Eigenfaces

Tag über Mathematik und Unterricht, Bellinzona

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Form and Structure

What?

Manual for implementing a program for image compression and face recoginition in `Python`.

Who?

Single person or group work for a whole school class.

How?

Text with theoretical and practical exercises, including solutions. Template codes are provided and will be extended by the students.

Core question: How can a computer recognize faces?

Goals

Goals for the Students

- Generalize vector geometry from $\mathbb{R}^3$ to $\mathbb{R}^n$.
- Learn how to code in `Python`.

Goals for the Audience

- Explain to another teacher what eigenfaces are.
- Name one application of eigenfaces.
- Get to know an application of linear algebra accessible for students.
- Have fun and look at a lot of pictures.

Training Set

The code learns how to classify from given training images.

Class	Image 1	Image 2	Image 3	Image 4	⋯
Adam Sandler					⋯
Emma Watson					⋯
Natalie Portman					⋯
⋮	⋮	⋮	⋮	⋮	⋱

Training Set

The code learns how to classify from given training images.

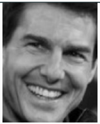
Class	Image 1	Image 2	Image 3	Image 4	...
male					...
female					...

Image as Matrix

Representation of Grayscale Images

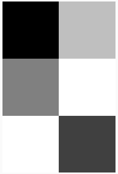
Map pixels to values between 0 (black) and 1 (white) and represent as $M \times N$ matrix with entries $p_{ij} \in [0, 1]$.

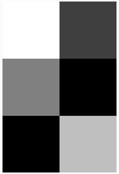
$$\longleftrightarrow \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MN} \end{pmatrix}$$

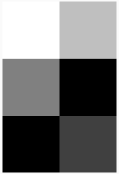
Example

Which image on the right is represented by the following matrix?

$$\begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{2} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$







Images as Vectors

$$\longleftrightarrow \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MN} \end{pmatrix} \longleftrightarrow \begin{pmatrix} p_{11} \\ \vdots \\ p_{M1} \\ p_{12} \\ \vdots \\ p_{M2} \\ \vdots \\ p_{1N} \\ \vdots \\ p_{MN} \end{pmatrix}$$

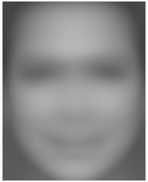
Question: What is the following code doing to the image $\vec{p}$?

```python
def get_negative(p):
    MN = len(p)
    for i in range(MN):
        p[i] = 1.0 - p[i]
    return p
```

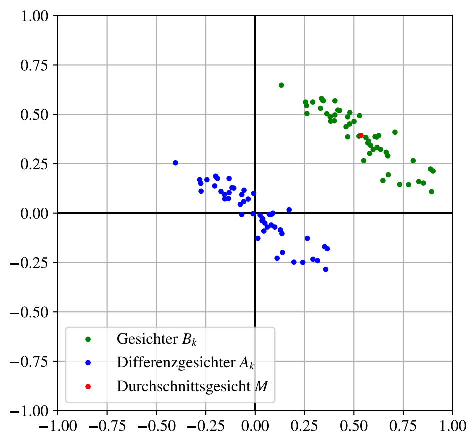
Mean Face and Difference Faces

We can center the training images around the origin by subtracting the mean face.

$$\longrightarrow \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{MN} \end{pmatrix}$$

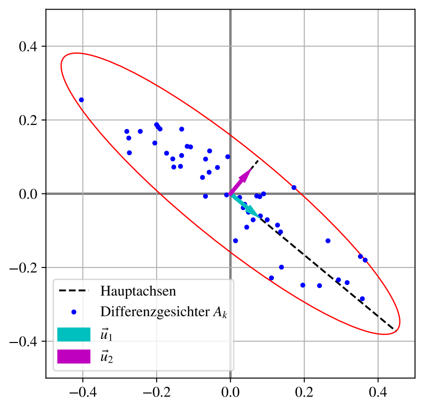


$$\longrightarrow \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{MN} \end{pmatrix}$$



Eigenfaces

Principal component analysis using singular value decomposition yields **eigenfaces**.

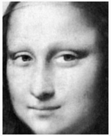


Eigenfaces as images: $\vec{p}_k = \sigma_k \, \vec{u}_k + \vec{m}$

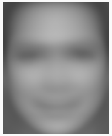
Transition $\mathbb{R}^3 \longrightarrow \mathbb{R}^{M \cdot N}$ (projection onto eigenfaces)

New face image as supersposition of mean face and eigenfaces: $c_k = \vec{u}_k \cdot (\vec{p} - \vec{m})$

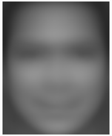
$\vec{p}$



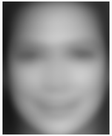
$=$ $\vec{m}$



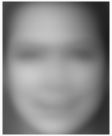
$+ c_1$ $\vec{u}_1$



$+ c_2$ $\vec{u}_2$



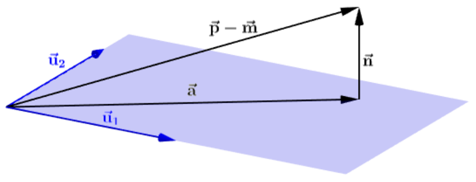
$+ c_3$ $\vec{u}_2$



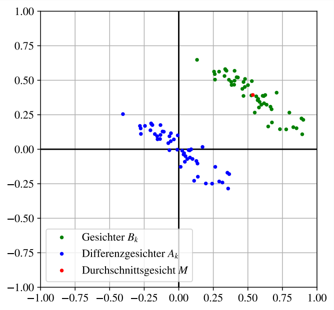
$+ \cdots$

Use prior knowledge in $\mathbb{R}^3$

- linear combination
- scalar product
- orthogonality



Eigenfaces vs. Test Images



Observation

This can be done with any other (sufficiently large) set of images!

Question

Then what distinguishes the eigenfaces?

Expansion w.r.t. the training images:

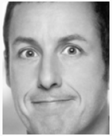
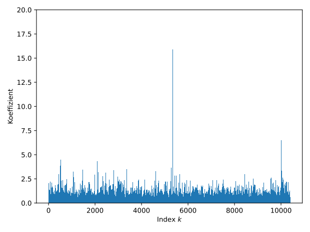
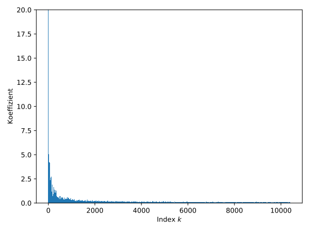
 $= c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $+ \cdots$

Image Compression

Absolute values $|c_k|$ of the **coefficients** of the linear combination w.r.t. the training images and eigenfaces.



training images



eigenfaces

Question 1

What are the differences?

Question 2

Are the eigenfaces better?

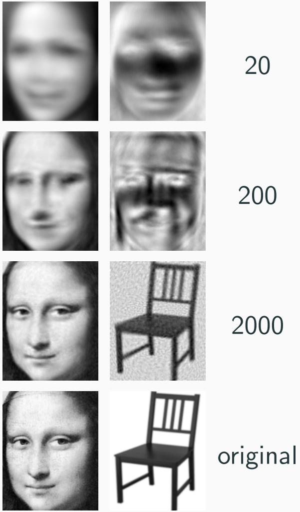


Image Recognition

Literature

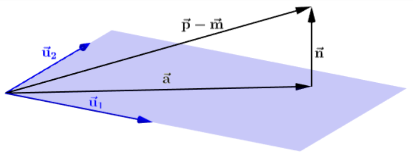
Turk, Pentland, *Face Recognition Using Eigenfaces*, 1991

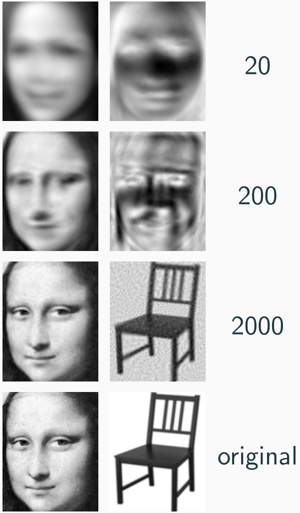
Task

For a given image $\vec{p}$, decide if it shows a face.

Idea

If $\vec{p} - \vec{m}$ lies almost in the subspace spanned by the first $K \approx 2000$ eigenfaces, then it is probably a face.





Gender Recoginition

Task

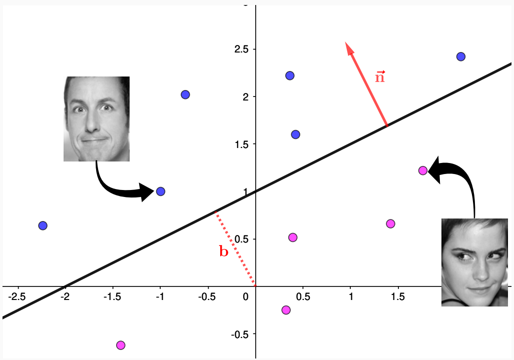
Given an image of a face (any person), determine the gender of the person.

Structure

This is a binary classification problem.

Approach

Use a separating hyperplane: Male faces on one side and female faces on the other side.



Gender Recoginition

Step 1

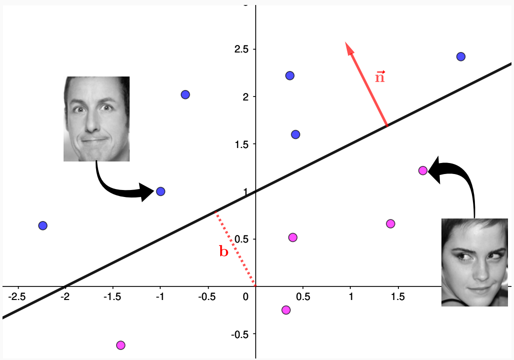
Compute $\vec{n} \in \mathbb{R}^{M \cdot N}$ and b, such that

$$\{\vec{x} \mid \vec{x} \cdot \vec{n} + b = 0\}$$

optimally separates the genders on the training images.

Step 2

Given a new image $\vec{p}$, check on which side of the plain it is.



Problem

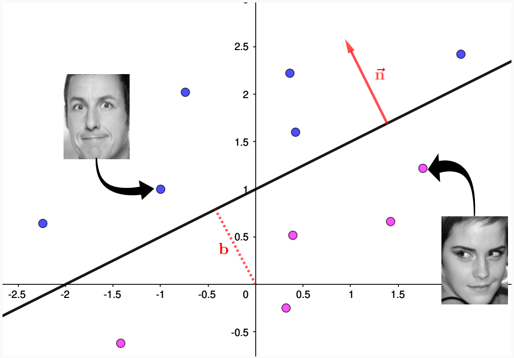
Normal $\vec{n} \in \mathbb{R}^{M \cdot N}$ has too may parameters to optimize for. Recall: $M = 180, N = 144$

Solution

Represent each image by its coefficient vector

$$\vec{c} = (c_1, \dots, c_K)^T$$

w.r.t. the first $K \approx 2000$ eigenfaces.



Didactical Aspects:
Some Examples

Transition $\mathbb{R}^3 \longrightarrow \mathbb{R}^{M \cdot N}$ (Distance Point-Line)

Distance Point-Line in $\mathbb{R}^2$

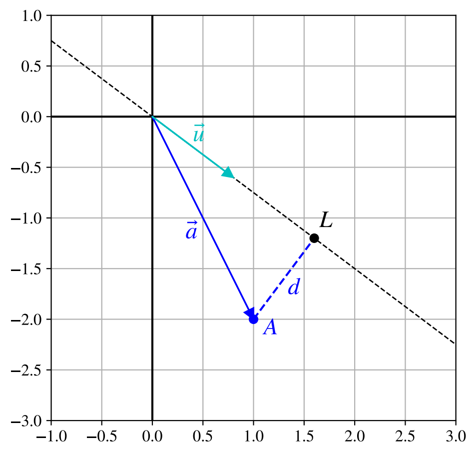
- draw picture
- concrete Numbers

Distance Point-Line in $\mathbb{R}^4$

- no picture
- concrete numbers

Distance Point-Line in $\mathbb{R}^{M \cdot N}$

- picture impossible
- variables instead of numbers



Group work (Setting up the training images)

Positive Interdependence

The more training images, the better the result.

Individual Accountability

If somebody provides wrong image files, the whole program won't work.

Promotive Interaction

The database is ready only when everyone is done. Hence faster students should help the slower students.

Foster Interpersonal Skills

Collaboration is only necessary when the results are assembled.

Group Processing

Cutting pictures to the right resolution helps to see pictures as $M \times N$ Matrix of pixels.

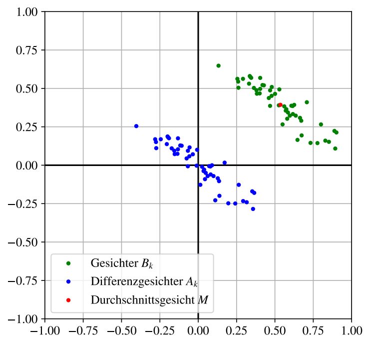
Self-Explanation (Images as Vectors)

Question

Name two differences and one similarity of the simplified picture on the right and the real situation (i.e. resolution $M = 180$ and $N = 144$).

Question

Can the difference-faces (blue) be rendered to images?



Further Didactical Aspects

Scaffolding

- Computing eigenfaces by SVD is too advanced $\rightarrow$ eigenfaces as blackbox.
- Loading and saving images has nothing to do with mathematics $\rightarrow$ code-templates.

Interleaved practice

- blocked: Develop whole theory first, then write code.
- interleaved: Alternate between theory and programming.

Holistic mental model confrontation

Compare simplified pictures in 3 dimensions with $N \cdot M$ dimensions.

Summary

Eigenfaces ...

1. ... build on vector geometry in $\mathbb{R}^3$.
2. ... can visualize linear algebra in higher dimensions.
3. ... can be used as blackbox.
4. ... allow to explore linear alebra in higher dimensions.

Links

https://educ.ethz.ch/unterrichtsmaterialien/mathematik/eigengesichter.html

https://github.com/OliverRietmann/eigenfaces_latex

https://github.com/OliverRietmann/eigenfaces