Folding Polygons and Knots

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Abstract

A large number of interesting geometric objects can be folded from a long strip of paper. In our talk, we discuss some known results concerning methods of folding certain regular polygons, and consider which knots can be folded in connection with these and what the relation is between them. We show some interesting mathematical activities that can be derived from these results as well as some problems as they have been and could be used in mathematics competitions.

1 Introduction

The worlds of knots and paper folding are mostly quite disparate. As practical practitioners, origamists have little connection to sailors as a rule. When we consider the two topics from a mathematical viewpoint, however, there is a rich intersection waiting to be discovered that can be derived from replacing the one-dimensional string usually considered as a medium for creating knots by a two-dimensional strip. Folding such a strip to create specific patterns and then knotting the results leads us to a world somewhere between the practical and abstract that throws up many interesting questions.

In this paper, we first present some problems in this area that were actually used in the international Kangaroo competition and then go on to suggest some potential competition problems derived from the properties of such strips. Mostly, these are presented in the form of challenge activities that can be offered to students to further their understanding of various elementary properties of euclidean geometry and knot theory in a creative way.

The ideas presented here all germinated from the idea of folding a strip of paper into a knot. As will become quite plain, this starting point can be devolved in many directions. The ideas presented here are certainly not the only available options.

The activities presented here require a long strip of paper with parallel sides, like a long, wide streamer or a roll of cash register paper. Even a roll of toilet paper will do in a pinch. In a number of steps, we will uncover some easy methods to fold regular polygons, polyhedra, stars and knots with this paper, and some of the surprising reasons these beautiful regular shapes result in such simple ways.

2 Warming up with a few Competition Problems

While not the most common topic for math competitions, the idea of folding a strip of paper has been used on occasion as the starting point for some competition problems. Here are a few from the Mathematical Kangaroo.

Problem 1 (Kangaroo Junior 2012, Problem 22)

A rectangular piece of paper $ABCD$ measuring 4 cm $\times$ 16 cm is folded along the line MN so that vertex C coincides with vertex A, as shown in figure 1. What is the area of the pentagon $ABNMD'$?

A) 17 B) 27 C) 37 D) 47 E) 57

The correct answer is D. The calculation becomes quite straight-forward when we realize that the folding crease NM is the perpendicular bisector of AC. This helps to determine the length of BN. We also note that quadrilateral $ANMD'$ is half of the rectangle, and therefore has the area 32. This makes the final steps simple enough, applying similar triangles.

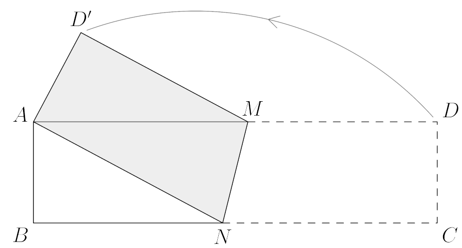


Figure 1: Junior 2012, Problem 22

Problem 2 (Kangaroo Cadet 2010, Problem 22)

A paper strip was folded three times in half and then completely unfolded so that you can still see the 7 folds going up or down. Which of the following views from the side cannot be obtained in this way, see figure 2?

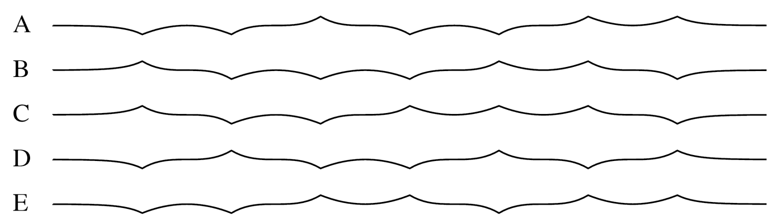


Figure 2: Cadet 2010, Problem 24

The correct answer is D. The solution to this problem requires some visualization, but no calculation at all. The first fold must bring the left and right halves together, and this is possible for all strips. The second fold must do the same, but restricted to either half of the strip. This is not possible for strip D.

Problem 3 (Kangaroo Student 2010, Problem 22)

The paper ribbon is folded three times as shown in figure 3. Find β if $\alpha = 70°$.

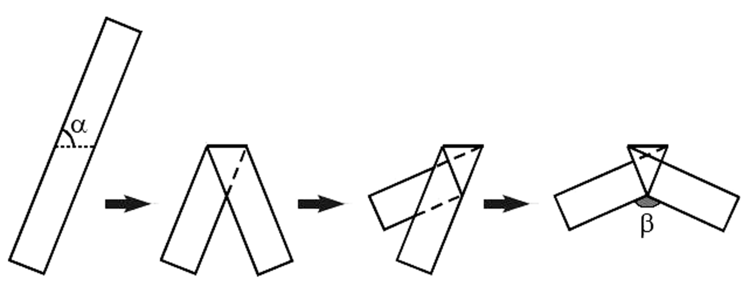


Figure 3: Student 2010, Problem 20

A) 140° B) 130° C) 120° D) 110° E) 100°.

The correct answer is C. The interior isosceles triangle we see in the second of the four steps has two angles of 70°, and the angle at its vertex is therefore 40°. Once we have noted this, the other angles are all quite easy to calculate.

As we see, there are several interesting aspects we can focus on in this context. While Problem 1 asks a question about area, Problem 2 is strictly about possible patterns and Problem 3 is concerned with angles. All of these topics will play a role in the following sections.

There are also a number of interesting problems that have been posed concerning knotting strings in some way. Here are a few examples.

Problem 4 (Kangaroo Pre-Ecolier 2016, Problem 2)

How many ropes are there in figure 4?

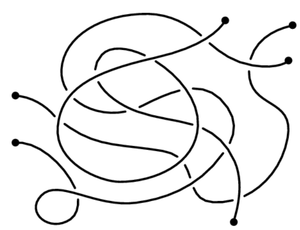


Figure 4: Pre-Ecolier 2016, Problem 2

A) 2 B) 3 C) 4 D) 5 E) 5

The correct answer is C. This problem was posed for students in grades 1 and 2, and is therefore quite elementary. It can be solved either by mentally unravelling the three strings, or by realizing that there are six string ends in the picture. Since every string has two ends, the number of strings must be half that number.

Problem 5 (Kangaroo Ecolier 2021, Problem 15)

Three ropes are laid down on the floor as shown. You can make one big, complete loop with three other pieces of rope. Which of the ropes shown will give you one big loop?

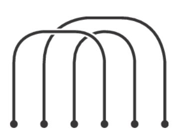
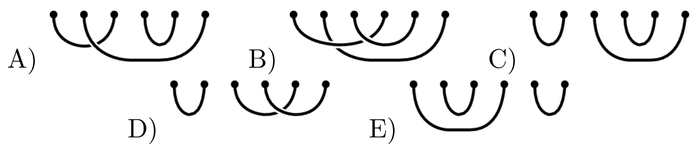


Figure 5: Ecolier 2021, Problem 15



The correct answer is C. There are several ways to solve this problem. It is possible to draw all five options and note which yields a single big loop, but this is a bit time consuming. With some reasoning and some more spatial recognition, we see that option A yields a small loop joining ends 2 and 6 (counting from the left). Similarly, both options B and E yield a closed loop joining ends 1 and 4, and option D yields a closed loop joining ends 3 and 5. This leaves C as the only remaining option.

Problem 6 (Kangaroo Student 2021, Problem 18)

A piece of string is lying on the table. It is partially covered by three coins as seen in the figure. Under each coin the string is equally likely to pass over

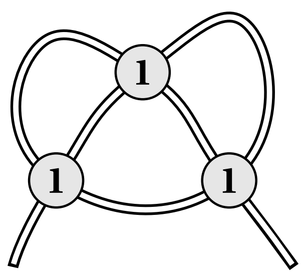
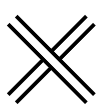
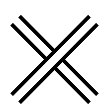


Figure 6: Student 2021, Problem 18

itself like this:  or like this: .

What is the probability that the string is knotted after its ends are pulled?

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{1}{8}$ D) $\frac{3}{4}$ E) $\frac{3}{8}$

The correct answer is B. Again, there are several ways to solve this. Perhaps the simplest is to just follow the string from one end to the other, noting at each crossing whether we are passing above or below the other part of the string. Of the eight possible above-below combinations, only the two above-below-above-below-above-below and below-above-below-above-below-above yield a knot, and the probability is therefore equal to $\frac{2}{8} = \frac{1}{4}$.

A common property of all three problems is the requirement of good visualisation skills to find an easy answer. We also note that these examples show us that questions about knots can be set at all age levels. Both of these facts will leave their mark throughout the following. Whereas problems 4 and 5 are purely topological, problem 6 combines knot theory with probability theory in a very unusual and surprising way, making the question so original. It is one of the attractive aspects of knot theory that it is a fairly new topic in the world of mathematics competitions and there are still many surprising aspects left to be discovered.

3 Folding a Strip of Squares and Half-Squares

In this section, we will consider ways of folding squares and half-squares (squares folded on the diagonal) on the paper strip. For each of the activities in this section, we assume that we are only allowed to fold the sides of squares perpendicular to the edges of the strip or the diagonals of the resulting squares. (For the purpose of this section, we will not allow any fancy origami moves like sink-folds or rabbit-ears.)

Activity 1: In the following figures, the strip of paper is cut off at right angles on the left, while the right side is as long as required, as denoted by the wavy lines.

As a first step (Fig. 7), we fold the lower left-hand corner of the strip onto the upper edge in such a way that the resulting crease passes through the upper left-hand corner of the strip. The resulting crease creates a 45° angle. Using the

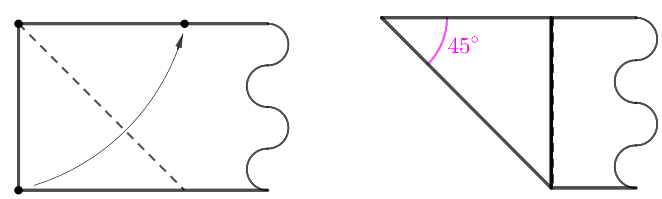


Figure 7: Half of a right angle

vertical edge as a guide, we can therefore fold the missing side of the left-most square of the strip.

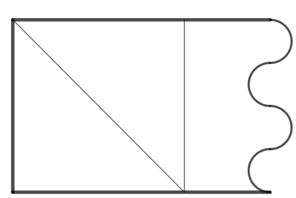


Figure 8: folding the first square

We can then use the first square as a guide for the next, and so on, creating a strip of any number of identical squares.

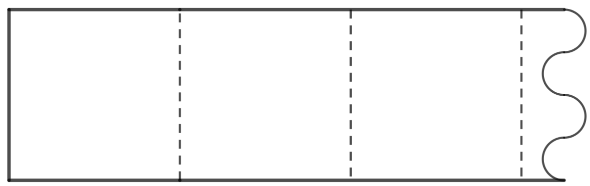


Figure 9: several adjacent squares

Challenge 1a: As we see in the following figures, folding a diagonal in an inner square of the strip results in an external angle that is not a right angle.

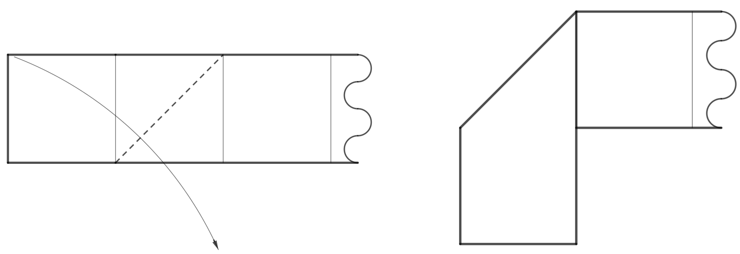


Figure 10: the result of one diagonal fold

The challenge is to fold the figure 11 from a strip made up of the smallest possible number of squares. We already know that three squares will not be enough. Can you do it with four?

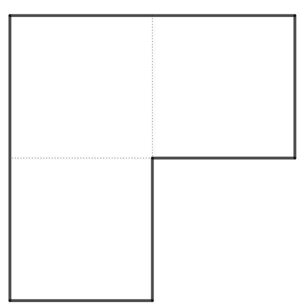


Figure 11: L-shape

Solution to Challenge 1a: Yes, it can be done with a strip of four squares. One method is shown in figure 12:

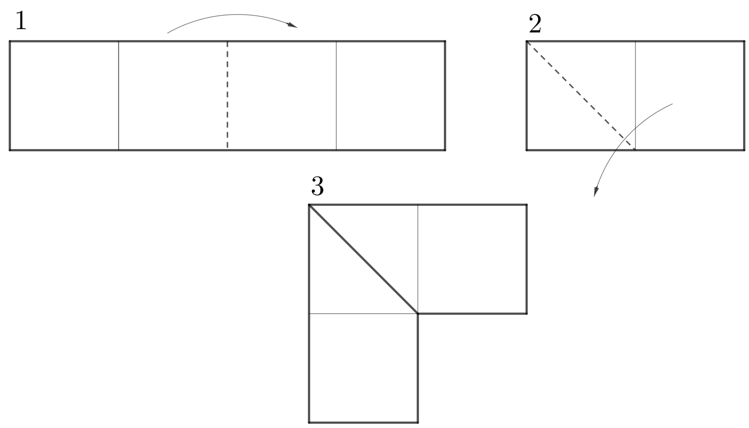


Figure 12: solution to the L-shape

Note that this is not the only possible way to do this. Any solution will, however, have one diagonal fold and one horizontal fold.

Challenge 1b: So, that one was easy. Now what about this? Can you fold a square with sides that are twice the width of the paper strip with a strip made up of as few squares as possible?

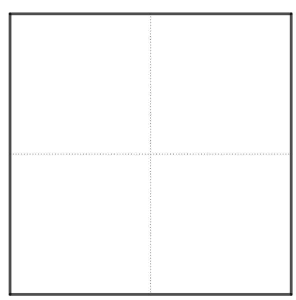


Figure 13: a bigger square

Solution to Challenge 1b: In figure 14 we see a solution that uses a strip of six squares:

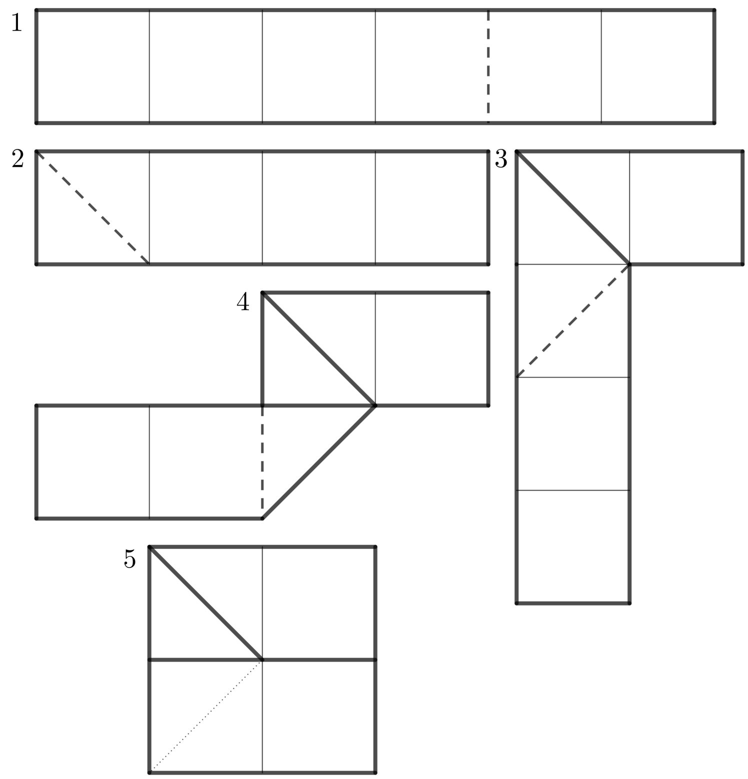


Figure 14: solution to a bigger square

Less than six small squares cannot result in a big square. This is due to the fact that a change in direction of the strip always requires a diagonal fold, and such a fold will always require at least one extra square. Since folding the big square certainly requires at least two changes of direction, folding the big square certainly requires a strip composed of at least $4 + 2 = 6$ small squares.

Challenge 1c: This next one is a little more challenging. Can you fold a cross composed of five small squares, each of which is the same width as the strip of paper, as shown in figure 15? Can you do this using a strip made up of the smallest number of squares?

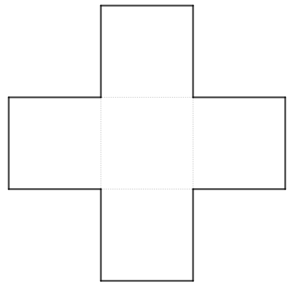


Figure 15: a cross

Solution to Challenge 1c: In figure 16 we see a solution that uses a strip of nine squares:

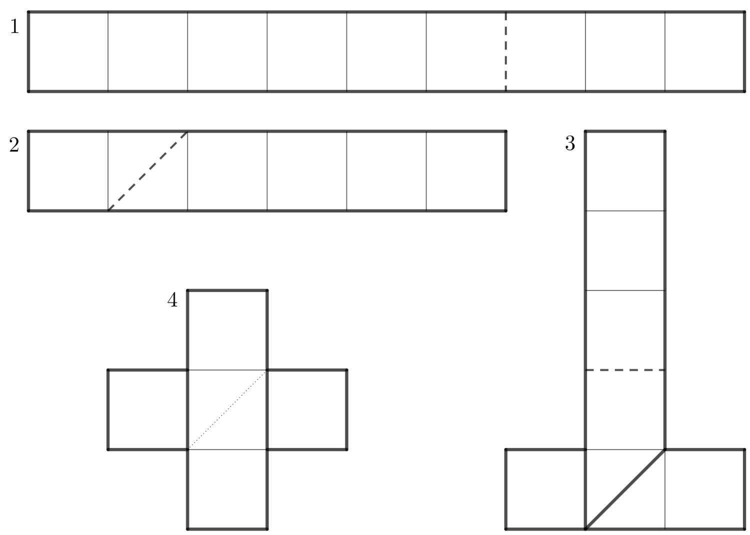


Figure 16: solution to a cross

Challenge 1d: Now, try to fold each of the four shapes shown in 17. Try once again to do this using a strip made up of the smallest number of squares in each case.

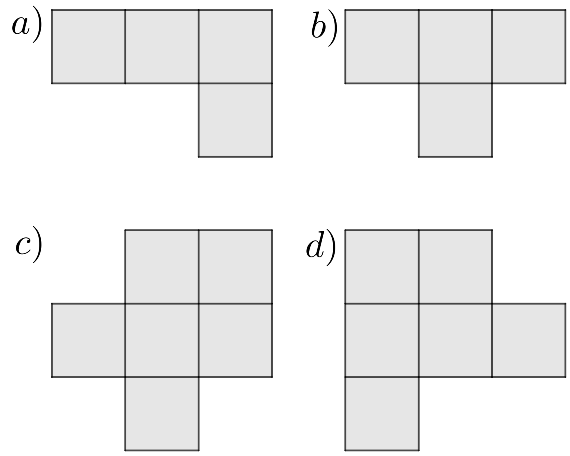


Figure 17: now fold these!

Note that solutions are not given for these shapes. After all, we have to leave something for you to discover for yourself!

Extra Challenge 1e: What is the smallest number of squares required to fold a strip to cover the faces of a cube, whose edges are the same length as the width of the strip? What about a cube whose edges are the same length as the diagonals of the squares on the strip?

Solution to Challenge 1e: In figure 18 we see a solution to the first part that uses a strip of eight squares:

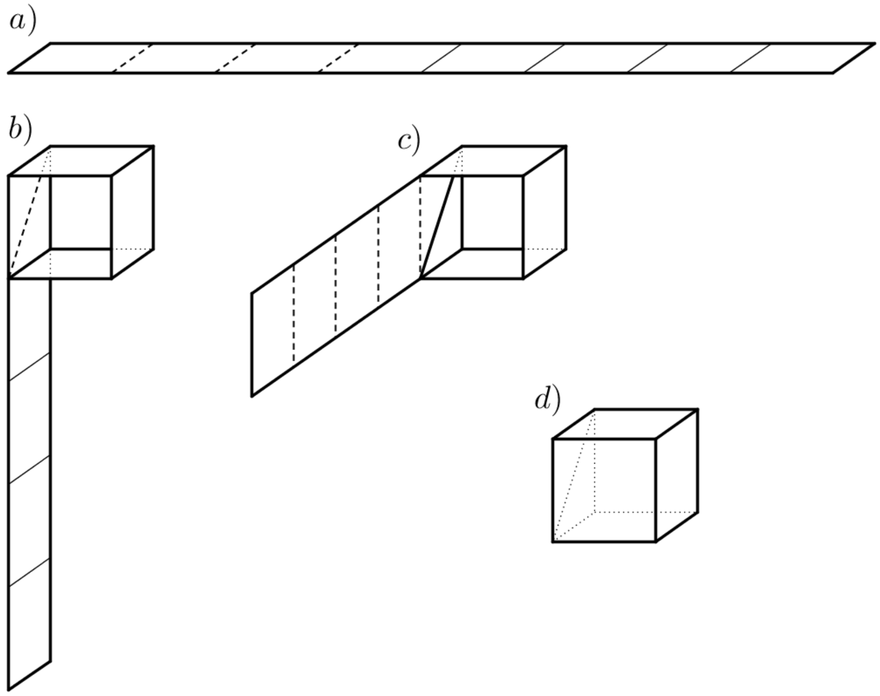


Figure 18: a cube from 8 squares

This is reasonably easy to find. If you have tried your hand at it, you will have noticed that the second part is quite a bit more difficult, however. A possible path to a solution is as follows:

In the following figures, the steps toward a solution are shown using successive squares on a strip. We start with a first square, shown in brown. In the next step. this square will turn pink and the edges of the square in which the strip is continued are highlighted by thick black lines. The squares that were already in their proper place previously are alternately colored red and blue, while a square that has been added in its proper spot will be green. If such a square has been added but must be altered by some folding maneuver, it will be colored brown, and will change to green when it reaches its final resting spot, after which it turns red or blue.

In Figure 19, we see what happens if we simply place one square with its diagonal on an edge of the cube, and then fold the adjoining squares onto the surface of the cube directly. In this case, the squares form a ring of six, that dies not cover the complete surface if the cube, but leaves two vertices of the cube completely untouched.

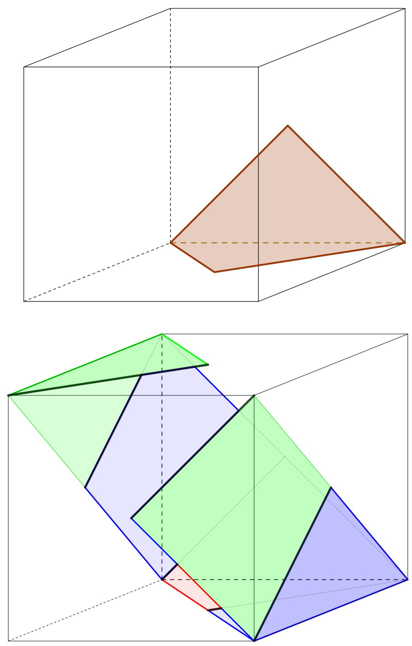


Figure 19: covering a centered ring

In order to cover these corners, we must change directions somehow. In 20, we start with the same preliminary square and the same edge on its upper left, and cover the top right-hand back corner of the cube in successive steps in such a way that the bottom half of the final brown square can be folded up, allowing us to continue with the next available continuation edge in the same spot we started from.

We note that we have used six squares to cover the three half faces of the cube. Since each square in the strip has an area equal to half of a cube face, this means that we require three squares more that would strictly be necessary just from the point of view of the area to be covered.

We can combine the ring from 19 with two such corner coverings, resulting in a solution of the problem requiring a total of 18 squares on the strip to cover the six square faces of the cube.

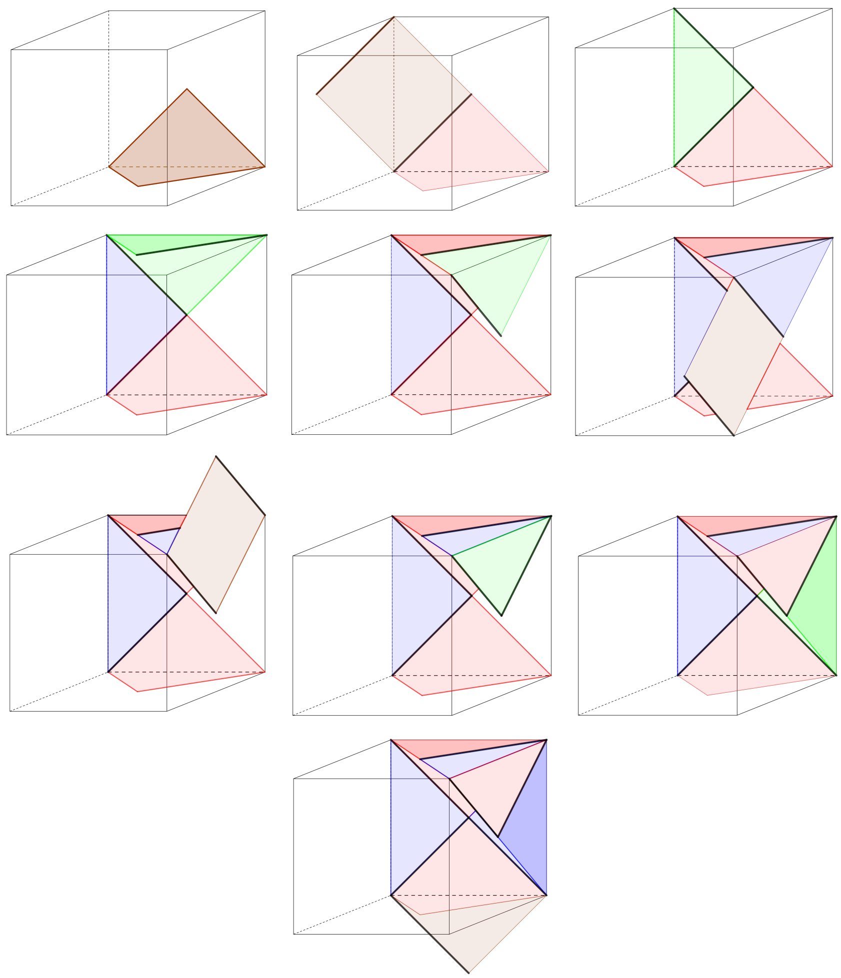


Figure 20: covering a corner

In case you want to try your hand at actually folding this, you will find it quite difficult to actually wrangle the long strip of paper appropriately. A good way to do this is to have a solid cube on hand whose edges are exactly the length of the diagonals of the squares on the paper strip. Covering this cube step by step is then actually quite straight-forward. Good luck!

Challenge 1f: Now we are ready to start thinking about knots. This is what a simple overhand knot looks like:

Figure 21: A typical overhand knot

Can you fold an overhand knot from a strip of paper, folding only the edges of the squares and some of their diagonals? Try to do this with as few squares as possible!

Solution to Challenge 1d: Here is a solution that uses a strip of 16 squares:

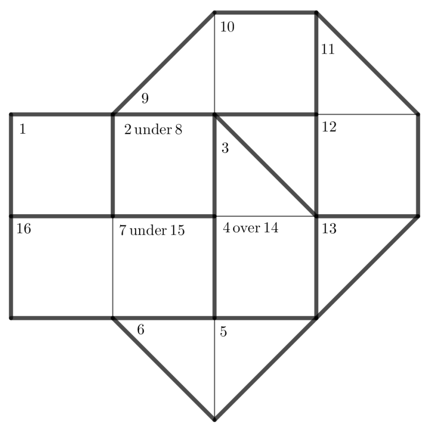


Figure 22: Solution: folding a knot

Just figuring out the notation used here is already a bit of a puzzle, but you might find it enjoyable to try to reproduce the folded knot shown here.

Of course, there are additional challenges. Can you find other knots produced in this way? Can you find a symmetric one? (Note that the pictured overhand knot tied in a string seems to be symmetric with respect to a vertical axis, if you ignore the over- and underpasses.) Can you find one that uses less squares? Once again, we will leave this up to the interested reader to discover.

4 Folding an equilateral triangle

First of all, we will discover an easy way to fold an equilateral triangle, that is, a regular triangle with three sides of equal length and three angles of equal size.

Activity 2: In the following figures, the strip of paper is once again cut off at right angles on the left, while the right side is as long as required, as again denoted by the wavy lines.

As a first step, we fold the strip length-wise, bringing the upper and lower left-hand corners of the strip together. This creates a horizontal crease, parallel to the edges of the strip.

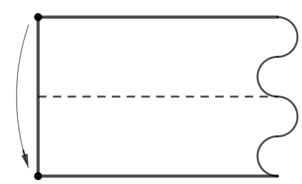


Figure 23: Folding the strip lengthwise

Next, we fold the upper left-hand corner onto the crease we just created in the first step. We do this in such a way that the new crease passes through the lower left-hand corner of the strip, see figure 24 on the left.

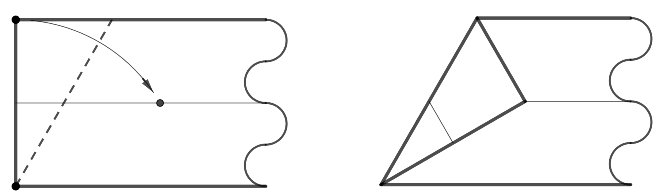


Figure 24: Further folding

This results in the situation we see in figure 24 on the right.

Challenge 2a: In figure 25, the crease on the right is at an angle of 60° to the bottom edge of the strip. Can you explain, why this angle must be 60°? Also, the angle at the top of the crease is also equal to 60°. Can you explain, why this must be the case?

Solution to Challenge 2a: This is quite easy to prove if we add a few lines to our figure as shown in Figure 26.

In this figure, AB is the left edge of the original strip, as it looked before we folded the triangle over. The line segment BC is the folding crease and D is the point on the mid-parallel of the edges of the strip on which A comes to

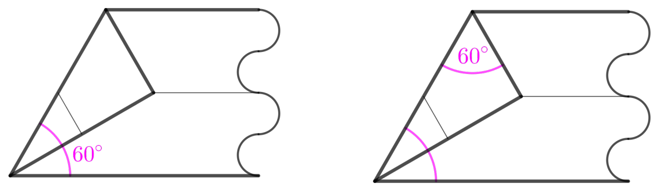


Figure 25: Two 60° angles

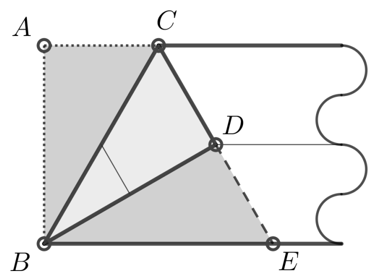


Figure 26: Why 60°?

lie after the fold. Finally, we extend segment CD beyond D, and let E denote the intercept of this extension with the bottom edge of the strip.

Since triangle CDB resulted by folding triangle CAB over, these two triangles are certainly congruent.

It is not difficult to see that triangle EDB is also congruent to triangle CDB. They have the side BD in common, and sides CD and ED must be of equal length since C, D and E are the points in which three parallel lines intersect a common line, with the middle line being the mid-parallel of the other two. Finally, since $\angle CAB = 90°$ and $\angle CAB = \angle CDB = 90°$, we also have $\angle EDB = 180° - \angle CDB = 90°$. This gives us $\angle CDB = \angle EDB$, and triangles CDB and EDB are congruent by the side-angle-side Theorem.

Now that we know that triangles CAB, CDB and EDB are all congruent, we obtain

$$\angle ABC = \angle CBD = \angle DBE,$$

and since their sum

$$\angle ABC + \angle CBD + \angle DBC = \angle ABE = 90°$$

is a right angle, we have

$$\angle ABC = \angle CBD = \angle DBE = 30°.$$

Having found this, we are essentially done. We now have

$$\angle CBE = \angle CBD + \angle DBE = 30° + 30° = 60°$$

and

$$\angle DCB = 90^\circ - \angle CBD = 90^\circ - 30^\circ = 60^\circ,$$

completing the proof. □

Having shown this, we can now put our preliminary fold to good use. We have just shown that triangle CBE is equilateral, since it has two angles of 60°. This means that we can fold the strip once more, using the edge of the triangular flap as a guide-line, resulting in an equilateral triangle, see Figure 27. This triangle will prove quite useful in the next section.

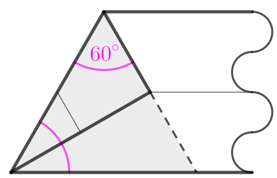


Figure 27: An equilateral triangle

5 Folding a Strip of Equilateral Triangles

Using the method developed in the previous section, we can easily fold a series of equilateral triangles on our strip of paper by using the outer edge if each triangle as a guide for the next crease. This results in a strip as shown in figure 28.

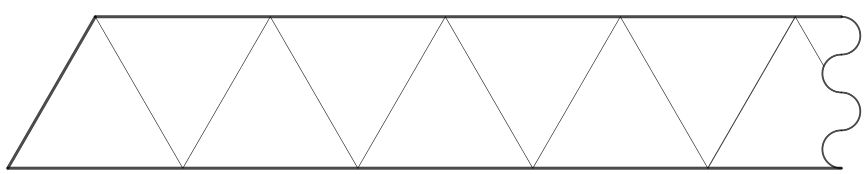


Figure 28: A strip of equilateral triangles

It is now interesting to consider what kinds of things we can fold with such strips by only folding on the edges of these triangles. In other words, we are only allowed folds of the type shown in Figure 29.

We can get started on our path by folding a larger triangle.

Challenge 2b: Fold an equilateral triangle with sides twice the length of the sides of the triangles on the strip, as shown in Figure 30. What is the smallest number of small triangles the strip can have in order for this to be possible?

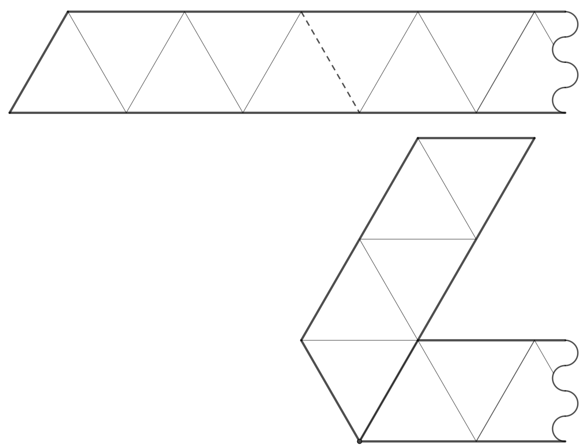


Figure 29: Folding along triangles

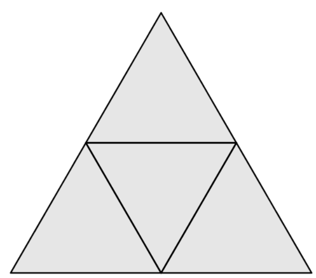


Figure 30: Folding a slightly larger triangle

Solution to Challenge 2b: The triangle can be folded with a strip made of seven small triangles, as shown in Figure 31.

Challenge 2c: Fold an equilateral triangle with sides three times the length of the sides of the triangles on the strip, as shown in Figure 32. What is the smallest number of small triangles the strip can have in order for this to be possible?

No solution is offered to this challenge. If you have been paying attention so far, folding this triangle should be a snap!

Challenge 2d: Fold the four shapes shown in the following figure. In each case, use a strip of triangles, only folding the strip along sides of the triangles. It is easy to count the number of triangles that make up the final shape in each case, namely 4, 6, 5 and 12 respectively. How many triangles will the shortest strip have that you can fold these shapes from in each case?

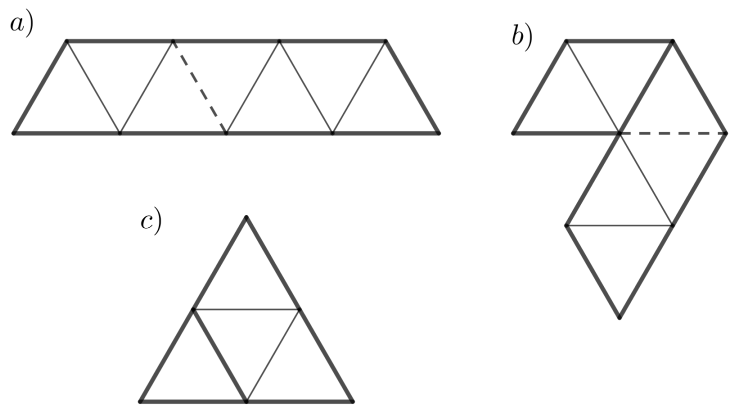


Figure 31: Solution: Folding a slightly larger triangle

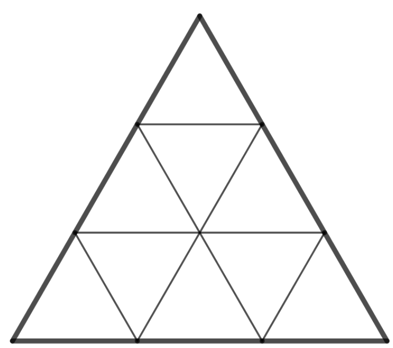


Figure 32: Folding an even larger triangle

Solution to Challenge 2d: The smallest numbers of triangles in the strips for these four shapes are

a) 5,   b) 12,   c) 8,   d) ??? (you will have to figure this one out yourself)

If you want to know what the folded strips look like in each case, you will have to get folding.

Extra Challenge 2e: There are three platonic solids with triangular faces: the regular tetrahedron with four, the regular octahedron with eight and the regular icosahedron with twenty.

Fold the three platonic solids with triangular faces shown in the figure. In each case, use a strip of triangles, only folding the strip along sides of the triangles. We know that the number of triangles that make up the surface in each case is 4, 8 and 20 respectively. How many triangles will the shortest strip have that you can fold these solids from in each case?

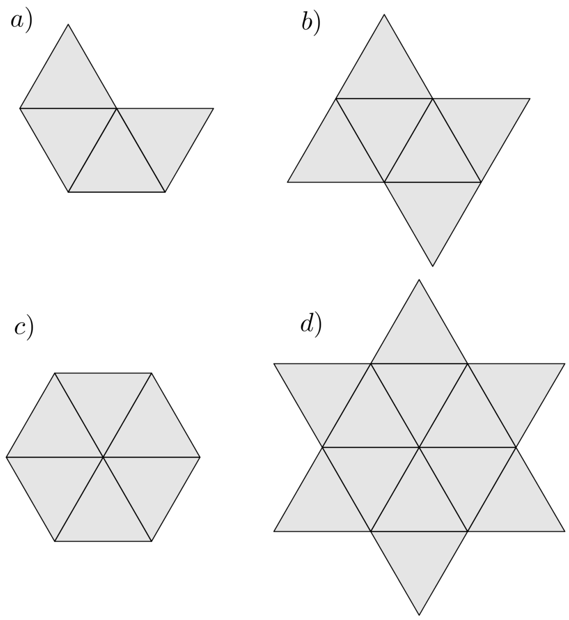


Figure 33: Folding along triangles

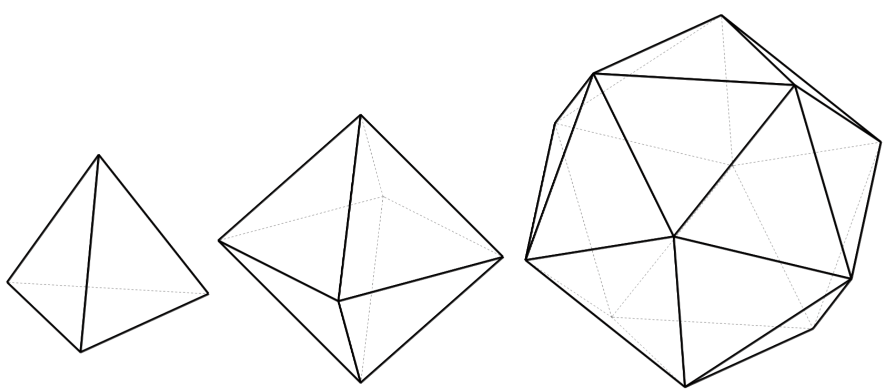


Figure 34: Folding along triangles

Partial Solution to Challenge 2d: Somewhat surprisingly, a strip of four equilateral triangles will fold to cover a regular tetrahedron. There is really no other way to fold this short strip, so solving this first part is easy.

The other two triangular platonic solids are a bit more challenging, but since there are only ever two options to place any triangle next to another, and one of these folds back onto the previous one, it is not so hard to find the solutions by trial and error. Giving this a go is once again left to the ambitious reader. If you find it hard to subjugate the strips, recall the tip from the cube. If you get a solid octahedron (or icosahedron) and a paper strip made of trinagles the same size as their faces, this will make things manageable.

Challenge 2f: Now we are ready to start thinking about knots that can be folded from strips of triangles. Recall the picture 37 of a simple overhand knot:

Can you fold an overhand knot from a strip of paper, folding only the edges of the triangles? Try to do this with as few squares as possible!

Solution to Challenge 2f: The figure on the left shows a solution that uses a strip of triangles. And on the right there is a solution that uses a shorter strip of triangles:

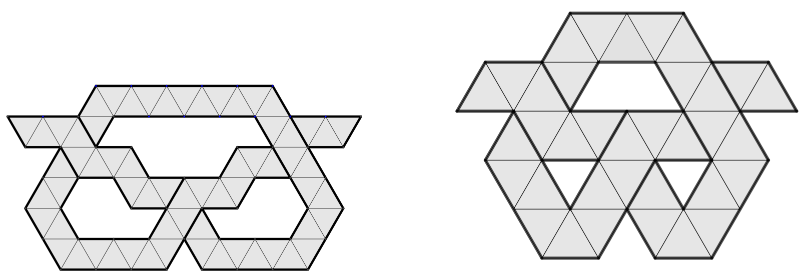


Figure 35: an overhand knot made up of (once with many and once with fewer)triangles

And on the left is a solution that uses even less triangles. On the right is a solution that uses the smallest number of triangles (34):

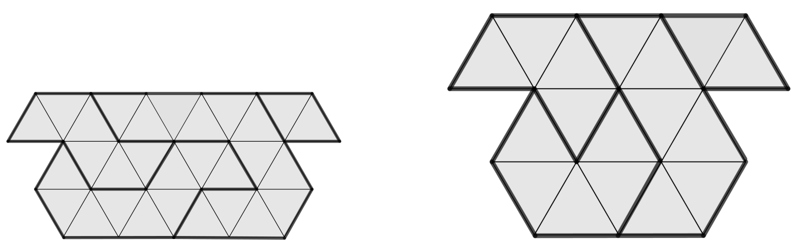


Figure 36: an overhand knot made up of triangles

Note that there are many ways to fold knots with triangle strips. If this has piqued your interest, you might like to try your hand at the following:

Can you fold an overhand knot using as few triangles as possible?

Can you fold an overhand knot that closes, and therefore yields a true trefoil knot if the ends are stuck together?

Can you fold a more complex knot than the trefoil, with more over-under passes of the folded strip?

6 Knotting a pentagon

In the above sections, We have already considered the overhand knot twice, once creating it by folding the paper strip in squares and half-squares and once by folding triangles. In this section, we will take a next step, and consider what happens if we remove these restrictions and just try to lay an overhand knot in the paper strip flat.

Before we do this, we note that such knots are not quite the same objects that mathematicians would generally refer to as "knots". In the strict topological sense, a knot is a configuration of a closed loop with certain crossings. If we close the overhand knot, for instance, by joining its two open ends in as direct a manner as possible, we obtain a knot known as a *trefoil knot*. This is derived from its resemblance to the clover leaf, as clover is a plant belonging to the botanical family known as trefoil. It is a knot with three crossings. Such a knot is shown in Figure 37.

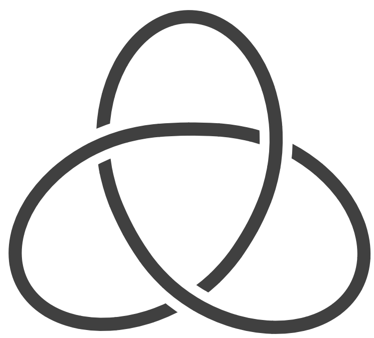


Figure 37: A trefoil knot

Keeping all this in mind, we are now ready to plunge into our next activity.

Activity 3: This will a slightly more open-ended activity than the first two. We will discover that there are a lot of ways to loop and flatten a strip of paper, and that such loopings result in knots with quite surprising properties.

Challenge 3a: Use a strip of paper and make an overhand knot. Be careful not to tear the paper. Try to tighten it as well as you can and then flatten it. What is your result?

Solution to Challenge 3a: Does the knotted section of your strip look like a pentagon? If you managed to pull it together really tightly, it might even appear to be a regular pentagon i.e. with all sides the same length and all angles equal. (This introduces an interesting question. Are these two properties actually equivalent? In general, the answer is no, but under the specific circumstances we are considering here, the answer is actually yes. You might want to give this some thought before you continue.)

With any luck, your strip should look something like Figure 38.

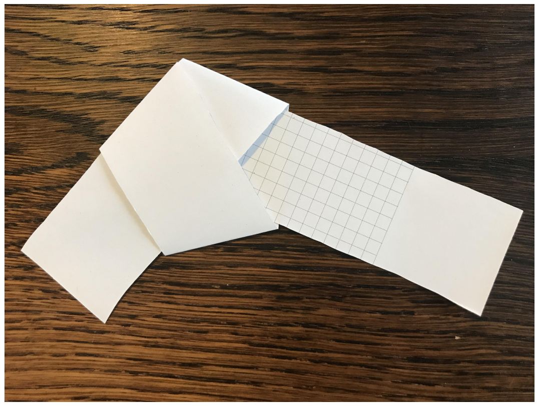


Figure 38: Pentagon folded from a strip of paper - paper version

It sure looks like a regular pentagon, doesn't it? Just compare it to the drawn version shown in Figure 39.

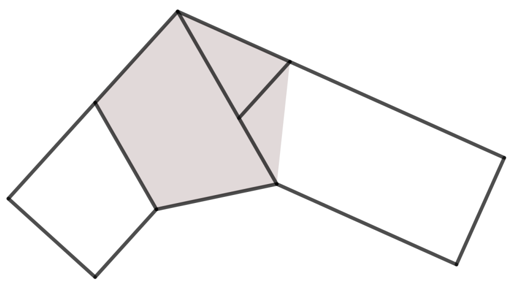


Figure 39: Pentagon folded from a strip of paper - graphics version

But how can we be sure? Well, let's give that some thought.

Challenge 3b: Prove that a tightly folded overhand knot in a strip of paper produces a regular pentagonal knot.

Solution to Challenge 3b: First of all, let's take a closer look at what happens when we fold a strip of paper just once.

In the top part of figure 40, we see a strip of paper, with the dashed line indicating where we intend to fold it. Since the opposite sides of the strip are parallel, the angles indicated by dots are equal. When we actually fold the strip along this line (as shown in the central part of the figure), the upper angle is folded down but does not change its size. This means that the grey triangle in the bottom part of the figure is isosceles.

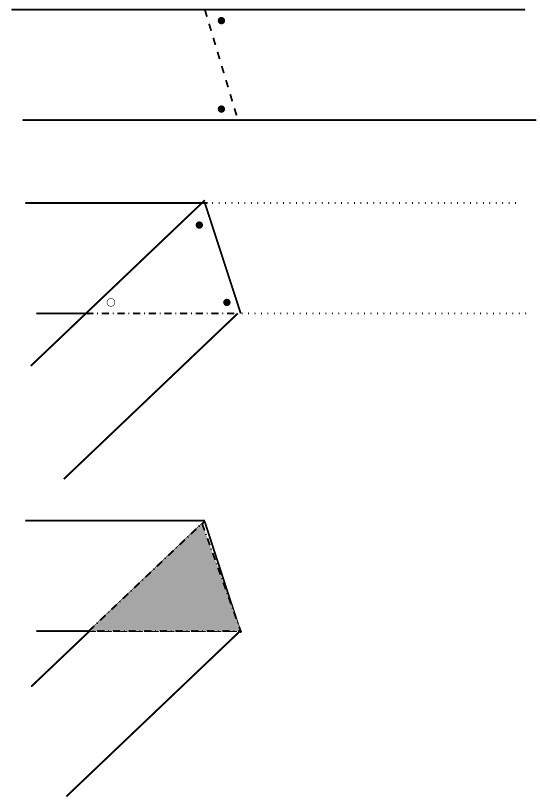


Figure 40: Folding a strip of paper produces an isosceles triangle

Now, in order to produce the overhand knot (which has three crossings) we need to fold three times. Each fold will create such an isosceles triangle. Furthermore, if this is going to be a regular pentagon we better make sure that the sides will have the same length. This will certainly be the case, as the dashed line (our original folding line) will be a side of the pentagon in each case, and we are folding by the same angle each time, resulting in sides of the same length. In practical terms, we had better be sure to fold in exactly the same way every time. We then produce three congruent isosceles triangles with our three folds, and as we shall see in the following, the angles will all be equal.

In Figure 41, we first look at one of the isosceles triangles, namely the red one. we have denoted the equal angles in the red triangle by filled circles and the third angle by an open circle. As this triangle was created by folding we have two pairs of parallel line segments, as the sides of the strip remain parallel. This means that the angles marked with open circles are all equal.

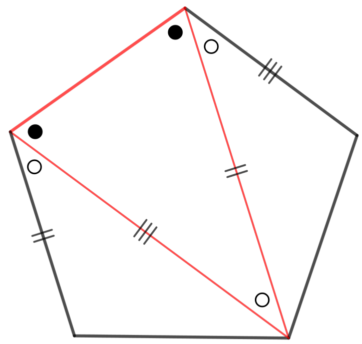


Figure 41: Studying the pentagon

Now we can simply repeat this observation. Every time we fold, we create a new isosceles triangle congruent to the previous one and hence we find the angle denoted by the open circle many more times, as we see in Figure 42.

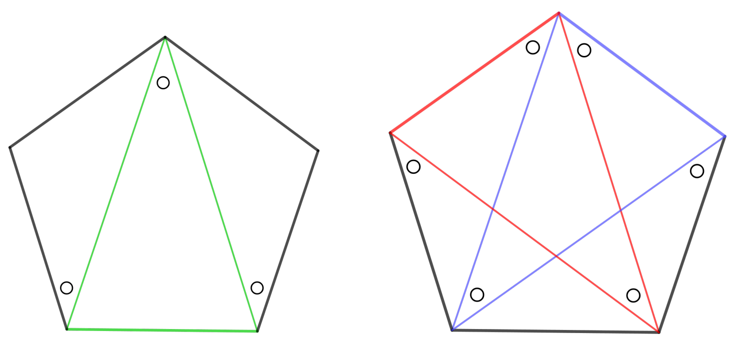


Figure 42: More pentagons

This means that the angle in a vertex of the pentagon is exactly three times as large as the angles denoted by an open circle, and also that the angles denoted by a closed circle are twice as large as the ones denoted by an open circle. Completing all the angles in this way, we deduce that all angles are equal, and hence this is indeed a regular pentagon, see Figure 43.

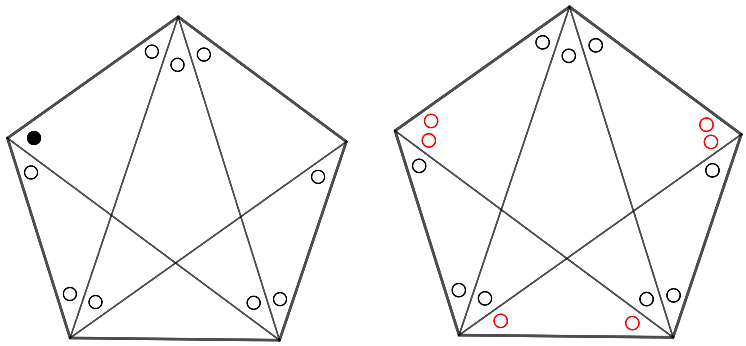


Figure 43: More pentagons and more angles

□

7 Knotting a heptagon

Can we actually also fold other polygons from a strip of paper? The answer is, yes, we can, but it is not always so easy. Let's take it step by step. This idea will motivate our final activity.

Activity 4: What other regular polygons can we fold with a strip of paper in this way?

Challenge 4a: Try to fold a heptagonal knot (remember, *hepta* means seven in Greek) from a strip of paper. Can you make a regular heptagon? Can you make the result look like Figure 44 and Figure 45?

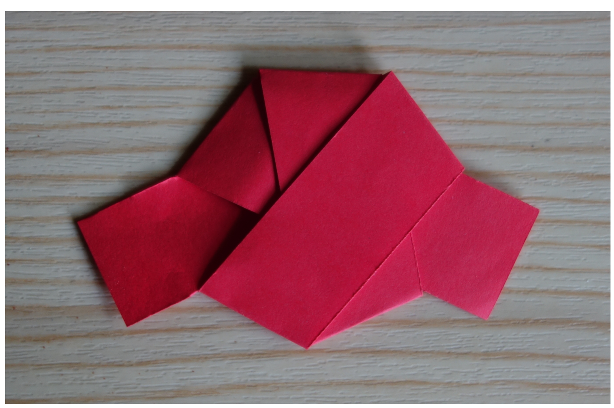


Figure 44: A heptagonal knot folded from a strip of paper - paper version

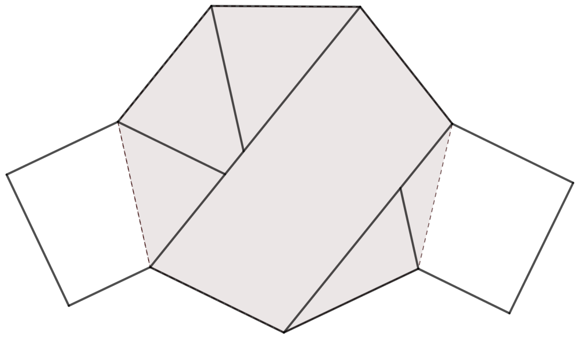


Figure 45: A heptagonal knot folded from a strip of paper - graphics version

A hint toward the solution to Challenge 4a: Did you work out how to do this? If not the following sequence of figures may be helpful – start with the simple knot and then make one extra turn:

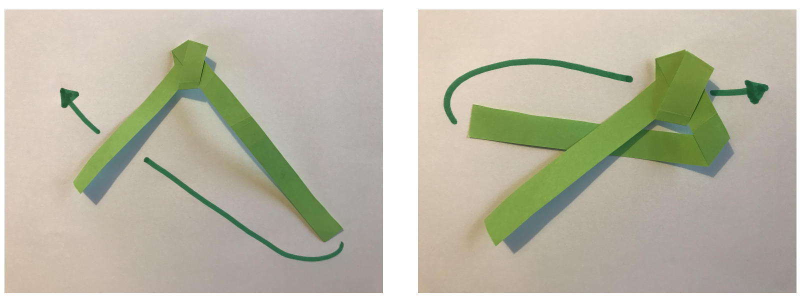


Figure 46: Instructions how to fold a heptagon

Finally, here are some more interesting questions to think about.

Challenge 4b:

- Why is the heptagon from Challenge 4a a regular heptagon if we fold it tightly? Can we apply the same arguments that we used for the pentagon here?
- What does the underlying knot look like here? How many crossings does the (closed) knot have, that we get by connecting the two loose ends?
- Is this heptagonal knot unique? In other words, is there more than one way to layer the strip when the knot is tied?
- Which other polygons can we create in this way? Does it make a difference whether the polygon has an even or an odd number of sides? Specifically, why is it not possible to create a regular hexagon in this way?

8 Summary

For further details regarding the folding of a heptagon or for knotting other polygons see [**M**]. If this topic has piqued your interest, all four references are heartily recommended for further inspiration.

References

[GGE] Ilan Garibi, David Goodman, Yossi Elran, *The Paper Puzzle Book: all you need is paper*, World Scientific, 2017.

[H] Thomas Hull *Project Origami: Activities for exploring mathematics*, Taylor & Francis Ltd., 2012.

[KMP] L. Christina Kinsey, Teresa E. Moore, Efstratios Prassidis *Geometry and Symmetry*, Wiley, 2010.

[M] Jun Maekawa, Chapter *Introduction to the Study of Tape Knots in Origami 5*, Fifth International Meeting of Origami Science, Mathematics, and Education, 1st Edition, 2011.