Development and Analysis of an Algorithm for the Linear Transport Equation

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This work is dedicated to my mentor R. Massjung, without whose help I would not have been able to write this paper.

Abstract

In my Extended Essay I will deal with a particular partial differential equation called the linear transport equation. This differential equation is very important in modeling the behavior of fluids using computer simulations. The exact solution of the differential equation is often not possible to calculate and thus an approximation has to be made by an algorithm. My Extended Essay will deal with the question: How does such an algorithm work, what are the requirements for such an algorithm and to what extent can I improve the algorithm. My scope will be the analysis of two algorithms, one using piecewise constant functions as an approximation and the other one using piecewise linear functions, whereupon I will try to develop an improved version of the algorithm. My algorithm will make use of piecewise quadratic functions, while keeping the additional requirements of such an algorithm in mind. After writing the algorithm I will compare the three different algorithms. I will show that the algorithm using piecewise linear functions is significantly better than the one using piecewise constant functions. My algorithm using piecewise quadratic functions works just as well as the one using piecewise linear functions. That means that I succeeded in developing an algorithm using piecewise quadratic functions while considering the additional requirements, but the algorithm has not augmented the quality of the results.

Contents

1 Introduction

My Extended Essay will deal with the linear transport equation which is a partial differential equation describing the transport and preservation of a quantity such as fluids. Computational methods are used to simulate such fluid flows. In my essay I will look at two algorithms for the simulation. My question is: How does such an algorithm work, what are the requirements for such an algorithm and to what extent can I improve the algorithm. The improvement of algorithms saves resources, which is why it is of great significance.

Throughout this essay I will mainly be relying on the works of Randall J. LeVeque [1], Nessyahu-Tadmor [3] and Ralf Massjung [2] (see Appendix D).

Due to the nature of my subject I expect my reader to have a knowledge of integration and differentiation.

2 The linear transport equation

2.1 Derivation of the linear transport equation

The linear transport equation is the following time-based partial differential equation

$$\frac{\partial u}{\partial t} + a \cdot \frac{\partial u}{\partial x} = 0 \tag{1}$$

where u is a function of x and time t. In this section we will see how this equation is derived. Usually partial differential equations (PDE) are gained from physical phenomena, so I will demonstrate the PDE with the aid of a physical example. Imagine a cylindrical glass tube filled with gas. The value of $x \in [x_1, x_2]$ describes a certain position in the tube, with x_1 and x_2 being the endpoints of the tube. If we look at a time t_0, then the function $\rho(x, t_0)$ describes the density of the gas at a certain position x. We get the mass of the gas in the tube by integrating.

$$m = \int_{x_1}^{x_2} \rho(x, t_0)\mathrm{d}x \tag{2}$$

The amount of gas in the tube can only change if a certain amount goes out of the tube or a certain amount comes in through the endpoints. For this example I will say that a constant amount of gas comes into the tube at x_1 with the velocity v_1 and leaves the tube at x_2 with the velocity v_2. Starting from the time t_0 if we want to find out how much more gas there is in the tube at time t_1, we have to integrate at x_1 and x_2 and consider the difference. Note that even though the incoming or outgoing gas has a constant velocity, the density of it might not be constant, which is why we have to consider the density function $\rho(x, t)$ when integrating.

$$\text{change in mass from time } t_0 \text{ to } t_1 = \underbrace{\int_{t_0}^{t_1} v_1 \cdot \rho(x_1, t)\mathrm{d}t}_{\text{gas coming in at } x_1} - \underbrace{\int_{t_0}^{t_1} v_2 \cdot \rho(x_2, t)\mathrm{d}t}_{\text{gas going out at } x_2} \tag{3}$$

If we want to know how much mass there is at time t_1 we can add the change in mass during the time period to the original amount of gas in the tube. This gives us an equation involving only integrals called the **integral form**.

$$\underbrace{\int_{x_1}^{x_2} \rho(x, t_1)\mathrm{d}x}_{\text{mass at time } t_1} = \underbrace{\int_{x_1}^{x_2} \rho(x, t_0)\mathrm{d}x}_{\text{mass at time } t_0} + \underbrace{\int_{t_0}^{t_1} v_1 \cdot \rho(x_1, t)\mathrm{d}t - \int_{t_0}^{t_1} v_2 \cdot \rho(x_2, t)\mathrm{d}t}_{\text{change in mass from time } t_0 \text{ to } t_1} \tag{4}$$

In this example I set the velocity to be constant. But this might not always be the case. Generally the velocity can be described as a function of the density, thus we can rewrite $v \cdot \rho = v(\rho) \cdot \rho = f(\rho)$. The integral form becomes

$$\int_{x_1}^{x_2} \rho(x, t_1)\mathrm{d}x = \int_{x_1}^{x_2} \rho(x, t_0)\mathrm{d}x + \int_{t_0}^{t_1} f(\rho(x_1, t))\mathrm{d}t - \int_{t_0}^{t_1} f(\rho(x_2, t))\mathrm{d}t. \tag{5}$$

If we look at any interval $I \subset [x_1, x_2]$, we can assert that the integral form holds for that interval too, since the mass at t_1 is the mass at t_0 plus the change in mass during the time period. Similarly we can extend our glass tube definition, and say that if the glass tube is in $\mathbb{R}$, then for any interval $I \subset \mathbb{R}$ the integral form holds. This equation is of fundamental importance, since it **preserves the given quantity**, in this case the mass. This means that the mass in $\mathbb{R}$ stays constant.

Since the integral form does not give us enough information, we can differentiate it to get the following differential equation (see Appendix A.1 for calculations).

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} f(\rho) = 0 \tag{6}$$

This form is called the **differential form**. This equation is very useful, since for every point (x, t) one can calculate the derivative in t-direction if one has the derivative in $x-$direction.

From this point on I will use a simpler notation for partial derivatives, namely $\frac{\partial \rho}{\partial t} = \rho_t$. Thus the differential form can be rewritten as

$$\rho_t + f(\rho)_x = 0 \tag{7}$$

The linear transport equation is a special case of this partial differential equation, namely when $f(\rho) = a \cdot \rho$. Beginning from the next section I will use the function u instead of ρ. Rewriting the integral form and generalizing it for any interval $[x_i, x_{i+1}]$ and time t_n yields

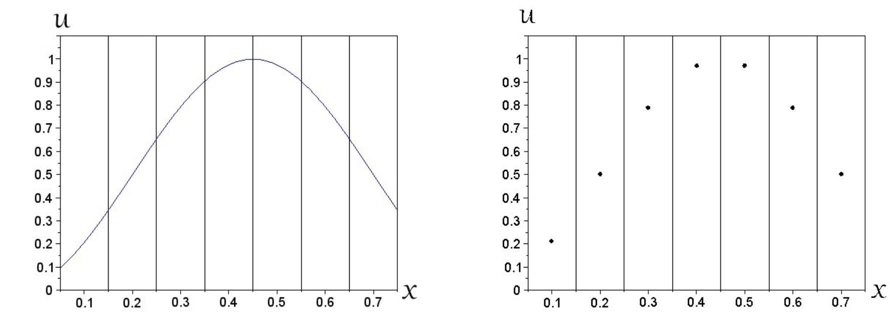
$$\int_{x_i}^{x_{i+1}} u(x, t_n)\mathrm{d}x = \int_{x_i}^{x_{i+1}} u(x, t_n)\mathrm{d}x + \int_{t_n}^{t_{n+1}} a \cdot u(x_i, t)\mathrm{d}t - \int_{t_n}^{t_{n+1}} a \cdot u(x_{i+1}, t)\mathrm{d}t. \tag{8}$$

For futher reading refer to LeVeque [1].

2.2 Description of algorithms for the partial differential equation

Now that we know the equation, we will look at the problem. If we get a starting state $u(x, t_0)$ and a function $f(u)$ which describes the velocity of each point in the x-axis, how will the function u develop?
Since it is often not possible to calculate the exact solution of this problem, an algorithm is used to approximate the solution. For that we first divide the x-axis with the function $u(x, t_0)$ into several segments or cells Ω_i, all having the width Δx. Then we calculate the intermediate values $\overline{u}_i$ of this segments, so that $\overline{u}_i \cdot \Delta x = \int_{\Omega_i} u(x, t_0)\mathrm{d}x$ (see Figure 1). It is easier to calculate with these $\overline{u}_i$-values than with the function as a whole. In the algorithm we have to compute how these $\overline{u}_i$-values develop. For that we set a time step Δt and look how the values in a cell Ω_i have progressed during that time.

Note that this means that the $\overline{u}_i$-values will change over time and thus require an index of time. For simplicity this index will be omitted and $\overline{u}_i$ will



(a) Original function $u(x, t_0)$ (b) Intermediate values $\overline{u}_i$

Figure 1: Distribution of $\overline{u}_i$-values

signify the value at the current time step. Where it is important to distinguish I will write $\overline{u}_i^n$ to be the intermediate value of u on the cell Ω_i at time t_n.

For my essay I will use the simple function $f(u) = a \cdot u$. The benefit of the function becomes evident once we think about its implications. Any value $\overline{u}_i$ will be transported with the velocity a to the right. This means that the function $u(x, t_0)$ as a whole will be shifted to the right as the time progresses. If we define the starting state as $u(x, t_0) = u_0(x)$, the solution of the PDE would be $u(x, t) = u_0(x - at)$. We can easily verify this by putting this function into the differential form. By using the chain rule we get

$$\begin{aligned}
&\frac{\partial}{\partial t} u_0(x - at) + a \frac{\partial}{\partial x} u_0(x - at) \\
&= u_0'(x - at) \cdot (-a) + a \cdot u_0'(x - at) \cdot 1 \\
&= (-a + a) \cdot u_0'(x - at) \\
&= 0.
\end{aligned} \tag{9}$$

Thus the benefit of this simple function f is that we know the exact solution. If we write an algorithm we can test it by comparing its results to the exact solution.

The study of simple cases such as this is often used in mathematical practice to gain insight about the problem and afterwards apply the knowledge to more complex versions of the problem.

A difficulty that we will face is that we only can calculate in a finite area. We can not calculate all the $\overline{u}_i$ values from $-\infty$ to ∞. Due to this we will define an interval where we will compute our $\overline{u}_i$ values and we will set up some conditions for the boundaries. I will say that left and right of the boundary the function value will be constant, namely c_l and c_r. If our interval is $[x_l, x_r]$ then

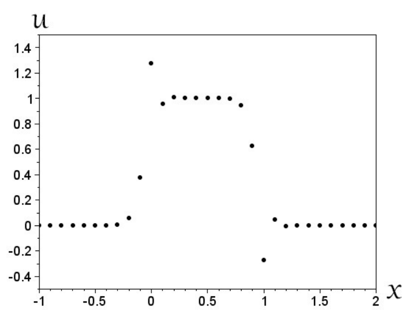


Figure 2: Algorithm with oscillations

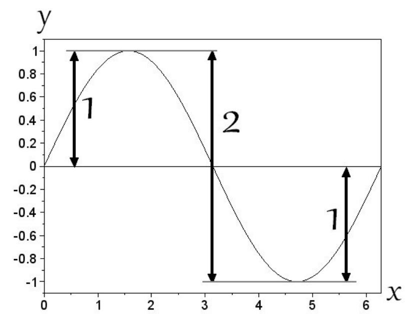


Figure 3: The TVD of the sine function

that means the function u will be defined as follows:

$$u = \begin{cases} c_l & \text{if } x < x_l \\ u(x,t) & \text{if } x \in [x_l, x_r] \\ c_r & \text{if } x > x_r \end{cases} \tag{10}$$

Since the transport of the quantity u has a certain known limited velocity, in our case a, we know how far the $\overline{u}_i$-values will have been transported at most in a given time frame. Thus even after a given time we can say in which interval the significant $\overline{u}_i$-values are located.

There is an additional condition that the algorithm has to fulfill. If an algorithm produces unrealistic results, that is if the function values get negative or if some values form unnatural peaks, then the validity of the solution will be impaired (see Figure 2). With time the algorithm will start to produce more and more unrealistic values and it will start to oscillate. But since this is against our interests the question is how can we supress such an oscillation. To do this we define the **Total Variation** of our function u at time t_n to be $\mathrm{TV}(u_n) = \sum_i |\overline{u}_{i+1} - \overline{u}_i|$. This sum describes the distance between the different $\overline{u}_i$-values in the u-axis, and if we sum it up it describes the distance between the extrema of the function. For example the total variation of the sine function in the interval $[0, 2\pi]$ is 4 (see Figure 3). The sum of the total variation is taken over all the i-values from $-\infty$ to ∞. However since the function becomes constant beyond the boundary, there only exists a finite amount of positive summands, so we can limit ourselves to the sum of all the $|u_{i+1} - u_i|$ within the boundary.

To avoid oscillations this total variation must not increase in the next time step. Thus if we take the next time step to be t_{n+1}, then the following must hold:

$$\mathrm{TV}(u_{n+1}) \leq \mathrm{TV}(u_n) \tag{11}$$

If we can find such an algorithm, then we say the algorithm is **total variation**

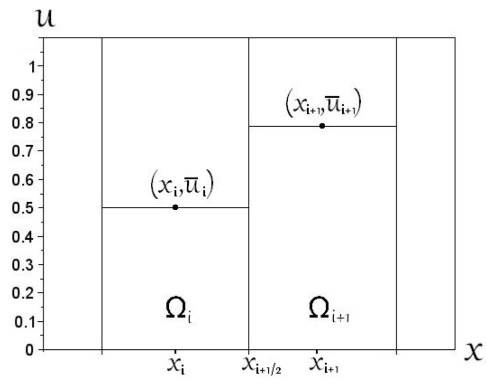


Figure 4: The cells Ω_i and Ω_{i+1}

diminishing (TVD).

Summing up an algorithm needs to preserve the quantity and it needs to be TVD.

3 Different algorithms

The following two algorithms have a similar method to calculate the new $\overline{u}_i$-values. I will first look at the simpler algorithm.

3.1 Staggered Lax-Friedrich's method

This algorithm uses **piecewise constant functions** to approximate the values in the integral form (8). With the time step $\Delta t = t_{n+1} - t_n$ one calculates the LHS, namely the integral at time t_{n+1}.

$$\int_{x_i}^{x_{i+1}} u(x, t_{n+1})\mathrm{d}x = \underbrace{\int_{x_i}^{x_{i+1}} u(x, t_n)\mathrm{d}x}_{\substack{\text{approximation using} \\ x_i\text{-values}}} + \underbrace{\int_{t_n}^{t_{n+1}} a \cdot u(x_i, t)\mathrm{d}t - \int_{t_n}^{t_{n+1}} a \cdot u(x_{i+1}, t)\mathrm{d}t}_{\substack{\text{approximation using } x_i\text{-values} \\ \text{and the differential form of the equation}}} \tag{12}$$

To describe the algorithm we will look at two neighboring cells Ω_i and Ω_{i+1}. These two cells have the intermediate values $\overline{u}_i$ and $\overline{u}_{i+1}$ (see Figure 4). We will approximate the integrals in the integral form (12) to find the $\overline{u}_i$-values in the next time step. First we can calculate the integral from x_i to x_{i+1}.

$$\int_{x_i}^{x_{i+1}} u(x, t_n)\mathrm{d}x = \frac{\overline{u}_i + \overline{u}_{i+1}}{2} \cdot \Delta x \tag{13}$$

On the LHS of (12) we will get an integral for the time t_{n+1}. As before we will describe this integral as

$$\int_{x_i}^{x_{i+1}} u(x, t_{n+1})\mathrm{d}x = \overline{u}_{i+\frac{1}{2}}^{n+1} \cdot \Delta x \tag{14}$$

where $\overline{u}_{i+\frac{1}{2}}^{n+1}$ describes the intermediate value at time t_{n+1} in the cell $\Omega_{i+\frac{1}{2}}$. Note that in a time step the cells get shifted by $\frac{\Delta x}{2}$, but that does not constitute a problem since after two steps the cells are back in their original positions. Plugging in the integrals (13) and (14) in (12) yields

$$\overline{u}_{i+\frac{1}{2}}^{n+1} \cdot \Delta x = \frac{\overline{u}_i + \overline{u}_{i+1}}{2} \cdot \Delta x + \int_{t_n}^{t_{n+1}} a \cdot u(x_i, t)\mathrm{d}t - \int_{t_n}^{t_{n+1}} a \cdot u(x_{i+1}, t)\mathrm{d}t. \tag{15}$$

Since we assumed that the value of u is constant on the cell Ω_i, we will likewise assume that the value of u is constant along the t-axis. So $u(x_i, t) = \overline{u}_i$. This means that

$$\begin{aligned} \int_{t_n}^{t_{n+1}} a \cdot u(x_i, t)\mathrm{d}t &= a \cdot \int_{t_n}^{t_{n+1}} \underbrace{u(x_i, t)}_{\text{is constant}} \mathrm{d}t \\ &= a \cdot \int_{t_n}^{t_{n+1}} \overline{u}_i \mathrm{d}t \\ &= a \cdot \overline{u}_i \cdot \Delta t. \end{aligned} \tag{16}$$

Analogously

$$\int_{t_n}^{t_{n+1}} a \cdot u(x_{i+1}, t)\mathrm{d}t = a \cdot \overline{u}_{i+1} \cdot \Delta t. \tag{17}$$

Again plugging it into the integral form (15) yields

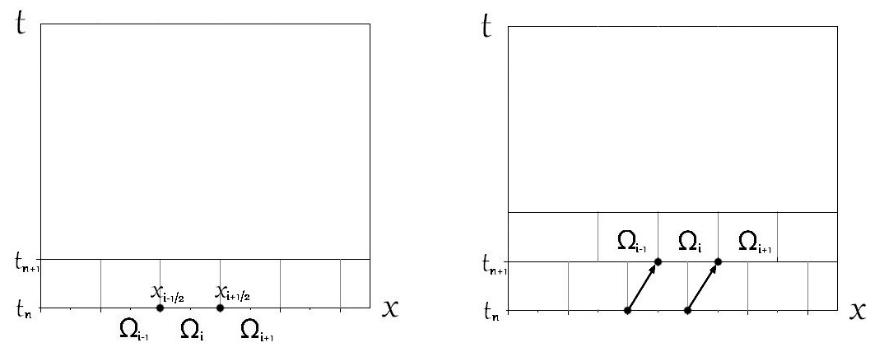
$$\overline{u}_{i+\frac{1}{2}}^{n+1} \cdot \Delta x = \frac{\overline{u}_i + \overline{u}_{i+1}}{2} \cdot \Delta x + a \cdot \overline{u}_i \cdot \Delta t - a \cdot \overline{u}_{i+1} \cdot \Delta t \tag{18}$$

Solving for $\overline{u}_{i+\frac{1}{2}}^{n+1}$ gives us

$$\overline{u}_{i+\frac{1}{2}}^{n+1} = \frac{\overline{u}_i + \overline{u}_{i+1}}{2} + a \cdot \frac{\Delta t}{\Delta x}(\overline{u}_i - \overline{u}_{i+1}). \tag{19}$$

Now that we have an expression for $\overline{u}_{i+\frac{1}{2}}^{n+1}$, the intermediate value in the next time step, the only remaining question is whether it fulfills the **TVD-property** (11). We will adjust the time step Δt so that the algorithm becomes TVD. For that we have to look at figure 5.

In (a) we see how the cell is at time t_n. At the point $x_{i+1/2}$ there is a discontinuity (compare with Figure 4). We can continue calculating as long as this discontinuity does not pass over to the next cell (b). If that were the case



(a) at time t_n (b) at time t_{n+1}

Figure 5: The cell grid from above

it could be that more quantity would go out of the cell than come in, resulting in negative values. But since we want to avoid that, we have to limit the time step. That means we have to see how long it takes for the point $x_{i+\frac{1}{2}}$ to cross the border of the cell at x_{i+1}, that is we have to see how long it takes for the point to travel the distance of $\frac{\Delta x}{2}$. This is easily calculated since we know the velocity to be a. Hence our time step can be at most

$$\Delta t \leq \frac{\Delta x}{2a} \tag{20}$$

This inequality is often expressed with a κ which limits the time step. Due to the TVD-condition the time step is often restricted further. In the case of the **piecewise constant method** with the function $f(u) = a \cdot u$, the time step limitation is

$$\Delta t = \kappa \frac{\Delta x}{2a} \qquad \text{with } k \leq 1. \tag{21}$$

The only thing remaining to show in this algorithm is that it is TVD if $\kappa \leq 1$. Therefore we have to show that $\mathrm{TV}(u_{n+1}) \leq \mathrm{TV}(u_n)$. Since I did not find a proof in my sources I set out to prove the following myself.

$$\begin{aligned}
&\mathrm{TV}(u_{n+1}) \\
&= \sum_i \left| \overline{u}^{n+1}_{i+\frac{1}{2}} - \overline{u}^{n+1}_{i-\frac{1}{2}} \right| \\
&= \sum_i \left| \frac{\overline{u}_i + \overline{u}_{i+1}}{2} + a\frac{\Delta t}{\Delta x}(\overline{u}_i - \overline{u}_{i+1}) - \frac{\overline{u}_{i-1} + \overline{u}_i}{2} - a\frac{\Delta t}{\Delta x}(\overline{u}_{i-1} - \overline{u}_i) \right|
\end{aligned}$$

Rearranging the terms $\frac{\overline{u}_i+\overline{u}_{i+1}}{2}$ and $\frac{\overline{u}_{i-1}+\overline{u}_i}{2}$ gives us

$$
\begin{aligned}
&= \sum_i \left| \frac{\overline{u}_i - \overline{u}_{i-1}}{2} + a\frac{\Delta t}{\Delta x}(\overline{u}_i - \overline{u}_{i-1}) + \frac{\overline{u}_{i+1} - \overline{u}_i}{2} - a\frac{\Delta t}{\Delta x}(\overline{u}_{i+1} - \overline{u}_i) \right| \\
&= \sum_i \left| (\overline{u}_i - \overline{u}_{i-1}) \left(\frac{1}{2} + a\frac{\Delta t}{\Delta x}\right) + (\overline{u}_{i+1} - \overline{u}_i) \left(\frac{1}{2} - a\frac{\Delta t}{\Delta x}\right) \right|
\end{aligned}
$$

Defining $\Delta\overline{u}_{i-\frac{1}{2}} = \overline{u}_i - \overline{u}_{i-1}$ and $\Delta\overline{u}_{i+\frac{1}{2}} = \overline{u}_{i+1} - \overline{u}_i$ and using the inequality $|a+b| \leq |a| + |b|$ yields

$$
\leq \sum_i \left| \Delta\overline{u}_{i-\frac{1}{2}} \left(\frac{1}{2} + a\frac{\Delta t}{\Delta x}\right) \right| + \left| \Delta\overline{u}_{i+\frac{1}{2}} \left(\frac{1}{2} - a\frac{\Delta t}{\Delta x}\right) \right|
$$

Assuming that the factors in the brackets are non-negative, we can take them out of the absolute value.

$$
= \sum_i |\Delta\overline{u}_{i-\frac{1}{2}}| \underbrace{\left(\frac{1}{2} + a\frac{\Delta t}{\Delta x}\right)}_{\text{assuming } \geq 0} + |\Delta\overline{u}_{i+\frac{1}{2}}| \underbrace{\left(\frac{1}{2} - a\frac{\Delta t}{\Delta x}\right)}_{\text{assuming } \geq 0}
$$

If we were to write the sum out and collect all the terms with the factor $\Delta\overline{u}_{i+\frac{1}{2}}$ we would see that what is left is

$$
\begin{aligned}
&= \sum_i |\Delta\overline{u}_{i+\frac{1}{2}}| \left(\frac{1}{2} - a\frac{\Delta t}{\Delta x} + \frac{1}{2} + a\frac{\Delta t}{\Delta x}\right) \\
&= \sum_i |\Delta\overline{u}_{i+\frac{1}{2}}| \\
&= \sum_i |\overline{u}_{i+1} - \overline{u}_i| \\
&= \mathrm{TV}(u_n). \qquad (22)
\end{aligned}
$$

Note that this only works since we have a finite amount of positive summands as described earlier. This means that the piecewise constant method is TVD, but only if our assumption of the factors in the bracket being larger or equal to zero is true. Thus it still remains to show that our assumption was correct. We have to show that

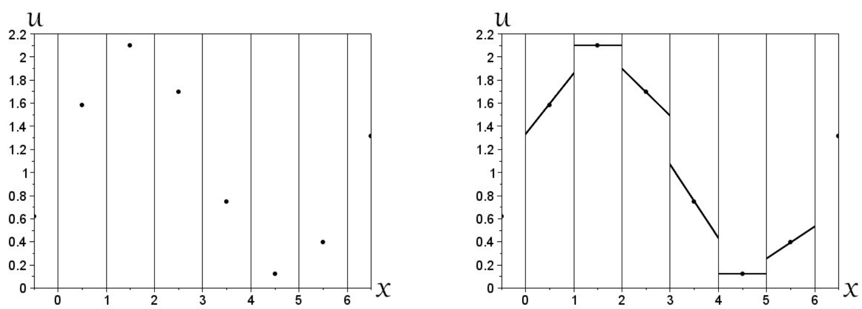
$$
0 \leq \frac{1}{2} \pm a\frac{\Delta t}{\Delta x} \qquad (23)
$$

or equivalently

$$
\pm a\frac{\Delta t}{\Delta x} \leq \frac{1}{2}. \qquad (24)
$$

If we look at our definition of Δt (21) we can see that the following holds.

$$
\pm a\frac{\Delta t}{\Delta x} \leq \left| a\frac{\Delta t}{\Delta x} \right| = \left| a \cdot \underbrace{\kappa\frac{\Delta x}{2a}}_{=\Delta t} \cdot \frac{1}{\Delta x} \right| = \left| \frac{\kappa}{2} \right| \leq \frac{1}{2} \qquad (25)
$$



(a) $\overline{u}_i$-values

(b) minmod slopes

Figure 6: The reconstruction of linear functions

Thus our assumption was correct and we have proved that the piecewise constant method is TVD if $\kappa \leq 1$.

3.2 Nessyahu-Tadmor method

The following algorithm will be a bit more complex than the previous one, because instead of using piecewise constant functions it will use piecewise linear functions. As a first step we must reconstruct a function in the x-axis. The only values given are the $\overline{u}_i$ values. The algorithm now reconstructs some linear functions on the cells Ω_i (see Figure 6). To do this the algorithm uses the *minmod* function. It is defined the following way:

$$\mathrm{minmod}(x, y) = \begin{cases} 0 & \text{if } x \cdot y \leq 0 \\ x & \text{if } |x| \leq |y| \\ y & \text{if } |y| < |x| \end{cases} \tag{26}$$

This means that whenever the input values have opposite signs, the function value is 0. However, if both x and y have the same sign, then the output of the function will be the value closer to 0.

The linear function is constructed in such a way, that the function passes through $\overline{u}_i$, and the slope is the smaller one between $\frac{\overline{u}_i - \overline{u}_{i-1}}{\Delta x} = \frac{\Delta \overline{u}_{i+1/2}}{\Delta x}$ and $\frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} = \frac{\Delta \overline{u}_{i-1/2}}{\Delta x}$. If we define the function on Ω_i to be $u_i(x)$, then we construct it the following way:

$$u_i(x) = \overline{u}_i + \frac{S_i}{\Delta x}(x - x_i) \qquad \text{where } S_i = \mathrm{minmod}(\Delta \overline{u}_{i+\frac{1}{2}}, \Delta \overline{u}_{i-\frac{1}{2}}) \tag{27}$$

If we insert $x = x_i$ into the function, we will get $\overline{u}_i$ as a result. However since we only want to calculate the integral we can shift the function by x_i to obtain

$$u_i(x) = \overline{u}_i + \frac{S_i}{\Delta x}x \qquad \text{where } S_i = \mathrm{minmod}(\Delta \overline{u}_{i+\frac{1}{2}}, \Delta \overline{u}_{i-\frac{1}{2}}) \tag{28}$$

This is done so that it is easier to calculate with the values, since one does not need to know where exactly in the x-axis the cell is located, but only that the cell boundary goes from $-\frac{\Delta x}{2}$ to $\frac{\Delta x}{2}$. If we calculate the integral in this cell, it still gives $\int_{\Omega_i} u_i(x)\mathrm{d}x = \overline{u}_i \cdot \Delta x$. This is essential since we do not want to increase the quantity in the cell with our reconstruction.

Further it needs to be remarked that our linear reconstruction of the u_i-values does not exceed the total variation of the values before reconstruction. This is crucial since an increase in total variation renders it impossible to show that the algorithm is total variation diminishing.

The next step would be to reconstruct the function in the t-axis. As discussed this algorithm uses linear functions, so the reconstruction in t-direction has to be linear too. To construct it we will use the differential equation

$$u_t + f(u)_x = 0 \tag{29}$$

or rewritten

$$\begin{aligned} u_t &= -f(u)_x \\ &= (-a \cdot u)_x \\ &= -a \cdot u_x. \end{aligned} \tag{30}$$

This means that the derivative or the slope in the t-axis is the slope in the x-axis multiplied by $-a$. Further we can assume that the function in t-axis will go through the point $\overline{u}_i$. With this information we can construct our function. I will call the function v_i.

$$\begin{aligned} v_i(t) &= \overline{u}_i + t \cdot u_t \\ &= \overline{u}_i + t \cdot (-a) \cdot u_x \\ &= \overline{u}_i - a \cdot t \cdot \frac{S_i}{\Delta x} \end{aligned} \tag{31}$$

Note that for every function $u_i(x)$ there will be a function $v_i(t)$. This function $v_i(t)$ is set so that for $t = 0$ we get $\overline{u}_i$. As in the previous algorithm, we have now both the function along the x- and the t-axis. This means that we can calculate its integrals and thus get the quantity at the next time step with the help of the integral form (12). The integral at time t_n is

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} u(x,t_n)\mathrm{d}x \\
&= \int_{x_i}^{x_{i+1/2}} u(x,t_n)\mathrm{d}x + \int_{x_{i-1/2}}^{x_{i+1}} u(x,t_n)\mathrm{d}x \\
&= \int_{0}^{\Delta x/2} u_i(x,t_n)\mathrm{d}x + \int_{-\Delta x/2}^{0} u_{i+1}(x,t_n)\mathrm{d}x \\
&= \left[\overline{u}_i \cdot x + \frac{S_i}{\Delta x}\frac{x^2}{2}\right]_0^{\Delta x/2} + \left[\overline{u}_{i+1} \cdot x + \frac{S_{i+1}}{\Delta x}\frac{x^2}{2}\right]_{-\Delta x/2}^{0} \\
&= \frac{\overline{u}_i \cdot \Delta x}{2} + \frac{S_i}{\Delta x}\frac{\Delta x^2}{8} + \frac{\overline{u}_{i+1} \cdot \Delta x}{2} - \frac{S_{i+1}}{\Delta x}\frac{\Delta x^2}{8} \\
&= \frac{\Delta x}{2}(\overline{u}_i + \overline{u}_{i+1}) + \frac{\Delta x}{8}(S_i - S_{i+1})
\end{aligned} \tag{32}$$

As a next step we have to evaluate the integrals at x_i and x_{i+1} in t-direction.

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} v_i(t)\mathrm{d}t &= \int_0^{\Delta t} \overline{u}_i - a \cdot t \cdot \frac{S_i}{\Delta x}\mathrm{d}t \\
&= \left[\overline{u}_i t - a\frac{S_i}{\Delta x}\frac{t^2}{2}\right]_0^{\Delta t} \\
&= \overline{u}_i \Delta t - a\frac{S_i}{\Delta x}\frac{\Delta t^2}{2}
\end{aligned} \tag{33}$$

and similarly

$$\int_{t_n}^{t_{n+1}} v_{i+1}(t)\mathrm{d}t = \overline{u}_{i+1}\Delta t - a\frac{S_{i+1}}{\Delta x}\frac{\Delta t^2}{2}. \tag{34}$$

Putting it all in the integral form yields

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} u(x,t_{n+1})\mathrm{d}x &= \int_{x_i}^{x_{i+1}} u(x,t_n)\mathrm{d}x + a \cdot \int_{t_n}^{t_{n+1}} v_i(t)\mathrm{d}t - a \cdot \int_{t_n}^{t_{n+1}} v_{i+1}(t)\mathrm{d}t \\
&= \frac{\Delta x}{2}(\overline{u}_i + \overline{u}_{i+1}) + \frac{\Delta x}{8}(S_i - S_{i+1}) \\
&\quad + a\left(\overline{u}_i \Delta t - a\frac{S_i}{\Delta x}\frac{\Delta t^2}{2} - \overline{u}_{i+1}\Delta t - a\frac{S_{i+1}}{\Delta x}\frac{\Delta t^2}{2}\right)
\end{aligned} \tag{35}$$

We can rewrite the LHS of the integral form as $\Delta x \cdot \overline{u}_{i+1/2}^{n+1}$ to obtain

$$\begin{aligned}
\Delta x \cdot \overline{u}_{i+1/2}^{n+1} &= \frac{\Delta x}{2}(\overline{u}_i + \overline{u}_{i+1}) + \frac{\Delta x}{8}(S_i - S_{i+1}) \\
&\quad + a\left(\overline{u}_i \Delta t - a\frac{S_i}{\Delta x}\frac{\Delta t^2}{2} - \overline{u}_{i+1}\Delta t + a\frac{S_{i+1}}{\Delta x}\frac{\Delta t^2}{2}\right) \\
\Leftrightarrow \quad \overline{u}_{i+1/2}^{n+1} &= \frac{\overline{u}_i + \overline{u}_{i+1}}{2} + \frac{S_i - S_{i+1}}{8} + a\frac{\Delta t}{\Delta x}(\overline{u}_i - \overline{u}_{i+1}) - \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}(S_i - S_{i+1})
\end{aligned} \tag{36}$$

Thus we now have an expression for $\overline{u}^{n+1}_{i+1/2}$. The only remaining thing to do in this algorithm is to limit the time step and show that with that time step the algorithm is TVD. This is done in Appendix A.2. By doing so we get a limitation for our time step, namely

$$\Delta t = \kappa \frac{\Delta x}{2a} \qquad \text{with } \kappa \leq \sqrt{2} - 1 \tag{37}$$

Refer to Nessyahu-Tadmor [3] for further reading.

4 Development of an algorithm

Now that I have listed two algorithms, I will start to analyze and compare them, and then try to develop an own improved version of the algorithm with the gained knowledge.

4.1 Comparison of the algorithms

As a first step both algorithms fit a function in the cells Ω_i, one being a constant function, and the other being a linear one. Both reconstructions preserve the quantity in the cell. In the Nessyahu-Tadmor method this linear function is constructed in such a way that the total variation of the reconstructed function is not bigger than the original total variation. Also in the Lax-Friedrich method this holds true, but here the total variation of the reconstruction is exactly as big as the original total variation. As a next step an approximation of the function is made in t-direction at the points $\overline{u}_i$. In both cases the function goes through the point $\overline{u}_i$, in the Lax-Friedrich method it is a constant function, in the Nessyahu-Tadmor method it is linear. The slope of the function is calculated using the differential form in the second algorithm. It is noteworthy that in the first algorithm the differential form holds too, since it states that the slope in t-direction is the same as the slope in x-direction, namely 0. After having reconstructed the function in x- and t-direction one can calculate the integral form. As a last step one has to show that the algorithm is TVD. To do this we have to limit the time step. If we succeed to show that the total variation does not increase with a certain time step, then we have shown that the algorithm does not produce any oscillations. So when developing an algorithm I have to consider the following steps:

1. Reconstruction of function in x-axis, with preservation of quantity
2. Limitation of the total variation in x-axis
3. Reconstruction of function in t-axis
4. Calculation of the integral form
5. Limit time step / Proof of TVD

4.2 Creating an own algorithm

In this section I will try to employ piecewise quadratic functions to improve the algorithm. To do this I will follow the steps from the previous section.

4.2.1 Reconstruction of function in x-axis

First I have to think about how I want to reconstruct the piecewise quadratic function. The function has to be reconstructed in the cell Ω_i using only the values $\overline{u}_{i-1}$, $\overline{u}_i$ and $\overline{u}_{i+1}$. Since a quadratic function has three parameters, I will have to find three conditions which our function has to fulfill. The first of these conditions is given by the fact that the quantity in the cell has to be preserved. This means that

$$\Delta x \cdot \overline{u}_i = \int_{\Omega_i} u_i(x)\mathrm{d}x \tag{38}$$

where u_i is our reconstructed quadratic function on the cell Ω_i. We have to set two more conditions for our function. We have not yet considered the values $\overline{u}_{i-1}$ and $\overline{u}_{i+1}$. When considering these values we have to find conditions which are symmetrical, this means that if we change the values $\overline{u}_{i-1}$ and $\overline{u}_{i+1}$, our function should only be mirrored but otherwise remain unchanged. If that were not the case we would give preference to a certain side which is to be avoided. After some consideration I came up with the condition that the slope at $\frac{\Delta x}{2}$ has to be the same as the slope $\frac{\overline{u}_{i+1}-\overline{u}_i}{\Delta x}$, and similarly the slope at $-\frac{\Delta x}{2}$ has to be $\frac{\overline{u}_i-\overline{u}_{i-1}}{\Delta x}$. This means that the three given conditions are the following:

- $\Delta x \cdot \overline{u}_i = \int_{\Omega_i} u_i(x)\mathrm{d}x$
- $u_i'(-\frac{\Delta x}{2}) = \frac{\overline{u}_i-\overline{u}_{i-1}}{\Delta x}$
- $u_i'(\frac{\Delta x}{2}) = \frac{\overline{u}_{i+1}-\overline{u}_i}{\Delta x}$

where

$$u_i(x) = \frac{q''}{2}x^2 + q'x + q \tag{39}$$

with q'', q' and q being the parameters of the quadratic function. Note that q, q' and q'' are different on each cell and thus require an index. For simplicity this index is omitted where it is not of necessity. Solving the conditions for the parameters yields (see Appendix A.3)

$$q = \overline{u}_i - \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24} \tag{40a}$$

$$q' = \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta x} \tag{40b}$$

$$q'' = \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{\Delta x^2} \tag{40c}$$

4.2.2 Limitation of the total variation in the x-axis

Next we have to consider whether our reconstruction has a bigger total variation. If that were the case, we would have to adapt our function so that the total variation does not increase. Recalling our definition of $\Delta\overline{u}_{1+1/2} = \overline{u}_{i+1} - \overline{u}_i$ and $\Delta\overline{u}_{i-1/2} = \overline{u}_i - \overline{u}_{i-1}$, we will consider the case when the two slopes have opposite signs, that is when $\Delta\overline{u}_{1+1/2} \cdot \Delta\overline{u}_{1-1/2} \leq 0$ (see Figure 7). This implies that $\overline{u}_i$ is an extrema (or a saddle point). If we reconstruct a quadratic function on this cell, it will either violate the extremum, leading to a bigger total variation, or it will violate the integral. So our only option is to say that our function on

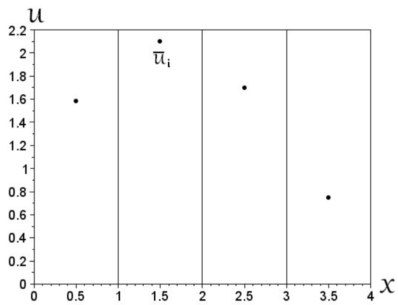


Figure 7: The value $\overline{u}_i$ is an extremum

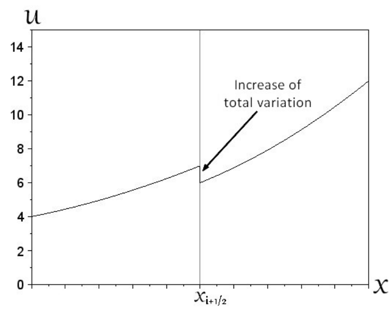


Figure 8: Increase of total variation

this cell will have to be constant, namely with the value $\overline{u}_i$.
The other case is when both slopes are positive. In this case the total variation might increase when at a point $x_{i+1/2}$ the functions u_{i-1} and u_i have two different values (see Figure 8). To limit this possible increase we will have to adapt our quadratic function. I will introduce a λ_i which will locally decrease the total variation on the cell Ω_i. This will be done the following way: First I will rewrite my function (39) as

$$u_i(x) = \lambda_i \frac{q''}{2} x^2 + \lambda_i q' x + q \tag{41}$$

The only thing that has changed is that I have inserted the λ_i in two places. The λ_i should have a value between 1 and 0. If the value is 1, then our original reconstruction remains unchanged, meaning that the total variation has not increased with our reconstruction. However if we choose a $\lambda_i < 1$ our total variation will decrease. It is easily seen that as λ_i tends to 0 our function u_i tends to a constant function with value q. This is not optimal since we want our function to tend to $\overline{u}_i$ as λ_i gets smaller. Else it would violate our condition of preservation of quantity. This means that I have to reconsider the integral condition

$$\Delta x \cdot \overline{u}_i = \int_{\Omega_i} u_i(x) \mathrm{d}x. \tag{42}$$

Recalculating q (see Appendix A.4) yields

$$q = \overline{u}_i - \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24}. \tag{43}$$

As can be seen now q also depends on λ_i. As a next step we can consider limiting our total variation with the aid of λ_i. To do that we will say that our function may not exceed the value of $\frac{\overline{u}_i - \overline{u}_{i-1}}{2}$ at $-\frac{\Delta x}{2}$, and similarly the value of $\frac{\overline{u}_{i+1} - \overline{u}_i}{2}$ may not be exceeded at $\frac{\Delta x}{2}$ (see Figure 17). Calculating the λ_i-values gives us 2 different values which are both smaller than one (see Appendix A.5), one for the right side, and one for the left. We will have to use the smaller of the values, with which we will limit the total variation of our reconstruction.

$$\lambda_i = \begin{cases} 0 & \text{when } \Delta\overline{u}_{i+1} \cdot \Delta\overline{u}_{i-1} \leq 0 \\ \min\left(3\frac{\overline{u}_{i-1}-\overline{u}_i}{2\overline{u}_{i-1}-\overline{u}_i-\overline{u}_{i+1}}, 3\frac{\overline{u}_{i+1}-\overline{u}_i}{2\overline{u}_{i+1}-\overline{u}_i-\overline{u}_{i-1}}\right) & \text{otherwise} \end{cases} \tag{44}$$

4.2.3 Reconstruction of function in t-axis

In the piecewise linear method we reconstructed the function in the t-axis with the help of the differential form, namely by calculating the slope of the function. In the case of piecewise quadratic functions the slope does not suffice but we need the second derivative too. To do this, we can differentiate the differential form (1) to obtain information about the second derivative (see Appendix A.6) and then reconstruct our function, which we will call $v_i(t)$. The function is an approximation by a Taylor-Polynomial of second order around the value $\overline{u}_i$. Doing this yields

$$v_i(t) = q - a\lambda_i q' t + a^2 \lambda_i q'' \frac{t^2}{2}. \tag{45}$$

4.2.4 Calculation of the integral form

This step is very tedious, but it does not need much explanation. That is why it is done in the Appendix A.7, and in this section only the results are presented.

$$\int_{x_i}^{x_{i+1}} u_i(x)\mathrm{d}x = \frac{\Delta x}{16}(\lambda_i\overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i\overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}) + \frac{\Delta x}{2}(\overline{u}_i + \overline{u}_{i+1}) \tag{46}$$

$$\begin{aligned} a\int_{t_n}^{t_{n+1}} v_i(t)\mathrm{d}t = \kappa\frac{\Delta x}{2}\overline{u}_i &+ (\kappa^3 - \kappa)\lambda_i\frac{\Delta x}{48}(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) \\ &- \kappa^2\lambda_i\frac{\Delta x}{16}(\overline{u}_{i+1} - \overline{u}_{i-1}) \end{aligned} \tag{47}$$

$$\begin{aligned} a\int_{t_n}^{t_{n+1}} v_{i+1}(t)\mathrm{d}t = \kappa\frac{\Delta x}{2}\overline{u}_{i+1} &+ (\kappa^3 - \kappa)\lambda_{i+1}\frac{\Delta x}{48}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1}) \\ &- \kappa^2\lambda_{i+1}\frac{\Delta x}{16}(\overline{u}_{i+2} - \overline{u}_i) \end{aligned} \tag{48}$$

This leads to the calculation of the value $\overline{u}^{n+1}_{i+1/2}$.

$$\begin{aligned} \overline{u}^{n+1}_{i+1/2} = \frac{\lambda_i\overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i\overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16} &+ \frac{\overline{u}_i + \overline{u}_{i+1}}{2} + \kappa\frac{\overline{u}_i - \overline{u}_{i+1}}{2} \\ + (\kappa^3 - \kappa)&\frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48} \\ - \kappa^2&\frac{\lambda_i\overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i\overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16} \end{aligned} \tag{49}$$

4.2.5 Limitation of time step and proof of TVD

This will be the most important step in the algorithm. Without the TVD-property the algorithm is inutile.
I will have to develop a strategy to prove that the algorithm is indeed TVD. The strategy to prove this will be very similar to the proof in the Nessyahu-Tadmor method. I will namely compare my expression of the total variation with their expression, and I will try to find similar terms. Then I will try to deal with those terms like they did (see Appendix A.8). Calculating this gives us the limitation for our time step.

$$\Delta t = \kappa \frac{\Delta x}{2a} \qquad \text{with } \kappa \leq 0.18144 \tag{50}$$

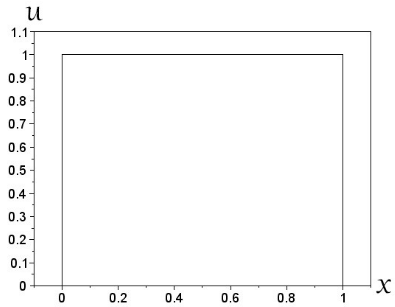


Figure 9: The function s_1

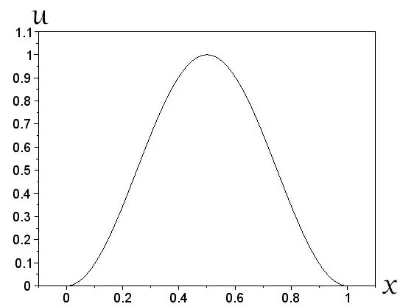


Figure 10: The function s_2

5 Comparison of the three algorithms

Now I will compare the three different algorithms. To do that I wrote a program which calculates the $\overline{u}_i$-values (see Appendix C). First I had to define a starting state $u(x_0, t)$. I chose the following two functions (see Figure 9 and 10)

$$s_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x > 1 \end{cases} \tag{51}$$

and

$$s_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin^2(x \cdot \pi) & \text{if } x \in [0,1] \\ 0 & \text{if } x > 1 \end{cases} \tag{52}$$

I had to consider functions which fulfilled property (10). Note that both functions have positive values only in the interval $[0,1]$. Next I could compare the different algorithms by looking at the different $\overline{u}_i$-values they produced after a certain amount of time. In the following page I evaluated the different algorithms with $\kappa = 0.25$, $\kappa = 0.15$, and $\kappa = 0.05$ (see Figures 11-16). I chose $\Delta x = \frac{1}{20}$. The amount of time steps was chosen in such a way so that the exact solution would be located between $x = 5$ and $x = 6$.

We see that there is a significant difference between the piecewise constant method and the piecewise linear method. However the piecewise quadratic method is only slightly better than the piecewise linear method. This means that my method does not improve the quality of solution significantly.

Instead of using the minmod function (26) in the piecewise linear method one can also use other so called **flux limiter** functions (see Nessyahu-Tadmor [3]). With these flux limiters a better result is achieved, however this comes at the cost of the unability to prove the TVD-property.

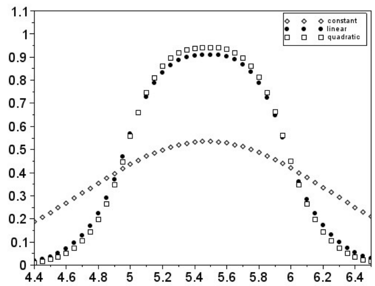


Figure 11: $s_1, \quad \kappa = 0.25$

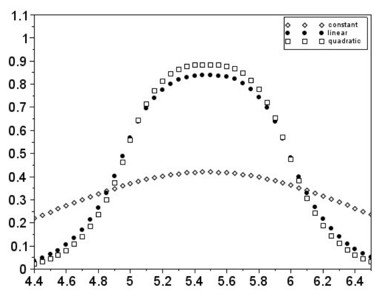


Figure 12: $s_1, \quad \kappa = 0.15$

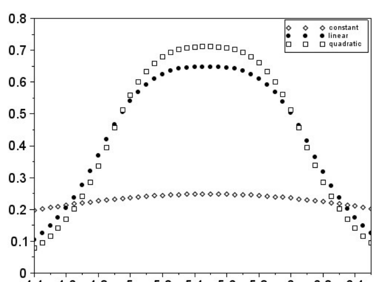


Figure 13: $s_1, \quad \kappa = 0.05$

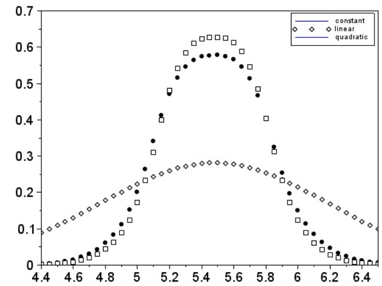


Figure 14: $s_2, \quad \kappa = 0.25$

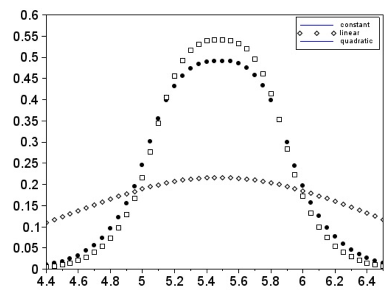


Figure 15: $s_2, \quad \kappa = 0.15$

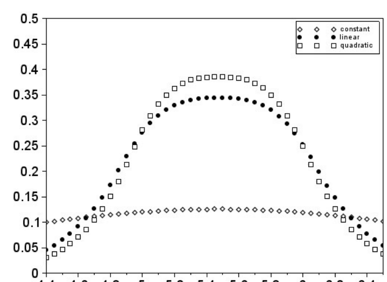


Figure 16: $s_2, \quad \kappa = 0.05$

In my algorithm I could rewrite (44) as

$$\lambda_i = \text{minmod}\left(3\frac{\overline{u}_{i-1} - \overline{u}_i}{2\overline{u}_{i-1} - \overline{u}_i - \overline{u}_{i+1}}, 3\frac{\overline{u}_{i+1} - \overline{u}_i}{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}}\right) \tag{53}$$

With this notation I have a minmod function too, and thus a better result might be achieved by using a different flux limiter function. However the analysis of other flux limiters would go beyond the scope of this essay.

6 Conclusion

In my essay I analyzed two methods and showed how they worked. I successfully managed to extend the existing algorithm, so that it works with piecewise quadratic functions. This was done under the required conditions of such an algorithm. However the extent of improvement of my algorithm was minimal. The question remains whether a different flux limiter would significantly increase the exactness of the algorithm. Further I only worked with the function $f(u) = a \cdot u$. It needs to be investigated how one can extend the existing algorithm to a more general function f.

A Calculations

A.1 Differentiating the Integral Form to obtain the Differential Form

$$\int_{x_1}^{x_2} \rho(x,t_1)\mathrm{d}x = \int_{x_1}^{x_2} \rho(x,t_0)\mathrm{d}x + \int_{t_0}^{t_1} f(\rho(x_1,t))\mathrm{d}t - \int_{t_0}^{t_1} f(\rho(x_2,t))\mathrm{d}t. \quad (54)$$

First we rewrite the equation as follows.

$$\int_{x_1}^{x_2} \rho(x,t_1)\mathrm{d}x - \int_{x_1}^{x_2} \rho(x,t_0)\mathrm{d}x = \int_{t_0}^{t_1} f(\rho(x_1,t))\mathrm{d}t - \int_{t_0}^{t_1} f(\rho(x_2,t))\mathrm{d}t. \quad (55)$$

To get the differential form, we will need to differentiate (partially) with respect to x_2 and t_1. For this purpose, we will interpret the values x_2 and t_1 as variables. In addition, I will assume the functions f and ρ to be differentiable. Let $\rho_t(x,t) = \frac{\partial \rho}{\partial t}$ and $f_x(x,t) = \frac{\partial f}{\partial x}$. Using the fundamental theorem of calculus, and the Schwarz theorem for mixed partial derivatives, we can evaluate the LHS and the RHS of the equation.
The LHS gives us

$$\begin{aligned}
&\frac{\partial^2}{\partial t_1 \partial x_2}\left(\int_{x_1}^{x_2} \rho(x,t_1)\mathrm{d}x - \int_{x_1}^{x_2} \rho(x,t_0)\mathrm{d}x\right) \\
&= \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial x_2}\int_{x_1}^{x_2} \rho(x,t_1)\mathrm{d}x - \frac{\partial}{\partial x_2}\int_{x_2}^{x_2} \rho(x,t_0)\mathrm{d}x\right) \\
&= \frac{\partial}{\partial t_1}(\rho(x_2,t_1) - \rho(x_2,t_0)) \qquad (56)
\end{aligned}$$

Rewriting this term using the fundamental theorem of calculus yields

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\underbrace{(\rho(x_2,t_1) - \rho(x_2,t_0))} \\
&= \frac{\partial}{\partial t_1}\int_{t_0}^{t_1} \rho_t(x_2,t)\mathrm{d}t \\
&= \rho_t(x_2,t_1) \qquad (57)
\end{aligned}$$

Analogously the RHS yields

$$\begin{aligned}
&\frac{\partial^2}{\partial x_2 \partial t_1}\left(\int_{t_0}^{t_1} f(\rho(x_1,t))\mathrm{d}t - \int_{t_0}^{t_1} f(\rho(x_2,t))\mathrm{d}t\right) \\
&= -f_x(\rho(x_2,t_1)) \qquad (58)
\end{aligned}$$

Note that in (56) and (58) the sequence of differentiation is different. However we can equate the two integrals according to Schwarz' Theorem (112). Equating both sides yields

$$\begin{aligned}
\rho_t(x_2,t_1) &= -f_x(\rho(x_2,t_1)) \\
\rho_t(x_2,t_1) + f_x(\rho(x_2,t_1)) &= 0 \qquad (59)
\end{aligned}$$

Since we know that this equation holds for any (x_2, t_1), we can generalize the equation to obtain the **differential form**.

$$\rho_t + f_x(\rho) = 0 \tag{60}$$

A.2 The proof of Nessyahu-Tadmor method being total variation diminishing

We want to prove that

$$\mathrm{TV}(u_{n+1}) \leq \mathrm{TV}(u_n) \tag{61}$$

Since from equation (36) I have given the expression for $\overline{u}^{n+1}_{i+1/2}$ I can write down $\mathrm{TV}(u_{n+1})$.

$$\begin{aligned}
&\mathrm{TV}(u_{n+1}) \\
&= \sum_i \left| \overline{u}^{n+1}_{i+1/2} - \overline{u}^{n+1}_{i-1/2} \right| \\
&= \sum_i \left| \frac{\overline{u}_i + \overline{u}_{i+1}}{2} - \frac{\overline{u}_{i-1} + \overline{u}_i}{2} + \frac{S_i - S_{i+1}}{8} - \frac{S_{i-1} - S_i}{8} \right. \\
&\qquad + a\frac{\Delta t}{\Delta x}(\overline{u}_i - \overline{u}_{i+1}) - a\frac{\Delta t}{\Delta x}(\overline{u}_{i-1} - \overline{u}_i) \\
&\qquad \left. - \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}(S_i - S_{i+1}) + \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}(S_{i-1} - S_i) \right|
\end{aligned}$$

Rearranging the terms and rewriting $\overline{u}_i - \overline{u}_{i-1} = \Delta\overline{u}_{i-1/2}$ and $\overline{u}_{i+1} - \overline{u}_i = \Delta\overline{u}_{i+1/2}$ gives us

$$\begin{aligned}
&= \sum_i \left| \Delta\overline{u}_{i+1/2} \cdot \left(\frac{1}{2} + \frac{1}{8}\frac{S_i - S_{i+1}}{\Delta\overline{u}_{i+1/2}} - a\frac{\Delta t}{\Delta x} - \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}\frac{1}{\Delta\overline{u}_{i+1/2}}(S_i - S_{i+1}) \right) \right. \\
&\qquad \left. + \Delta\overline{u}_{i-1/2} \cdot \left(\frac{1}{2} - \frac{1}{8}\frac{S_{i-1} - S_i}{\Delta\overline{u}_{i-1/2}} + a\frac{\Delta t}{\Delta x} + \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}\frac{1}{\Delta\overline{u}_{i-1/2}}(S_{i-1} - S_i) \right) \right|
\end{aligned}$$

Using the inequality $|a + b| \leq |a| + |b|$ we get

$$\begin{aligned}
&\leq \sum_i \left| \Delta\overline{u}_{i+1/2} \cdot \left(\frac{1}{2} + \frac{1}{8}\frac{S_i - S_{i+1}}{\Delta\overline{u}_{i+1/2}} - a\frac{\Delta t}{\Delta x} - \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}\frac{1}{\Delta\overline{u}_{i+1/2}}(S_i - S_{i+1}) \right) \right| \\
&\qquad + \left| \Delta\overline{u}_{i-1/2} \cdot \left(\frac{1}{2} - \frac{1}{8}\frac{S_{i-1} - S_i}{\Delta\overline{u}_{i-1/2}} + a\frac{\Delta t}{\Delta x} + \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}\frac{1}{\Delta\overline{u}_{i-1/2}}(S_{i-1} - S_i) \right) \right|
\end{aligned}$$

We define $E_{i+1/2} = \frac{1}{8}\frac{S_i - S_{i+1}}{\Delta\overline{u}_{i+1/2}} - a\frac{\Delta t}{\Delta x} - \frac{a^2}{2}\frac{\Delta t^2}{\Delta x^2}\frac{1}{\Delta\overline{u}_{i+1/2}}(S_i - S_{i+1})$.

$$\begin{aligned}
&\leq \sum_i \left| \Delta\overline{u}_{i+1/2} \cdot \left(\frac{1}{2} + E_{i+1/2} \right) \right| \\
&\qquad + \left| \Delta\overline{u}_{i-1/2} \cdot \left(\frac{1}{2} - E_{i-1/2} \right) \right|
\end{aligned}$$

We will assume that $|E_i| \leq \frac{1}{2}$, which means that the factor $\frac{1}{2} \pm E_i$ is non-negative and that we can take it out of the absolute value.

$$
\begin{aligned}
&= \sum_i \left|\Delta \overline{u}_{i+1/2}\right| \left(\frac{1}{2} + E_{i+1/2}\right) \\
&\qquad + \left|\Delta \overline{u}_{i-1/2}\right| \left(\frac{1}{2} - E_{i-1/2}\right)
\end{aligned}
$$

Writing the sum out and collecting only the $\Delta \overline{u}_{i+1/2}$ terms in a summand leaves us with

$$
\begin{aligned}
&= \sum_i \left|\Delta \overline{u}_{i+1/2}\right| \left(\frac{1}{2} + E_{i+1/2} + \frac{1}{2} - E_{i+1/2}\right) \\
&= \sum_i \left|\Delta \overline{u}_{i+1/2}\right| \\
&= \sum_i \left|\overline{u}_{i+1} - \overline{u}_i\right| \\
&= \mathrm{TV}(u_n) \qquad (62)
\end{aligned}
$$

But this only works if our assumption of $|E_i| \leq \frac{1}{2}$ is true, which means that we still have to prove that.

$$
|E_{i+1/2}| = \left| \frac{1}{8} \frac{S_i - S_{i+1}}{\Delta \overline{u}_{i+1/2}} - a \frac{\Delta t}{\Delta x} - \frac{a^2}{2} \frac{\Delta t^2}{\Delta x^2} \frac{1}{\Delta \overline{u}_{i+1/2}} (S_i - S_{i+1}) \right|
$$

First I will substitute $\Delta t = \kappa \frac{\Delta x}{2a}$.

$$
= \left| \frac{1}{8} \frac{S_i - S_{i+1}}{\Delta \overline{u}_{i+1/2}} - \frac{\kappa}{2} - \frac{\kappa^2}{8} \frac{1}{\Delta \overline{u}_{i+1/2}} (S_i - S_{i+1}) \right|
$$

Then I will use the inequality $|a + b| \leq |a| + |b|$ multiple times to get

$$
\begin{aligned}
&\leq \left| \frac{1}{8} \frac{S_i - S_{i+1}}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{\kappa}{2} \right| + \left| \frac{\kappa^2}{8} \frac{S_i - S_{i+1}}{\Delta \overline{u}_{i+1/2}} \right| \\
&\leq \left| \frac{1}{8} \frac{S_i}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{1}{8} \frac{S_{i+1}}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{\kappa}{2} \right| + \left| \frac{\kappa^2}{8} \frac{S_i}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{\kappa^2}{8} \frac{S_{i+1}}{\Delta \overline{u}_{i+1/2}} \right|
\end{aligned}
$$

Next we use the following inequalities,
$|S_i| = |\mathrm{minmod}(\Delta \overline{u}_{i+1/2}, \Delta \overline{u}_{i-1/2})| \leq |\Delta \overline{u}_{i+1/2}|$ and
$|S_{i+1}| = |\mathrm{minmod}(\Delta \overline{u}_{i+1/2}, \Delta \overline{u}_{i+3/2})| \leq |\Delta \overline{u}_{i+1/2}|$, to obtain

$$
\begin{aligned}
&\leq \left| \frac{1}{8} \frac{\Delta \overline{u}_{i+1/2}}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{1}{8} \frac{\Delta \overline{u}_{i+1/2}}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{\kappa}{2} \right| + \left| \frac{\kappa^2}{8} \frac{\Delta \overline{u}_{i+1/2}}{\Delta \overline{u}_{i+1/2}} \right| + \left| \frac{\kappa^2}{8} \frac{\Delta \overline{u}_{i+1/2}}{\Delta \overline{u}_{i+1/2}} \right| \\
&= \frac{1}{4} + \frac{\kappa}{2} + \frac{\kappa^2}{4}
\end{aligned}
\qquad (63)
$$

This expression still has to be smaller than $\frac{1}{2}$. Solving for κ yields.

$$\begin{aligned} \frac{1}{4} + \frac{\kappa}{2} + \frac{\kappa^2}{4} &\leq \frac{1}{2} \\ 1 + 2\kappa + \kappa^2 &\leq 2 \\ (\kappa + 1)^2 &\leq 2 \end{aligned}$$

Since $\kappa > 0$ we can take the root.

$$\begin{aligned} \kappa + 1 &\leq \sqrt{2} \\ \kappa &\leq \sqrt{2} - 1 \end{aligned} \tag{64}$$

This means that we have proved that this method is TVD if $\kappa \leq \sqrt{2} - 1$.

A.3 Reconstruction of quadratic function in x-axis

We have given the system of equations

$$\Delta x \cdot \overline{u}_i = \int_{\Omega_i} u_i(x) \mathrm{d}x \tag{65a}$$

$$u_i'(-\frac{\Delta x}{2}) = \frac{\overline{u}_i - \overline{u}_{i-1}}{\Delta x} \tag{65b}$$

$$u_i'(\frac{\Delta x}{2}) = \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} \tag{65c}$$

with $u_i(x) = \frac{q''}{2} x^2 + q' x + q$. This means that $u_i'(x) = q'' x + q'$. Solving (65c) for q' yields

$$\begin{aligned} u_i'(\frac{\Delta x}{2}) = q'' \frac{\Delta x}{2} + q' &= \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} \\ \Rightarrow \qquad q' &= \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} - q'' \frac{\Delta x}{2} \end{aligned} \tag{66}$$

Plugging this into (65b) yields

$$\begin{aligned} \frac{\overline{u}_i - \overline{u}_{i-1}}{\Delta x} &= -q'' \frac{\Delta x}{2} + q' \\ &= -q'' \frac{\Delta x}{2} + \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} - q'' \frac{\Delta x}{2} \\ &= -q'' \Delta x + \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} \\ q'' \Delta x &= \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} - \frac{\overline{u}_i - \overline{u}_{i-1}}{\Delta x} \\ \Rightarrow \qquad q'' &= \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{\Delta x^2} \end{aligned} \tag{67}$$

Put this back into (65c) to obtain

$$\begin{aligned} q' &= \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} - q'' \frac{\Delta x}{2} \\ &= \frac{\overline{u}_{i+1} - \overline{u}_i}{\Delta x} - \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{\Delta x^2} \cdot \frac{\Delta x}{2} \\ &= \frac{2\overline{u}_{i+1} - 2\overline{u}_i - \overline{u}_{i+1} - \overline{u}_{i-1} + 2\overline{u}_i}{2\Delta x} \\ \Rightarrow \qquad q' &= \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta x} \end{aligned} \tag{68}$$

Finally solving (65a) gives us

$$\begin{aligned} \Delta x \cdot \overline{u}_i &= \int_{\Omega_i} u_i(x) \mathrm{d}x \\ &= \int_{-\Delta x/2}^{\Delta x/2} u_i(x) \mathrm{d}x \\ &= \left[\frac{q''}{6} x^3 + \frac{q'}{2} x^2 + qx \right]_{-\Delta x/2}^{\Delta x/2} \\ &= \frac{q'' \Delta x^3}{48} + \frac{q' \Delta x^2}{8} + \frac{q\Delta x}{2} + \frac{q'' \Delta x^3}{48} - \frac{q' \Delta x^2}{8} + \frac{q\Delta x}{2} \\ &= \frac{q'' \Delta x^3}{24} + q\Delta x \\ \Delta x \cdot \overline{u}_i &= \frac{q'' \Delta x^3}{24} + q\Delta x \\ \Leftrightarrow \qquad \overline{u}_i &= \frac{q'' \Delta x^2}{24} + q \end{aligned}$$

or equivalently

$$q = \overline{u}_i - \frac{q'' \Delta x^2}{24} \tag{69}$$

A.4 Reconsidering the integral

The parameters q'' and q' will remain unchanged, however they will be tuned by a factor λ_i. That means that our function will gradually become a constant function as λ_i approaches 0. Since the function changes with the λ_i we will have to reevaluate whether the integral still holds, and possibly adapt it. So looking at condition (38), we see that

$$\begin{aligned}
\Delta x \cdot \overline{u}_i &= \int_{\Omega_i} u_i(x)\mathrm{d}x \\
&= \int_{-\Delta x/2}^{\Delta x/2} u_i(x)\mathrm{d}x \\
&= \left[\lambda_i \frac{q''}{6}x^3 + \lambda_i \frac{q'}{2}x^2 + qx\right]_{-\Delta x/2}^{\Delta x/2} \\
&= \lambda_i \frac{q''\Delta x^3}{48} + \lambda_i \frac{q'\Delta x^2}{8} + \frac{q\Delta x}{2} + \lambda_i \frac{q''\Delta x^3}{48} - \lambda_i \frac{q'\Delta x^2}{8} + \frac{q\Delta x}{2} \\
&= \lambda_i \frac{q''\Delta x^3}{24} + q\Delta x \\
\Delta x \cdot \overline{u}_i &= \lambda_i \frac{q''\Delta x^3}{24} + q\Delta x \\
\Leftrightarrow \qquad \overline{u}_i &= \lambda_i \frac{q''\Delta x^2}{24} + q
\end{aligned} \tag{70}$$

or equivalently

$$\begin{aligned}
q &= \overline{u}_i - \lambda_i \frac{q''\Delta x^2}{24} \\
&= \overline{u}_i - \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24}.
\end{aligned} \tag{71}$$

A.5 Calculation of the lambda values

We have to limit the function at the points $-\frac{\Delta x}{2}$ and $\frac{\Delta x}{2}$. I will begin with the right side first, namely at the point $\frac{\Delta x}{2}$. We said that the function value at that point may not exceed $\frac{\overline{u}_i + \overline{u}_{i+1}}{2}$ (see Figure 17). But we have to distinguish between the cases where the slope is positive and the one where it is negative. I will first begin with the case that the slope is positive. That means that $\Delta\overline{u}_{i+1/2} > 0$. So our restriction can be formulated as

$$\frac{\overline{u}_i + \overline{u}_{i+1}}{2} \geq u_i(\frac{\Delta x}{2}) \tag{72}$$

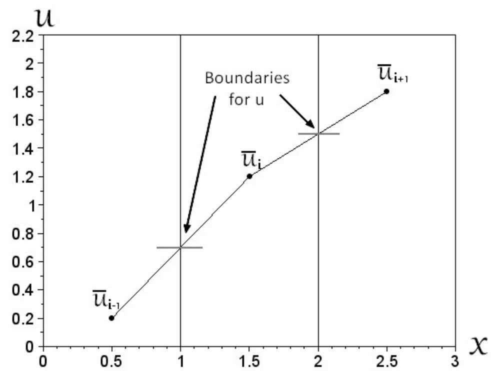


Figure 17: The boundaries of the function on the cell Ω_i

Calculating $u_i(\frac{\Delta x}{2})$ yields

$$\begin{aligned}
u_i(\frac{\Delta x}{2}) &= q + \lambda_i q' \frac{\Delta x}{2} + \lambda_i \frac{q''}{2}\frac{\Delta x^2}{4} \\
&= \overline{u}_i - \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24} + \lambda_i \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta x}\frac{\Delta x}{2} + \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{2 \cdot \Delta x^2}\frac{\Delta x^2}{4} \\
&= \overline{u}_i + \lambda_i \left(-\frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24} + \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{4} + \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{8} \right) \\
&= \overline{u}_i + \lambda_i \left(-\frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24} + \frac{6\overline{u}_{i+1} - 6\overline{u}_{i-1}}{24} + \frac{3\overline{u}_{i+1} + 3\overline{u}_{i-1} - 6\overline{u}_i}{24} \right) \\
&= \overline{u}_i + \lambda_i \left(\frac{8\overline{u}_{i+1} - 4\overline{u}_i - 4\overline{u}_{i-1}}{24} \right) \\
&= \overline{u}_i + \lambda_i \left(\frac{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}}{6} \right) \qquad (73)
\end{aligned}$$

Now we can solve (72) for λ_i.

$$\begin{aligned}
\frac{\overline{u}_i + \overline{u}_{i+1}}{2} &\geq u_i(\frac{\Delta x}{2}) \\
\frac{\overline{u}_i + \overline{u}_{i+1}}{2} &\geq \overline{u}_i + \lambda_i \left(\frac{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}}{6} \right) \\
\overline{u}_i + \overline{u}_{i+1} &\geq 2\overline{u}_i + \lambda_i \left(\frac{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}}{3} \right) \\
\overline{u}_{i+1} - \overline{u}_i &\geq \lambda_i \left(\frac{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}}{3} \right) \\
3(\overline{u}_{i+1} - \overline{u}_i) &\geq \lambda_i (2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}) \qquad (74)
\end{aligned}$$

The next step would be to take the factor on the RHS to the left, but for that we have to consider that it could be negative, which in turn could reverse the inequality sign. But we can easily verify that the factor is positive. Since we know that $\Delta\overline{u}_{i+1/2}$ is positive, we can assume that $\Delta\overline{u}_{i-1/2}$ is positive too. If that were not the case λ_i would be 0 consequently. This implies that

$$2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1} = 2(\overline{u}_{i+1} - \overline{u}_i) + (\overline{u}_i - \overline{u}_{i-1}) = 2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2} > 0 \quad (75)$$

That means we can take the factor in (74) to the left to obtain

$$\begin{aligned} 3(\overline{u}_{i+1} - \overline{u}_i) &\geq \lambda_i(2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}) \\ 3\frac{\overline{u}_{i+1} - \overline{u}_i}{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}} &\geq \lambda_i \end{aligned} \quad (76)$$

We can calculate the restriction for a negative slope in a similar fashion, because $u_i(\frac{\Delta x}{2})$ will still give the same value as in (73). The only difference is that now $\Delta\overline{u}_{i+1/2}$ and $\Delta\overline{u}_{i-1/2}$ are both negative. So our restriction (with same steps as in (74)) simplifies to

$$\begin{aligned} \frac{\overline{u}_i + \overline{u}_{i+1}}{2} &\leq u_i\left(\frac{\Delta x}{2}\right) \\ \frac{\overline{u}_i + \overline{u}_{i+1}}{2} &\leq \overline{u}_i + \lambda_i\left(\frac{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}}{6}\right) \\ 3(\overline{u}_{i+1} - \overline{u}_i) &\leq \lambda_i(2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}) \end{aligned} \quad (77)$$

Since the factor on the RHS is now negative, the inequality sign changes when taking the factor to the left, yielding

$$3\frac{\overline{u}_{i+1} - \overline{u}_i}{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}} \geq \lambda_i \quad (78)$$

This is the exact same as in (76).

Now I have to calculate the restriction for the point $-\frac{\Delta x}{2}$. Since this situation is symmetric to the situation at point $\frac{\Delta x}{2}$ I will use a simple trick to solve this problem. If I change the values of $\overline{u}_{i+1}$ and $\overline{u}_{i-1}$ I will have the same situation as before, but instead of at the point $-\frac{\Delta x}{2}$ it will be at the point $\frac{\Delta x}{2}$. Since we have already calculated the solution to this one, we can simply switch back the values $\overline{u}_{i+1}$ and $\overline{u}_{i-1}$ to get the desired restriction at $-\frac{\Delta x}{2}$. This means that the restriction there is

$$3\frac{\overline{u}_{i-1} - \overline{u}_i}{2\overline{u}_{i-1} - \overline{u}_i - \overline{u}_{i+1}} \geq \lambda_i \quad (79)$$

Now the question is which of the two λ_i I have to take. The answer is the smaller one. Both conditions must be fulfilled, and that can only be achieved by taking the smaller λ_i value. However if both of these values are greater than 1, then we have to take $\lambda_i = 1$, since we want λ_i to be in the interval $[0, 1]$. But as the following calculation shows, that can not be the case.

$$3\frac{\overline{u}_{i+1}-\overline{u}_i}{2\overline{u}_{i+1}-\overline{u}_i-\overline{u}_{i-1}} \leq 1 \qquad 3\frac{\overline{u}_{i-1}-\overline{u}_i}{2\overline{u}_{i-1}-\overline{u}_i-\overline{u}_{i+1}} \leq 1$$

$$3\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}} \leq 1 \qquad 3\frac{\Delta\overline{u}_{i-1/2}}{2\Delta\overline{u}_{i-1/2}+\Delta\overline{u}_{i+1/2}} \leq 1$$

$$3\Delta\overline{u}_{i+1/2} \leq 2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2} \qquad 3\Delta\overline{u}_{i-1/2} \leq 2\Delta\overline{u}_{i-1/2}+\Delta\overline{u}_{i+1/2}$$

$$\Delta\overline{u}_{i+1/2} \leq \Delta\overline{u}_{i-1/2} \qquad \Delta\overline{u}_{i-1/2} \leq \Delta\overline{u}_{i+1/2} \tag{80}$$

Since one of the above inequalities will be true, it implies that one of the λ_i values will be less or equal 1. Note that in the above equalities it is assumed that $\Delta\overline{u}_{i+1/2}$ and $\Delta\overline{u}_{i-1/2}$ are both positive. This follows from the fact that else the λ_i-value would be 0 to avoid an increase in the total variation (see Figure 7). The other case is when both of the values are negative. In that case (80) would have the inequality sign in the other direction, but one of the two inequalities would still be true. That means that λ_i will be defined the following way.

$$\lambda_i = \begin{cases} 0 & \text{when } \Delta\overline{u}_{i+1}\cdot\Delta\overline{u}_{i-1} \leq 0 \\ \min\left(3\frac{\overline{u}_{i-1}-\overline{u}_i}{2\overline{u}_{i-1}-\overline{u}_i-\overline{u}_{i+1}}, 3\frac{\overline{u}_{i+1}-\overline{u}_i}{2\overline{u}_{i+1}-\overline{u}_i-\overline{u}_{i-1}}\right) & \text{otherwise} \end{cases} \tag{81}$$

A.6 Calculation of slopes in t-direction

If we want to reconstruct the function in t-direction we will need the first and second derivative at the point $\overline{u}_i$. The first derivative is easily given by (1), namely

$$u_t = -a \cdot u_x \tag{82}$$

Now we want to have the second derivative. We can differentiate (1) to obtain

$$u_{tt} = -a \cdot u_{xt} \tag{83}$$

Note that I am using the notation $\frac{\partial^2 u}{\partial t \partial x} = u_{xt}$. The problem with (83) is that we dont know the value of u_{xt}. But we know from Theorem (112) that $u_{xt} = u_{tx}$, if both u_{xt} and u_{tx} are continuous in an open disk R containing our x and t. We assume that to be true. So if we differentiate (82) with respect to x we get the value of u_{tx}.

$$u_{tx} = -a \cdot u_{xx} \tag{84}$$

Here we can calculate u_{xx}. If we plug (84) into (83) we obtain

$$\begin{aligned} u_{tt} &= -a \cdot u_{xt} \\ &= -a \cdot u_{tx} \\ &= -a \cdot (-a \cdot u_{xx}) \\ &= a^2 \cdot u_{xx} \end{aligned} \tag{85}$$

Since we know the first and second derivative, we can reconstruct a quadratic function around the point x_i, which has the value q. I will call this reconstruction $v_i(t)$.

$$v_i(t) = q + u_t \cdot t + \frac{u_{tt}}{2} \cdot t^2 \tag{86}$$

All that remains now is to calculate the exact values of u_t and u_{tt}.

$$\begin{aligned} u_t &= -a \cdot u_x \\ &= -a \cdot u_i'(0) \\ &= -a \cdot (\lambda_i q'' \cdot 0 + \lambda_i q') \\ &= -a\lambda_i q' \end{aligned} \tag{87}$$

$$\begin{aligned} u_{tt} &= a^2 \cdot u_{xx} \\ &= a^2 \cdot u_i''(0) \\ &= a^2 \lambda_i q'' \end{aligned} \tag{88}$$

This means our function in t-axis is

$$v_i(t) = q - a\lambda_i q' \cdot t + a^2 \lambda_i q'' \cdot \frac{t^2}{2} \tag{89}$$

A.7 Calculation of the integral form

First I will calculate the integral in the x-axis.

$$\int_{x_i}^{x_{i+1}} u(x, t_n)\mathrm{d}x = \int_0^{\Delta x/2} u_i(x)\mathrm{d}x + \int_{-\Delta x/2}^{0} u_{i+1}(x)\mathrm{d}x \tag{90}$$

The first integral is equivalent to

$$\begin{aligned} \int_0^{\Delta x/2} u_i(x)\mathrm{d}x &= \left[\frac{1}{6}\lambda_i q_i'' x^3 + \frac{1}{2}\lambda_i q_i' x^2 + q_i x\right]_0^{\Delta x/2} \\ &= \frac{1}{48}\lambda_i q_i'' \Delta x^3 + \frac{1}{8}\lambda_i q_i' \Delta x^2 + \frac{1}{2} q_i \Delta x \\ &= \frac{\Delta x^3}{48}\lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{\Delta x^2} + \frac{\Delta x^2}{8}\lambda_i \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta x} \\ &\quad + \frac{\Delta x}{2}\left(\overline{u}_i - \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24}\right) \\ &= \frac{\Delta x}{48}\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) + \frac{\Delta x}{16}\lambda_i(\overline{u}_{i+1} - \overline{u}_{i-1}) \\ &\quad - \frac{\Delta x}{48}\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) + \frac{\Delta x}{2}\overline{u}_i \\ &= \frac{\Delta x}{16}\lambda_i(\overline{u}_{i+1} - \overline{u}_{i-1}) + \frac{\Delta x}{2}\overline{u}_i \end{aligned} \tag{91}$$

Similarly the second integral yields

$$
\begin{aligned}
\int_{-\Delta x/2}^{0} u_{i+1}(x)\mathrm{d}x &= \left[\frac{1}{6}\lambda_{i+1}q''_{i+1}x^3 + \frac{1}{2}\lambda_{i+1}q'_{i+1}x^2 + q_{i+1}x\right]_{-\Delta x/2}^{0} \\
&= \frac{1}{48}\lambda_{i+1}q''_{i+1}\Delta x^3 - \frac{1}{8}\lambda_{i+1}q'_{i+1}\Delta x^2 + \frac{1}{2}q_{i+1}\Delta x \\
&= \frac{\Delta x^3}{48}\lambda_{i+1}\frac{\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1}}{\Delta x^2} - \frac{\Delta x^2}{8}\lambda_{i+1}\frac{\overline{u}_{i+1} - \overline{u}_i}{2\Delta x} \\
&\quad - \frac{\Delta x}{2}\left(\overline{u}_{i+1} - \frac{\overline{u}_{i+1} + \overline{u}_i - 2\overline{u}_{i+1}}{24}\right) \\
&= \frac{\Delta x}{48}\lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1}) - \frac{\Delta x}{16}(\overline{u}_{i+2} - \overline{u}_i) \\
&\quad - \frac{\Delta x}{48}\lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1}) + \frac{\Delta x}{2}\overline{u}_{i+1} \\
&= -\frac{\Delta x}{16}\lambda_{i+1}(\overline{u}_{i+2} - \overline{u}_i) + \frac{\Delta x}{2}\overline{u}_{i+1}
\end{aligned} \tag{92}
$$

Adding up (91) and (92) yields

$$
\int_{x_i}^{x_{i+1}} u(x,t_n)\mathrm{d}x = \frac{\Delta x}{16}\big(\lambda_i(\overline{u}_{i+1}-\overline{u}_{i-1})-\lambda_{i+1}(\overline{u}_{i+2}-\overline{u}_i)\big)+\frac{\Delta x}{2}(\overline{u}_i+\overline{u}_{i+1}) \tag{93}
$$

Next I will calculate the integral in t-direction.

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} u(x_i,t)\mathrm{d}t &= \int_0^{\Delta t} v_i(t)\mathrm{d}t \\
&= \int_0^{\Delta t} q_i - a\lambda_i q'_i t + \frac{1}{2}a^2\lambda_i q''_i t^2 \mathrm{d}t \\
&= \left[q_i t - \frac{1}{2}a\lambda_i q'_i t^2 + \frac{1}{6}a^2\lambda_i q''_i t^3\right]_0^{\Delta t} \\
&= q_i\Delta t - \frac{1}{2}a\lambda_i q'_i\Delta t^2 + \frac{1}{6}a^2\lambda_i q''_i\Delta t^3 \\
&= \left(\overline{u}_i - \lambda_i\frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24}\right)\Delta t - \frac{1}{2}a\lambda_i\frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta x}\Delta t^2 \\
&\quad + \frac{1}{6}a^2\lambda_i\frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{\Delta x^2}\Delta t^3
\end{aligned}
$$

Setting our time step as $\Delta t = \kappa \frac{\Delta x}{2a}$ (see (21)) yields

$$\begin{aligned}
&= \left(\overline{u}_i - \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{24}\right) \kappa \frac{\Delta x}{2a} - \frac{1}{2} a \lambda_i \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta x} \kappa^2 \frac{\Delta x^2}{4a^2} \\
&\quad + \frac{1}{6} a^2 \lambda_i \frac{\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i}{\Delta x^2} \kappa^3 \frac{\Delta x^3}{8a^3} \\
&= \kappa \frac{\Delta x}{2a} \overline{u}_i - \kappa \lambda_i \frac{\Delta x}{48a} (\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \kappa^2 \lambda_i \frac{\Delta x}{16a} (\overline{u}_{i+1} - \overline{u}_{i-1}) \\
&\quad + \kappa^3 \lambda_i \frac{\Delta x}{48a} (\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i)
\end{aligned} \tag{94}$$

Since we want to have this integral multiplied by the velocity we get

$$\begin{aligned}
a \int_{t_n}^{t_{n+1}} u(x_i, t) \mathrm{d}t = \kappa \frac{\Delta x}{2} \overline{u}_i + (\kappa^3 - \kappa) \lambda_i \frac{\Delta x}{48} (\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) \\
- \kappa^2 \lambda_i \frac{\Delta x}{16} (\overline{u}_{i+1} - \overline{u}_{i-1})
\end{aligned} \tag{95}$$

By incrementing the index in (95) we get

$$\begin{aligned}
a \int_{t_n}^{t_{n+1}} u(x_{i+1}, t) \mathrm{d}t = \kappa \frac{\Delta x}{2} \overline{u}_{i+1} + (\kappa^3 - \kappa) \lambda_{i+1} \frac{\Delta x}{48} (\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1}) \\
- \kappa^2 \lambda_{i+1} \frac{\Delta x}{16} (\overline{u}_{i+2} - \overline{u}_i)
\end{aligned} \tag{96}$$

A.8 Proof of TVD

We have the expression for $\overline{u}_{i+1/2}^{n+1}$, namely

$$\begin{aligned}
\overline{u}_{i+1/2}^{n+1} = \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1} \overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1} \overline{u}_{i+2}}{16} + \frac{\overline{u}_i + \overline{u}_{i+1}}{2} + \kappa \frac{\overline{u}_i - \overline{u}_{i+1}}{2} \\
+ (\kappa^3 - \kappa) \frac{\lambda_i (\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1} (\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48} \\
- \kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1} \overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1} \overline{u}_{i+2}}{16}
\end{aligned} \tag{97}$$

Thus we can calculate the total variation in the $(n+1)$-th time step.

$$
\begin{aligned}
&\mathrm{TV}(u_{n+1}) \\
&= \sum_i |\overline{u}_{i+1/2}^{n+1} - \overline{u}_{i-1/2}^{n+1}| \\
&= \sum_i \Bigg| \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16} + \frac{\overline{u}_i + \overline{u}_{i+1}}{2} + \kappa \frac{\overline{u}_i - \overline{u}_{i+1}}{2} \\
&\quad + (\kappa^3 - \kappa) \frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48} \\
&\quad - \kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16} \\
&\quad - \frac{\lambda_{i-1}\overline{u}_i + \lambda_i \overline{u}_{i-1} - \lambda_{i-1}\overline{u}_{i-2} - \lambda_i \overline{u}_{i+1}}{16} - \frac{\overline{u}_{i-1} + \overline{u}_i}{2} - \kappa \frac{\overline{u}_{i-1} - \overline{u}_i}{2} \\
&\quad - (\kappa^3 - \kappa) \frac{\lambda_{i-1}(\overline{u}_i + \overline{u}_{i-2} - 2\overline{u}_{i-1}) - \lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i)}{48} \\
&\quad + \kappa^2 \frac{\lambda_{i-1}\overline{u}_i + \lambda_i \overline{u}_{i-1} - \lambda_{i-1}\overline{u}_{i-2} - \lambda_i \overline{u}_{i+1}}{16} \Bigg|
\end{aligned}
$$

Rearranging the terms $\frac{\overline{u}_i + \overline{u}_{i+1}}{2}$ and $\frac{\overline{u}_{i-1} + \overline{u}_i}{2}$ yields

$$
\begin{aligned}
&= \sum_i \Bigg| \frac{\overline{u}_{i+1} - \overline{u}_i}{2} + \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16} + \kappa \frac{\overline{u}_i - \overline{u}_{i+1}}{2} \\
&\quad + (\kappa^3 - \kappa) \frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48} \\
&\quad - \kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16} \\
&\quad + \frac{\overline{u}_i - \overline{u}_{i-1}}{2} - \frac{\lambda_{i-1}\overline{u}_i + \lambda_i \overline{u}_{i-1} - \lambda_{i-1}\overline{u}_{i-2} - \lambda_i \overline{u}_{i+1}}{16} - \kappa \frac{\overline{u}_{i-1} - \overline{u}_i}{2} \\
&\quad - (\kappa^3 - \kappa) \frac{\lambda_{i-1}(\overline{u}_i + \overline{u}_{i-2} - 2\overline{u}_{i-1}) - \lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i)}{48} \\
&\quad + \kappa^2 \frac{\lambda_{i-1}\overline{u}_i + \lambda_i \overline{u}_{i-1} - \lambda_{i-1}\overline{u}_{i-2} - \lambda_i \overline{u}_{i+1}}{16} \Bigg|
\end{aligned}
$$

Similarly to the proof of Nessyahu-Tadmor, I will rewrite $\overline{u}_i - \overline{u}_{i-1} = \Delta\overline{u}_{i-1/2}$ and $\overline{u}_{i+1} - \overline{u}_i = \Delta\overline{u}_{i+1/2}$ and factor these out.

$$\begin{aligned}
= \sum_i \Bigg| & \Delta\overline{u}_{i+1/2} \Bigg(\frac{1}{2} + \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta\overline{u}_{i+1/2})} + \kappa \frac{1}{2} \\
& + (\kappa^3 - \kappa) \frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48(\Delta\overline{u}_{i+1/2})} \\
& - \kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta\overline{u}_{i+1/2})} \Bigg) \\
& + \Delta\overline{u}_{i-1/2} \Bigg(\frac{1}{2} - \frac{\lambda_{i-1}\overline{u}_i + \lambda_i \overline{u}_{i-1} - \lambda_{i-1}\overline{u}_{i-2} - \lambda_i \overline{u}_{i+1}}{16(\Delta\overline{u}_{i-1/2})} - \kappa \frac{1}{2} \\
& - (\kappa^3 - \kappa) \frac{\lambda_{i-1}(\overline{u}_i + \overline{u}_{i-2} - 2\overline{u}_{i-1}) - \lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i)}{48(\Delta\overline{u}_{i-1/2})} \\
& + \kappa^2 \frac{\lambda_{i-1}\overline{u}_i + \lambda_i \overline{u}_{i-1} - \lambda_{i-1}\overline{u}_{i-2} - \lambda_i \overline{u}_{i+1}}{16(\Delta\overline{u}_{i-1/2})} \Bigg) \Bigg|
\end{aligned}$$

Next I will define the terms in the brackets to be $\frac{1}{2} + E_{i+1/2}$ and $\frac{1}{2} - E_{i-1/2}$ to simplify.

$$\begin{aligned}
&= \sum_i \left| \Delta\overline{u}_{i+1/2} \cdot \left(\frac{1}{2} + E_{i+1/2} \right) + \Delta\overline{u}_{i-1/2} \cdot \left(\frac{1}{2} - E_{i-1/2} \right) \right| \\
&\leq \sum_i \left| \Delta\overline{u}_{i+1/2} \cdot \left(\frac{1}{2} + E_{i+1/2} \right) \right| + \left| \Delta\overline{u}_{i-1/2} \cdot \left(\frac{1}{2} - E_{i-1/2} \right) \right|
\end{aligned}$$

Assuming the values in the brackets are non-negative I take them out of the absolute value.

$$\leq \sum_i |\Delta\overline{u}_{i+1/2}| \left(\frac{1}{2} + E_{i+1/2} \right) + |\Delta\overline{u}_{i-1/2}| \cdot \left(\frac{1}{2} - E_{i-1/2} \right)$$

As in the previous proof, I will write the sum out and collect all terms with $\Delta\overline{u}_{i+1/2}$ to obtain

$$\begin{aligned}
&= \sum_i |\Delta\overline{u}_{i+1/2}| \cdot \left(\frac{1}{2} + E_{i+1/2} + \frac{1}{2} - E_{i+1/2} \right) \\
&= \sum_i |\Delta\overline{u}_{i+1/2}| \cdot \left(\frac{1}{2} + \frac{1}{2} \right) \\
&= \sum_i |\Delta\overline{u}_{i+1/2}| \\
&= \sum_i |\overline{u}_{i+1} - \overline{u}_i| \\
&= \mathrm{TV}(u_n) \qquad (98)
\end{aligned}$$

Again as before I am left to prove that the values $|E_i| \leq \frac{1}{2}$, since only then the proof works. I will divide $E_{i+1/2}$ into several terms and estimate each of the terms individually.

$$\begin{aligned}
|E_{i+1/2}| = &\left| \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})} + \kappa \frac{1}{2} \right. \\
&+ (\kappa^3 - \kappa) \frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48(\Delta \overline{u}_{i+1/2})} \\
&\left. - \kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})} \right| \\
\leq &\left| \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})} \right| \qquad (99a) \\
+ &\left| \kappa \frac{1}{2} \right| \qquad (99b) \\
+ &\left| (\kappa^3 - \kappa) \frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48(\Delta \overline{u}_{i+1/2})} \right| \qquad (99c) \\
+ &\left| \kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})} \right| \qquad (99d)
\end{aligned}$$

First I will evaluate (99a).

$$\begin{aligned}
&\left| \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})} \right| \\
= &\left| \frac{\lambda_i(\overline{u}_{i+1} - \overline{u}_{i-1}) + \lambda_{i+1}(\overline{u}_i - \overline{u}_{i+2})}{16(\Delta \overline{u}_{i+1/2})} \right| \\
= &\left| \frac{\lambda_i(\overline{u}_{i+1} - \overline{u}_i + \overline{u}_i - \overline{u}_{i-1}) + \lambda_{i+1}(\overline{u}_i - \overline{u}_{i+1} + \overline{u}_{i+1} - \overline{u}_{i+2})}{16(\Delta \overline{u}_{i+1/2})} \right| \\
= &\left| \frac{\lambda_i(\Delta \overline{u}_{i+1/2} + \Delta \overline{u}_{i-1/2}) - \lambda_{i+1}(\Delta \overline{u}_{i+1/2} + \Delta \overline{u}_{i+3/2})}{16(\Delta \overline{u}_{i+1/2})} \right| \\
\leq &\left| \frac{\lambda_i(\Delta \overline{u}_{i+1/2} + \Delta \overline{u}_{i-1/2})}{16(\Delta \overline{u}_{i+1/2})} \right| + \left| \frac{\lambda_{i+1}(\Delta \overline{u}_{i+1/2} + \Delta \overline{u}_{i+3/2})}{16(\Delta \overline{u}_{i+1/2})} \right| \qquad (100)
\end{aligned}$$

Next I will use the inequalities

$$\begin{aligned}
|\lambda_i| &= \left| \min\left(3\frac{\overline{u}_{i-1} - \overline{u}_i}{2\overline{u}_{i-1} - \overline{u}_i - \overline{u}_{i+1}}, 3\frac{\overline{u}_{i+1} - \overline{u}_i}{2\overline{u}_{i+1} - \overline{u}_i - \overline{u}_{i-1}} \right) \right| \\
&= \left| \min\left(3\frac{\Delta \overline{u}_{i-1/2}}{2\Delta \overline{u}_{i-1/2} + \Delta \overline{u}_{i+1/2}}, 3\frac{\Delta \overline{u}_{i+1/2}}{2\Delta \overline{u}_{i+1/2} + \Delta \overline{u}_{i-1/2}} \right) \right| \\
&\leq \left| 3\frac{\Delta \overline{u}_{i+1/2}}{2\Delta \overline{u}_{i+1/2} + \Delta \overline{u}_{i-1/2}} \right| \qquad (101)
\end{aligned}$$

and

$$\begin{aligned}
|\lambda_{i+1}| &= \left|\min\left(3\frac{\overline{u}_i - \overline{u}_{i+1}}{2\overline{u}_i - \overline{u}_{i+1} - \overline{u}_{i+2}}, 3\frac{\overline{u}_{i+2} - \overline{u}_{i+1}}{2\overline{u}_{i+2} - \overline{u}_{i+1} - \overline{u}_i}\right)\right| \\
&= \left|\min\left(3\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{1+1/2} + \Delta\overline{u}_{i+3/2}}, 3\frac{\Delta\overline{u}_{i+3/2}}{2\Delta\overline{u}_{i+3/2} + \Delta\overline{u}_{i+1/2}}\right)\right| \\
&\leq \left|3\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}\right| \qquad (102)
\end{aligned}$$

Note that I can obtain inequalities (101) and (102) only because I know that the λ_i-values are non-negative. Plugging this into (100) gives us

$$\begin{aligned}
&\left|\lambda_i\frac{(\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2})}{16(\Delta\overline{u}_{i+1/2})}\right| + \left|\lambda_{i+1}\frac{(\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2})}{16(\Delta\overline{u}_{i+1/2})}\right| \\
\leq &\left|\frac{3}{16}\cdot\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2}}\cdot\frac{\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2}}{\Delta\overline{u}_{i+1/2}}\right| \\
+ &\left|\frac{3}{16}\cdot\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}\cdot\frac{\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}{\Delta\overline{u}_{i+1/2}}\right| \\
= &\left|\frac{3}{16}\cdot\frac{\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2}}\right| + \left|\frac{3}{16}\cdot\frac{\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}\right|
\end{aligned}$$

We know that $\Delta\overline{u}_{i+1/2}$ and $\Delta\overline{u}_{i-1/2}$ have the same sign, since otherwise the λ_i-value would be 0. This means that $\left|\frac{\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}}\right| \leq 1$. Similarly $\left|\frac{\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}}\right| \leq 1$. Using this yields

$$\begin{aligned}
&\left|\frac{3}{16}\cdot\frac{\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i-1/2}}\right| + \left|\frac{3}{16}\cdot\frac{\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}{2\Delta\overline{u}_{i+1/2} + \Delta\overline{u}_{i+3/2}}\right| \\
\leq &\frac{3}{16} + \frac{3}{16} = \frac{3}{8} \qquad (103)
\end{aligned}$$

I can not simplify (99b) so I will move on to (99c). Factoring out $|\kappa^3 - \kappa|$ and using the inequalities (101) and (102) for the λ_i-values yields

$$\begin{aligned}
&\left|(\kappa^3-\kappa)\frac{\lambda_i(\overline{u}_{i+1}+\overline{u}_{i-1}-2\overline{u}_i)-\lambda_{i+1}(\overline{u}_{i+2}+\overline{u}_i-2\overline{u}_{i+1})}{48(\Delta\overline{u}_{i+1/2})}\right| \\
=&|\kappa^3-\kappa|\left|\frac{\lambda_i(\Delta\overline{u}_{i+1/2}-\Delta\overline{u}_{i-1/2})-\lambda_{i+1}(\Delta\overline{u}_{i+3/2}-\Delta\overline{u}_{i+1/2})}{48(\Delta\overline{u}_{i+1/2})}\right| \\
\leq&|\kappa^3-\kappa|\left(\left|\lambda_i\frac{(\Delta\overline{u}_{i+1/2}-\Delta\overline{u}_{i-1/2})}{48(\Delta\overline{u}_{i+1/2})}\right|+\left|\lambda_{i+1}\frac{(\Delta\overline{u}_{i+3/2}-\Delta\overline{u}_{i+1/2})}{48(\Delta\overline{u}_{i+1/2})}\right|\right) \\
\leq&|\kappa^3-\kappa|\left(\left|\frac{3}{48}\cdot\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}}\cdot\frac{\Delta\overline{u}_{i+1/2}-\Delta\overline{u}_{i-1/2}}{\Delta\overline{u}_{i+1/2}}\right|\right. \\
&\qquad\left.+\left|\frac{3}{48}\cdot\frac{\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}}\cdot\frac{\Delta\overline{u}_{i+3/2}-\Delta\overline{u}_{i+1/2}}{\Delta\overline{u}_{i+1/2}}\right|\right) \\
=&|\kappa^3-\kappa|\left(\left|\frac{3}{48}\cdot\frac{\Delta\overline{u}_{i+1/2}-\Delta\overline{u}_{i-1/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}}\right|+\left|\frac{3}{48}\cdot\frac{\Delta\overline{u}_{i+3/2}-\Delta\overline{u}_{i+1/2}}{2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}}\right|\right) \\
\leq&\frac{3}{48}|\kappa^3-\kappa|\left(\frac{|\Delta\overline{u}_{i+1/2}|+|\Delta\overline{u}_{i-1/2}|}{|2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}|}+\frac{|\Delta\overline{u}_{i+3/2}|+|\Delta\overline{u}_{i+1/2}|}{|2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}|}\right) \qquad (104)
\end{aligned}$$

Again using that $\Delta\overline{u}_{i+1/2}$ and $\Delta\overline{u}_{i-1/2}$ have the same signs, I can say that $\frac{|\Delta\overline{u}_{i+1/2}|+|\Delta\overline{u}_{i-1/2}|}{|2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}|} \leq 1$ and analogously $\frac{|\Delta\overline{u}_{i+3/2}|+|\Delta\overline{u}_{i+1/2}|}{|2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}|} \leq 1$. This means that

$$\begin{aligned}
&\frac{3}{48}|\kappa^3-\kappa|\left(\frac{|\Delta\overline{u}_{i+1/2}|+|\Delta\overline{u}_{i-1/2}|}{|2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i-1/2}|}+\frac{|\Delta\overline{u}_{i+3/2}|+|\Delta\overline{u}_{i+1/2}|}{|2\Delta\overline{u}_{i+1/2}+\Delta\overline{u}_{i+3/2}|}\right) \\
\leq&\frac{3}{48}|\kappa^3-\kappa|(1+1)=\frac{1}{8}|\kappa^3-\kappa| \qquad (105)
\end{aligned}$$

The last term (99d) is exactly the same as the first term (99a), if we factor out $|\kappa^2|$. This means that

$$\left|\kappa^2\frac{\lambda_i\overline{u}_{i+1}+\lambda_{i+1}\overline{u}_i-\lambda_i\overline{u}_{i-1}-\lambda_{i+1}\overline{u}_{i+2}}{16(\Delta\overline{u}_{i+1/2})}\right| \leq |\kappa^2|\frac{3}{8} \qquad (106)$$

Putting all estimations into (99) yields

$$\begin{aligned}
&\left|\frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})}\right| \\
&+ \left|\kappa \frac{1}{2}\right| \\
&+ \left|(\kappa^3 - \kappa)\frac{\lambda_i(\overline{u}_{i+1} + \overline{u}_{i-1} - 2\overline{u}_i) - \lambda_{i+1}(\overline{u}_{i+2} + \overline{u}_i - 2\overline{u}_{i+1})}{48(\Delta \overline{u}_{i+1/2})}\right| \\
&+ \left|\kappa^2 \frac{\lambda_i \overline{u}_{i+1} + \lambda_{i+1}\overline{u}_i - \lambda_i \overline{u}_{i-1} - \lambda_{i+1}\overline{u}_{i+2}}{16(\Delta \overline{u}_{i+1/2})}\right| \\
&\leq \frac{3}{8} + |\kappa|\frac{1}{2} + |\kappa^3 - \kappa|\frac{1}{8} + |\kappa^2|\frac{3}{8}
\end{aligned} \tag{107}$$

This term still has to be smaller than $\frac{1}{2}$. Since $\kappa \in [0, 1]$ we know that $|\kappa^3 - \kappa| = \kappa - \kappa^3$. Solving the inequality gives us

$$\begin{aligned}
\frac{3}{8} + \frac{1}{2}\kappa + \frac{1}{8}(\kappa - \kappa^3) + \frac{3}{8}\kappa^2 &\leq \frac{1}{2} \\
\frac{3}{8} + \frac{4}{8}\kappa + \frac{1}{8}\kappa - \frac{1}{8}\kappa^3 + \frac{3}{8}\kappa^2 &\leq \frac{4}{8} \\
3 + 4\kappa + \kappa - \kappa^3 + 3\kappa^2 &\leq 4 \\
-\kappa^3 + 3\kappa^2 + 5\kappa - 1 &\leq 0
\end{aligned}$$

This gives us the value

$$\kappa \leq 0.18144 \tag{108}$$

B Used Theorems

First fundamental theorem of calculus

Let f be continuous on the closed interval $[a, b]$ and let F be the indefinite integral of f on $[a, b]$. Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \tag{109}$$

Second fundamental theorem of calculus

Let f be a continuous function on an open interval I. Let a be any point in I. Let

$$F(x) := \int_a^x f(t)\mathrm{d}t \tag{110}$$

then

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x). \tag{111}$$

Schwarz' Theorem - equality of mixed partials

Let f be a function of x and y. Let f_{xy} and f_{yx} both be continuous in an open disk R. Then

$$f_{xy}(x, y) = f_{yx}(x, y) \qquad \forall (x, y) \in R \tag{112}$$

C Source code

The following is the code from Scilab which was used to compute the development of the $\overline{u}_i$-values. The code was written by myself.

```
//settings
clear all;
clf;
n=10; //Number of cells between two integers
a=-1; //Interval boundary
b=11;
v=3; //velocity
iter=100; //iterations
version=1; //start distribution
algorithm=3; //1-constant, 2-linear, 3-quadratic, 4-oscillations
k=0.2; //kappa-value


function f=minmod(a,b)
    if a*b<=0 then f=0
    elseif abs(a)>=abs(b) then f=b
    else f=a
    end
endfunction

delx=1/n;
amount=n*(b-a)+1; //amount of intervals
xdata = linspace(a,b,amount) //Even distribution of intervals in [a,b]
Un=zeros(amount,1);
U=zeros(amount,1);




//Defining start values
select version
    case 1 then
        U(n+1:2*n)=1; //f(x)=1 when 0<x<1, else 0
    case 2 then
        function y = anything(x)
        y(find(x < 0)) = 0;

        x2 = x(0 <= x & x <= 1);
        y(find(0 <= x& x<= 1)) = sin(x2*%pi)^2;

        y(find(1 < x)) = 0;
```

```
        endfunction
        //plot(xdata,anything)

        function g=ig(x)
            g= x/2 - sin(2*%pi*x)/(4*%pi);
        endfunction
        //plot(xdata,ig)

        for i=n+1:1:2*n
            U(i)=ig((i-n)*delx)-ig((i-n-1)*delx);
            U(i)=U(i)/delx;
        end
end

plot(xdata, U,'.')


//Start of algorithm
select algorithm
    case 1 then
        //1. using constant functions
        delt=k*delx/(2*v);

        for j=1:iter
            for i=1:amount-1
                Un(i)=(U(i)+U(i+1))/2 + v*delt*(U(i)-U(i+1))/delx;
            end

            delete();
            plot(xdata+delx/2, Un,'g.');


            for i=2:amount
                U(i)=(Un(i-1)+Un(i))/2 + v*delt*(Un(i-1)-Un(i))/delx;
            end

            delete();
            plot(xdata, U,'g.');

        end
    case 2 then
        //2. using linear functions
        delt=k*delx/(2*v);

        for j=1:iter
            S=zeros(amount,1);
```

```
            for i=2:amount-1
                S(i)=minmod(U(i)-U(i-1) , U(i+1)-U(i));
            end
            for i=2:amount-2
                Un(i)=0.5*(U(i)+U(i+1)) + 0.125*(S(i)-S(i+1))
- delt*v*(U(i+1)-U(i))/delx - (delt^2)*(v^2)*(S(i)-S(i+1))
*0.5/(delx^2);
            end

            delete();
            plot(xdata+delx/2,Un,'.')


            for i=2:amount-1
                S(i)=minmod(Un(i)-Un(i-1) , Un(i+1)-Un(i));
            end

            for i=2:amount-1
                U(i)=0.5*(Un(i-1)+Un(i)) + 0.125*(S(i-1)-S(i))
 - delt*v*(Un(i)-Un(i-1))/delx - (delt^2)*(v^2)*(S(i-1)-S(i))
*0.5/(delx^2);
            end

            delete();
            plot(xdata,U,'.');
        end
    case 3 then
        //3. using quadratic functions
        for j=1:iter

            L=zeros(amount,1);

            for i=2:amount-1
                if (U(i)-U(i-1))*(U(i+1)-U(i))<=0 then
                    L(i)=0;
                else
                    L(i)=3*min((U(i+1)-U(i))/(2*U(i+1)-U(i)-U(i-1)),
(U(i-1)-U(i))/(2*U(i-1)-U(i)-U(i+1)));
                end
            end

            for i=2:amount-2
                Un(i)=0.0625*(L(i)*U(i+1)+L(i+1)*U(i)-L(i)*U(i-1)
-L(i+1)*U(i+2))+0.5*(U(i)+U(i+1))+ k*0.5*(U(i)-U(i+1))
```

```
+(k^3-k)*(L(i)*(U(i+1)+U(i-1)-2*U(i)) - L(i+1)*(U(i+2)
+U(i)-2*U(i+1)))/48 -k^2*0.0625*(L(i)*U(i+1)+L(i+1)*U(i)
-L(i)*U(i-1)-L(i+1)*U(i+2));
            end

            delete();
            plot(xdata+delx/2,Un,'r.');



            for i=2:amount-1
                if (Un(i)-Un(i-1))*(Un(i+1)-Un(i))<=0 then
                    L(i)=0;
                else
                    L(i)=3*min((Un(i+1)-Un(i))/(2*Un(i+1)-Un(i)-Un(i-1)),
(Un(i-1)-Un(i))/(2*Un(i-1)-Un(i)-Un(i+1)));
                end
            end


            for i=3:amount-1
                U(i)=0.0625*(L(i-1)*Un(i)+L(i)*Un(i-1)-L(i-1)*Un(i-2)
-L(i)*Un(i+1))+0.5*(Un(i-1)+Un(i)) + k*0.5*(Un(i-1)-Un(i))
+(k^3-k)*(L(i-1)*(Un(i)+Un(i-2)-2*Un(i-1)) - L(i)*(Un(i+1)
+Un(i-1)-2*Un(i)))/48 -k^2*0.0625*(L(i-1)*Un(i)+L(i)
*Un(i-1)-L(i-1)*Un(i-2)-L(i)*Un(i+1));
            end

            delete();
            plot(xdata,U,'r.');
        end
    case 4 then

        for j=1:iter

         delt=k*delx/(2*v);
            for i=2:amount-1
                Un(i)=-a*delt*0.5*(U(i+1)-U(i-1))/delx+U(i);
            end
            delete();
            plot(xdata,U,'k.');
            for i=2:amount-1
                U(i)=-a*delt*0.5*(Un(i+1)-Un(i-1))/delx+Un(i);
            end

            delete();
```

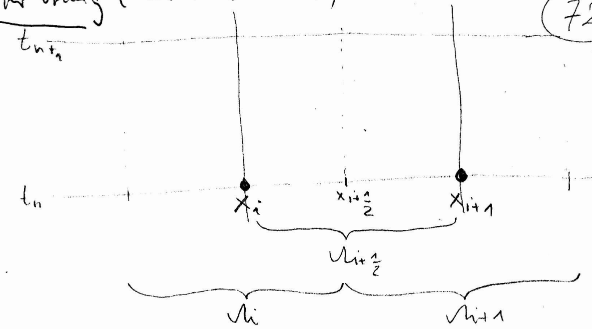
```
            plot(xdata,U,'k.');
        end
end

disp("U");
disp(U);
```

D Lecture notes

Version erster Ordnung (Lax-Friedrichs) 72

Herleitung:



Integralform auf $[t_n, t_{n+1}] \times \Omega_{i+\frac12}$

$\Downarrow$

$$0 = \int_{\Omega_{i+\frac12}} u(t_{n+1},x)\,dx - \int_{\Omega_{i+\frac12}} u(t_n,x)\,dx + \int_{t_n}^{t_{n+1}} f(u(t,x_{i+1})) - f(u(t,x_i))\,dt$$

- $$\Rightarrow 0 = \Delta x \cdot \left(u_{i+\frac12}^{n+1} - \frac{u_i^n + u_{i+1}^n}{2}\right) + \Delta t \underbrace{\left(f(u_{i+1}^n) - f(u_i^n)\right)}_{\text{unter CFL-Bed.}}$$

$$\Rightarrow \boxed{u_{i+\frac12}^{n+1} = \frac12\left(u_i^n+u_{i+1}^n\right) - \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^n)-f(u_i^n)\right)}$$

... ein konservatives Verfahren

- Version zweiter Ordnung: (Nessyahu-Tadmor)

$\hat u^n(x) = u_i^n + S_i \cdot \frac{x-x_i}{\Delta x}$ für $x \in \Omega_i$, $\hat u(t,x)$ ex. Lösung zu diesen AW'en

... wieder Integralform auf $[t_n, t_{n+1}] \times \Omega_{i+\frac12}$

$\Downarrow$

$$0 = \Delta x \cdot u_{i+\frac12}^{n+1} - \underbrace{\int_{\Omega_{i+\frac12}} \hat u^n(x)\,dx}_{\text{exakt berechnen}} + \int_{t_n}^{t_{n+1}} \underbrace{f(\hat u(t,x_{i+1})) - f(\hat u(t,x_i))\,dt}_{\text{Approx. durch Mittelpunktregel}}$$

$$\Rightarrow 0 = \Delta x \cdot u_{i+\frac12}^{n+1} - \frac{\Delta x}{2}\left(u_i^n+u_{i+1}^n\right) - \frac{\Delta x}{8}\left(S_i - S_{i+1}\right) + \Delta t\left(f(u_{i+1}^{n+\frac12}) - f(u_i^{n+\frac12})\right)$$

Klar: $\quad \ldots$ Approximation von $u_i^{n+1/2}$

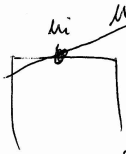
$f(u) = \frac{u^2}{2}$

$= \frac{u}{2} \cdot u$

(für gewisse Zeit ist Lösung glatt in x_i):

$$\Rightarrow \quad u_i^{n+\frac{1}{2}} \approx u_i^n + \frac{\Delta t}{2} \cdot \hat{u}_t$$

$$= u_i^n - \frac{\Delta t}{2} \frac{d}{dx} f(\hat{u}(t_n, x_i))$$



$f(u) = a \cdot u$, $a = \text{const.}$

$$= u_i^n - \frac{\Delta t}{2} f'(u_i^n) \cdot \hat{u}_x^n(x_i)$$

$$u_t = -f(u)_x = -a\, u_x \qquad = u_i^n - \frac{\Delta t}{2} \cdot f'(u_i^n) \cdot \frac{\delta_i}{\Delta x}$$

$= -f'(u) \cdot u_x$

$\Rightarrow$ Verf. von Nessyahu-Tadmor:

$u_t \approx -f'(u_i) \cdot u_i'$ bzw. $\approx -a \cdot u_i'$

$$\boxed{u_{i+\frac{1}{2}}^{n+1} = \frac{1}{2}(u_i^n + u_{i+1}^n) + \frac{1}{8}(\delta_i - \delta_{i+1}) + \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^{n+\frac{1}{2}}) - f(u_i^{n+\frac{1}{2}})\right) \quad \text{mit} \quad u_i^{n+\frac{1}{2}} = u_i^n - \frac{\Delta t}{2} \cdot \frac{\delta_i}{\Delta x} \cdot f'(u_i^n)}$$

$$u_i(t) = u_i + \left(-f'(u_i) \cdot u_i'\right) \cdot (t - t_n)$$

Satz 8.4 Das Verfahren von Nessyahu-Tadmor ist ein TVD-Verf., wenn

$$\int_{t_n}^{t_{n+1}} a \cdot u \, dt$$

$$\delta_i = \text{minmod}\left(\Delta u_{i-\frac{1}{2}}^n, \Delta u_{i+\frac{1}{2}}^n\right)$$

und die CFL-Bed.

$$\int_{t_n}^{t_{n+1}} f(u_i(t))\, dt$$

$$t_{n+1} - t_n \le \varkappa \cdot \frac{\Delta x}{\max_{u \in V^n} |f'(u)|} \quad \text{mit } V^n \equiv \left[\min_i u_i^n, \max_i u_i^n\right]$$

mit $0 < \varkappa \le \frac{1}{2}(\sqrt{2} - 1) = 0.21\ldots$

$\approx \Delta t \cdot f(u_i(t_n + \frac{\Delta t}{2}))$ erfüllt ist.

(74)

Beweis:

$$TV(u^{n+1}) = \sum \left| u^{n+1}_{i+\frac12} - u^{n+1}_{i-\frac12} \right|$$

$$= \sum \Big| \tfrac12 (u_{i+1} - u_i) + \tfrac12 (u_i - u_{i-1}) + \tfrac18 (S_{i+1} - S_i) - \tfrac18 (S_i - S_{i-1}) - \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^{n+\frac12}) - f(u_i^{n+1/2})\right) + \frac{\Delta t}{\Delta x}\left(f(u_i^{n+\frac12}) - f(u_{i-1}^{n+\frac12})\right)\Big|$$

$|a+b| = |a|+|b|$ iff $a \cdot b \geq 0$ ~~...~~ $\Rightarrow$ $a \geq 0$ and $b \geq 0$

$$\leq \sum |\Delta u_{i+\frac12}| \cdot \left\{ \frac12 + \frac18 \frac{S_{i+1}-S_i}{\Delta u_{i+\frac12}} - \frac{\Delta t}{\Delta x}\left(\frac{f(u_{i+1}^{n+\frac12}) - f(u_i^{n+\frac12})}{\Delta u_{i+\frac12}}\right)\right\}$$

$$+ |\Delta u_{i-\frac12}| \cdot \left\{ \frac12 - \frac18 \frac{S_i - S_{i-1}}{\Delta u_{i-\frac12}} + \frac{\Delta t}{\Delta x}\left(\frac{f(u_i^{n+\frac12}) - f(u_{i-1}^{n+\frac12})}{\Delta u_{i-\frac12}}\right)\right\}$$

falls $\boxed{\left| \frac{1}{\Delta u_{i+\frac12}} \left(\frac18 (S_{i+1} - S_i) - \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^{n+\frac12}) - f(u_i^{n+\frac12})\right)\right)\right| =: |E_{i+\frac12}| \leq \frac12 \quad \forall i}$

Index in (einer) Summe verschoben

$$= \sum |\Delta u_{i+\frac12}| \cdot \left\{ \frac12 + \frac12 \right\}$$ $|E_{i+\frac12}|$ kürzt sich weg + addiert

$$= \sum |u^n_{i+1} - u^n_i| = TV(u^n)$$

D.h. wir müssen noch $|E_{i+\frac12}| \leq \frac12 \ \forall i$ zeigen.

$|a-b| \leq |a| + |b|$

$$|E_{i+\frac12}| \leq \frac18 \left| \frac{S_{i+1} - S_i}{\Delta u_{i+\frac12}} \right| + \frac{\Delta t}{\Delta x} \underbrace{\left| \frac{f(u_{i+1}^{n+\frac12}) - f(u_i^{n+\frac12})}{u_{i+1}^{n+\frac12} - u_i^{n+\frac12}} \right|}_{= |f'(u^*)|} \cdot \left| \frac{u_{i+1}^{n+\frac12} - u_i^{n+\frac12}}{\Delta u_{i+\frac12}} \right|$$

75

mit $u^* \in \left[u_{i+1}^{n+\frac12},\ u_i^{n+\frac12}\right]$

$$u_i^{n+1/2} = u_i^n - \delta_i \cdot \frac{\Delta t}{2\cdot\Delta x} f'(u_i^n) = \hat{u}^n(x^*)$$

$$\left|\frac{\Delta t}{2\cdot\Delta x} f'(u_i^n)\right| \le \frac{\varkappa}{2} \le \frac{1}{2}$$

mit $\delta_i = \text{minmod}(\dots)$ folgt $\hat{u}^n(x) \in \hat{V}^n$ $\Rightarrow$ $u_i^{n+\frac12} \in \hat{V}^n \Rightarrow u^* \in \hat{V}^n$

$$\Rightarrow |f'(u^*)| \le \varkappa \cdot \frac{\Delta x}{\Delta t}$$

$$\Rightarrow |E_{i+\frac12}| \le \frac{1}{8}\left|\frac{\delta_{i+1}-\delta_i}{\Delta u_{i+\frac12}}\right| + \varkappa \left|\frac{u_{i+1}^{n+\frac12} - u_i^{n+\frac12}}{\Delta u_{i+\frac12}}\right|$$

$$|\delta_i| = |\text{minmod}(\Delta u_{i-\frac12}, \Delta u_{i+\frac12})| \le \min\left\{|\Delta u_{i-\frac12}|, |\Delta u_{i+\frac12}|\right\} \le |\Delta u_{i+\frac12}|$$

$$|\delta_{i+1}| = |\text{minmod}(\Delta u_{i+\frac12}, \Delta u_{i+\frac32})| \le |\Delta u_{i+\frac12}|$$

$$\left|\frac{u_{i+1}^{n+\frac12} - u_i^{n+\frac12}}{\Delta u_{i+\frac12}}\right| \quad \left|\frac{\Delta u_{i+\frac12} - \frac{\Delta t}{2\cdot\Delta x}\left(\delta_{i+1} f'(u_{i+1}^n) - \delta_i f'(u_i)\right)}{\Delta u_{i+\frac12}}\right|$$

$$\le 1 + \frac{1}{2}\varkappa \cdot \frac{|\delta_{i+1}| + |\delta_i|}{|\Delta u_{i+\frac12}|} \le 1 + \varkappa$$

$$\Rightarrow |E_{i+\frac12}| \le \frac14 + \varkappa(1+\varkappa) = \boxed{\frac14 + \varkappa + \varkappa^2 \overset{!}{\le} \frac12 \ \underline{\text{und}}\ \varkappa > 0}$$

$$\Leftrightarrow \boxed{\varkappa + \varkappa^2 \le \frac14,\ \varkappa > 0}$$

$$\Leftrightarrow 0 < \varkappa \le -\frac12 + \sqrt{\frac12} = 0.21\dots$$

Figures

All Figures in my Extended Essay were made by myself using the free softwares *Scilab* and *PhotoFiltre*.

References

[1] R. LeVeque, *Numerical Methods for Conservation Laws.* Birkhäuser Verlag, Basel (1992)

[2] R. Massjung, *Numerik von Transportgleichungen.* Unpublished lecture notes. RWTH Aachen (2003) [see Appendix D]

[3] H. Nessyahu, E. Tadmor, *Non-oscillatory Central Differencing for Hyperbolic Conservation Laws.* Journal of Computational Physics, Vol. 87, 408–463 (1990)