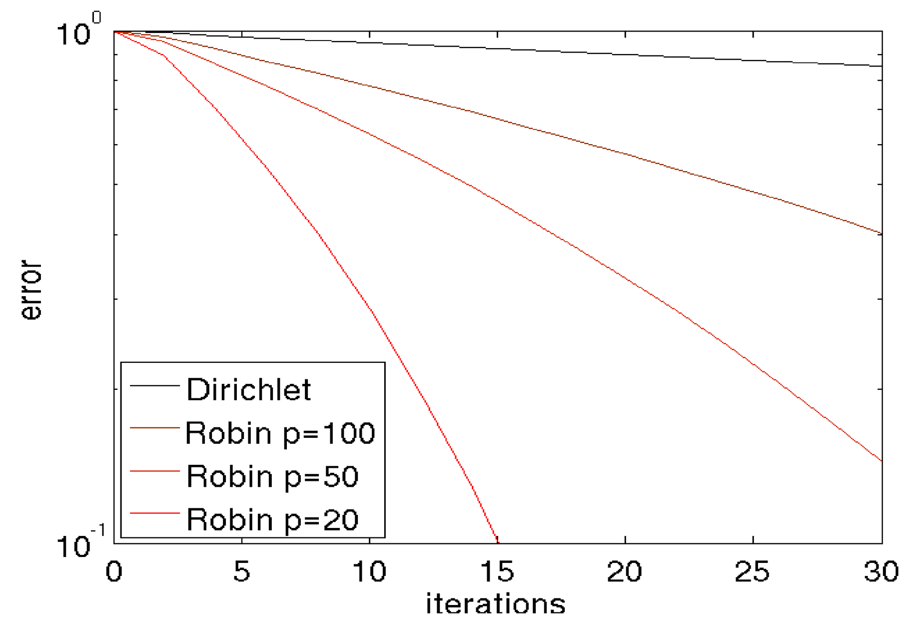


Figure 1: *Comparing Dirichlet with Robin*

Remark 2. *The algorithm described here, applied to many subdomains, is naturally parallel.*

We will simplify the problem and apply the Domain Decomposition Method to the one dimensional heat equation over $\mathbb{R}$:

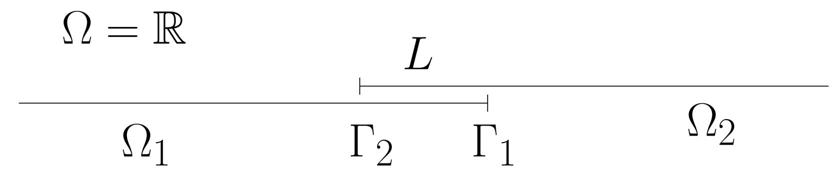


Figure 2: *The decomposition of* $\mathbb{R}$ *in two subdomains*

2. Effect of time on convergence

IN 2001 Gander and Zhao (see [1]) were working on domain decomposition methods for the n-dimensional heat equation using Dirichlet transmission conditions. They analyzed the error and showed that the convergence is linear for a large time interval and superlinear for a small time interval. The theoretical results they proofed are the following.

For the n-dimensional heat equation with Dirichlet transmission conditions, the error satisfies:

- **on an infinite time interval:**

$$\max_j \|e_j^{k(m+2)}\|_\infty \le (\gamma(m,L))^k \max_j \|e_j^0\|_\infty,$$

with $e_j^n := u - u_j^n$ the error of the heat equation for subdomain j at iteration n and $\gamma \le 1$.

- **on a short time interval:**

$$\max_j \|e_j^k\|_T \le (2n)^k \operatorname{erfc}\left(\frac{k\delta}{2\sqrt{\pi T}}\right) \max_j \|e_j^0\|_T.$$

The next plot illustrates these results. The upper curve has a time interval of length $T = 2$ and the lower one has an interval of length $T = 0.1$.

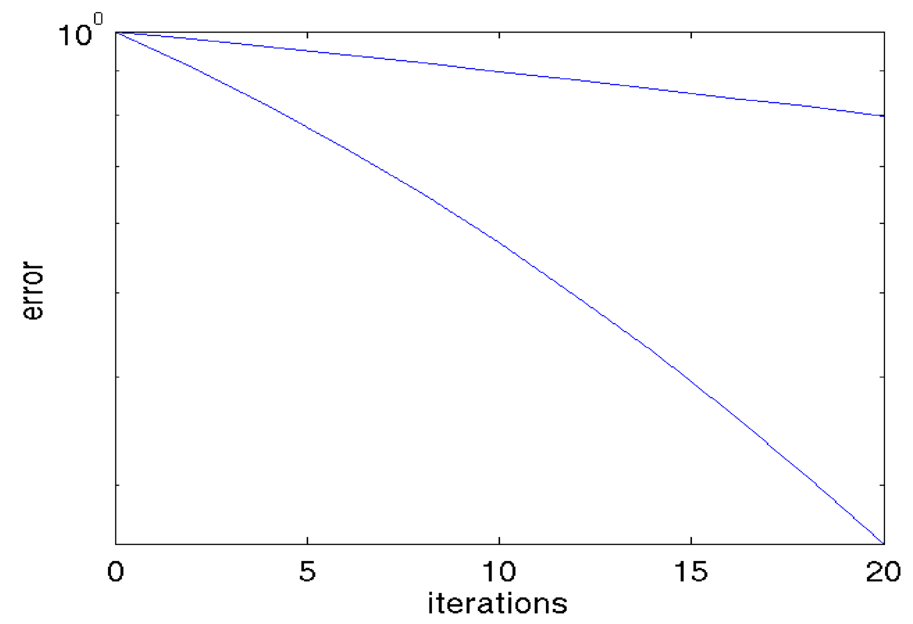


Figure 3: *Linear/superlinear convergence with T=2/0.1*

Questions:

1. **Can we find an analytic error estimate for Robin transmission conditions?**
2. **Has time the same influence when using Robin transmission conditions?**
3. **How can we make use of the short time behaviour to improve the method?**
4. **Can we find the optimal parameter p for this problem?**

3. Error estimates for Robin transmission conditions

WE answer the first question in the restricted context of $\mathbb{R}$. The error estimate of this problem is obtained using the Laplace transform. Once in the frequency domain, the error equation has a solution of the form

$$\hat{e}_i^n(\Gamma_j, s) = e^{-2nL\sqrt{s}} \left(\frac{p-\sqrt{s}}{p+\sqrt{s}}\right)^{2n} \hat{e}_i^0(\Gamma_j, s), \quad i \ne j, \qquad (1)$$

where s is the frequency variable and $\hat{e}_i^0$ the Laplace transform of the initial guess.

Proposition 1. *The DDM with Robin conditions applied to the unbounded domain* $\mathbb{R}$ *subdivided into two subdomains as illustrated in figure 2 has an error over the interface given by*

$$e_i^n(\Gamma_j, t) = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^{2n-k} (2p)^k \int_0^t P_k^{2n}(L,\tau) e_i^0(\Gamma_j, t-\tau) d\tau,$$

where

$$\begin{aligned} P_0^{2n}(L,t) &= \tfrac{2nLe^{-(2n)^2L^2/4t}}{2\sqrt{\pi}t^{3/2}}, \\ P_1^{2n}(0,t) &= \tfrac{1}{\sqrt{\pi t}} - pe^{p^2t} \operatorname{erfc}(p\sqrt{t}), \\ P_1^{2n}(L,t) &= e^{2npL}\left(P_1^{2n}(0) - 2n\textstyle\int_0^L e^{-2npx} \tfrac{2nxe^{-2n^2x^2/4t}}{2\sqrt{\pi}t^{3/2}} dx\right), \\ P_{k+1}^{2n}(L,t) &= \tfrac{(-1)^k}{k!} \tfrac{d^k}{dp^k} P_1^{2n}(L,t), \quad k \in 1, \dots, 2n-1. \end{aligned}$$

Proof. The proof consists of a judiciously chosen transformation of (1) which permits the calculation of the inverse Laplace transform using differential equations. □

The special case without overlap, i.e. $L = 0$, leads to a simpler expression. If we define the function

$$f(\tau) := \frac{1}{\sqrt{\pi}\tau} - e^{\tau^2} \operatorname{erfc}(\tau),$$

then we get the error estimate

$$\begin{aligned} e_i^n(\Gamma_j, t) = {} & (-1)^{2n} e_i^0(\Gamma_j, t) + \\ & \sum_{k=1}^{2n-1} C(k,n) \int_0^{p\sqrt{t}} f^{k-1}(\tau)\tau^k e_i^0(\Gamma_j, t-\tau^2/p^2) d\tau + \\ & \frac{(-1)^{2n-1}2^{2n+1}}{(2n-1)!} \int_0^{p\sqrt{t}} f^{2n-1}(\tau)\tau^{2n} e_i^0(\Gamma_j, t-\tau^2/p^2) d\tau. \end{aligned}$$

where the coefficients of the sum are given by

$$C(k,n) = \binom{2n}{k+1} \frac{(-1)^{2n-1}2^{k+1}(4n-k+1)}{(k-1)!(2n-k)}.$$

This result aims to answer the questions introduced in the next section.

4. Robin and short times

WE study the behaviour of the method with Robin transmission conditions applied to a short time interval. Numerical computations lead to the following error plot for a final time of $T = 0.01$ and an overlap of four mesh points.

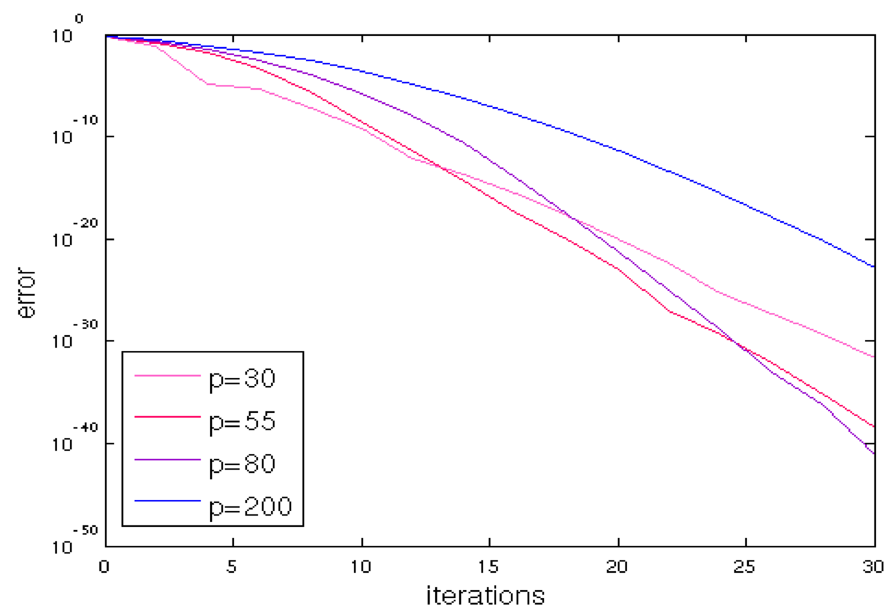


Figure 4: *Complexity of a short time situation*

This figure illustrates all the interesting behaviours of the method we are working on:

- **Find the solution of the min-max problem:**

$$\min_p \left(\max_{t \in (0,T)} (\|e_i^n(\Gamma_j, t)\|) \right),$$

- **Predetermine the parameters n and p for a choosen tolerance,**
- **Explain the cusps of the curves around the optimal p.**

References

[1] Martin J. Gander and Hongkai Zhao, *Overlapping Schwarz Waveform Relaxation for the Heat Equation in n-Dimension*, BIT 2000, Vol 40, No. 4, pp. 001-004.

[2] Martin J. Gander and Laurence Halpern, *Méthodes de relaxation d'ondes (SWR) pour l'équation de la chaleur en dimension 1*, C. R. Acad. Sci. Paris, t., Série I, p.1-4, Analyse Numérique.