4th degree polynomials
and the Golden Section

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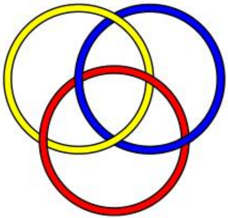
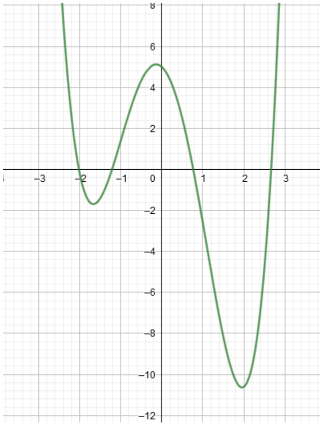
ETH 12 September 2018

About me

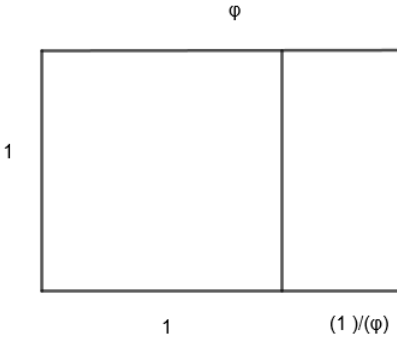
PhD Mathematics, Algebraic Topology
University of Illinois at Chicago Circle 1980
Teaching at Math Dept, Göteborg, 1980-2012
Mostly teacher training, both prospective and in-service
Now happily retired
Love music and the Borromean rings



Quartic

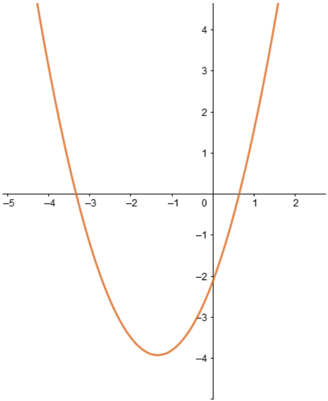


Golden Rectangle

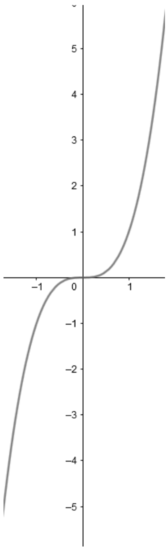


$$\varphi = \frac{\sqrt{5}+1}{2} = \frac{2}{\sqrt{5}-1}$$

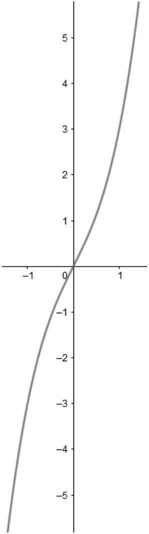
Quadratic



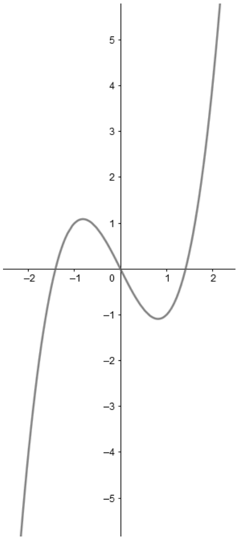
Cubic



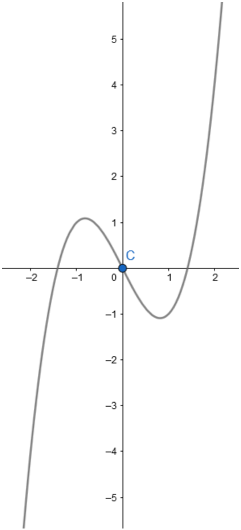
$y = x^3 + 2x$



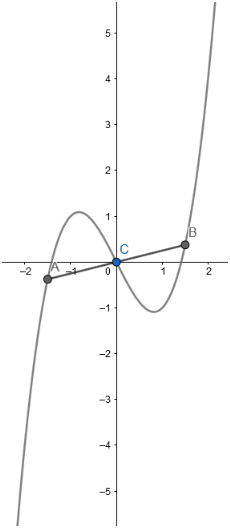
$y = x^3 - 2x$



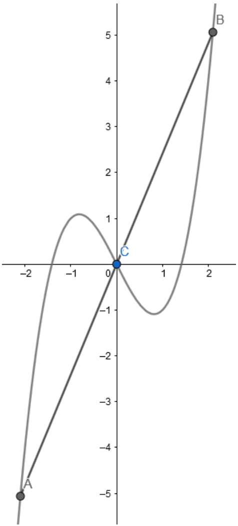
Inflection point



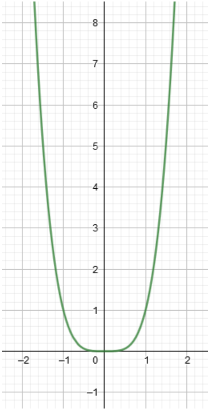
Symmetry



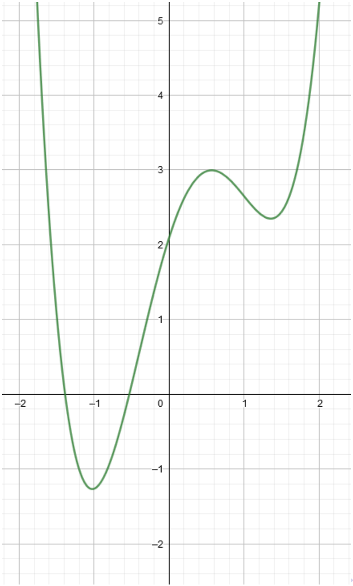
Symmetry



$y = x^4$



General quartic



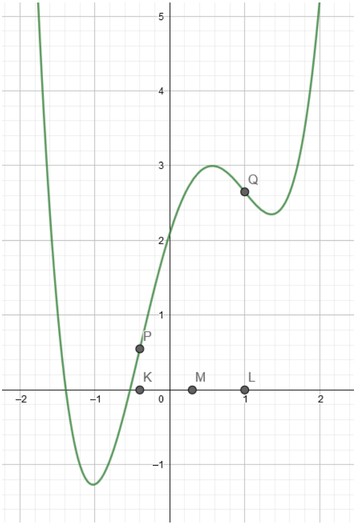
Quartic with two inflections

$f(x) = x^4 +$ lower terms
$f'(x) = 4x^3 +$ lower terms
$f''(x) = 12x^2 +$ lower terms
We want inflection points $k < l$, so
$f''(x) = 12(x-k)(x-l)$
which will be easier to handle if we complete the square:

$$\text{let} \begin{cases} m &= \frac{k+l}{2} \quad \text{(midpoint)} \\ r &= \frac{l-k}{2} \quad \text{(radius)} \end{cases}$$

$$\text{so} \begin{cases} k &= m-r \\ l &= m+r \end{cases}$$

Inflection points and midpoint



Quartic with two inflections

$$\begin{aligned} \text{Then } f''(x) &= 12(x-k)(x-l) \\ &= 12(x-m+r)(x-m-r) \\ &= 12((x-m)^2 - r^2) \\ &= 12(x-m)^2 - 12r^2 \end{aligned}$$

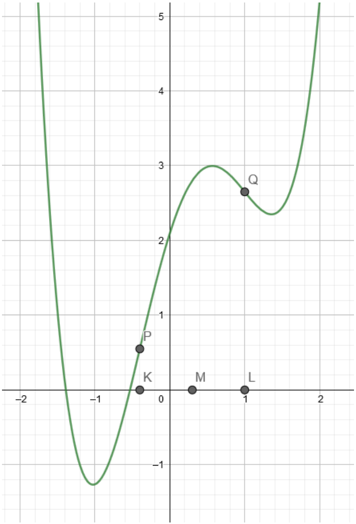
so $f'(x) = 4(x-m)^3 - 12r^2(x-m) + c$

(or even better $+\ cr^3$, but it doesn't matter)

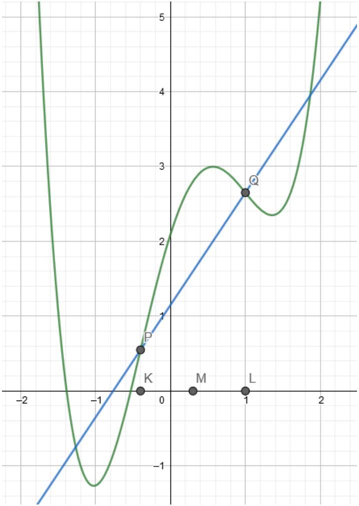
and $f(x) = (x-m)^4 - 6r^2(x-m)^2 + c(x-m) + d$

(or $+\ cr^3(x-m) + dr^4$)

Inflection points and midpoint



Inflection line



Inflection line

$f(x) = (x-m)^4 - 6r^2(x-m)^2 + c(x-m) + d$
Now, the inflection line is given by $y = g(x)$, where

$$g(x) = \frac{f(l) - f(k)}{l-k}(x-m) + \frac{f(l)+f(k)}{2}$$

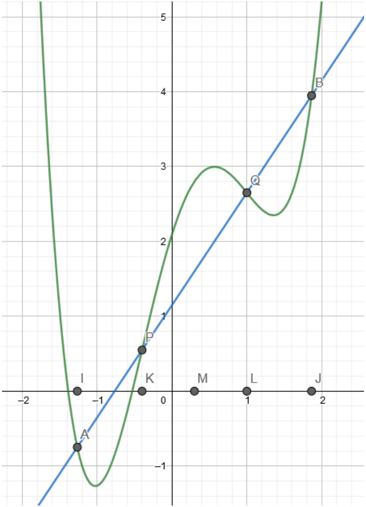
Since $l - m = r = -(k-m)$ and $l - k = 2r$, we have

$$\begin{cases} f(l) &= r^4 - 6r^4 + cr + d \\ f(k) &= r^4 - 6r^4 - cr + d \end{cases}$$

and $\quad g(x) = c(x-m) - 5r^4 + d$
so $\quad f(x) - g(x) = (x-m)^4 - 6r^2(x-m)^2 + 5r^4$

$$= ((x-m)^2 - r^2)((x-m)^2 - 5r^2)$$

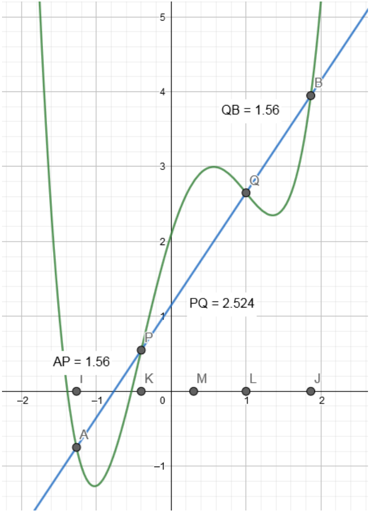
Intersections



Lengths

$f(x) - g(x) = ((x-m)^2 - r^2)((x-m)^2 - 5r^2)$
so the inflection line intersects the quartic for
$x = m \pm r$ and $x = m \pm r\sqrt{5}$, that is at
$i = m - r\sqrt{5}, k = m - r, l = m + r, j = m + r\sqrt{5}$, so
$j - l = k - i$ and $\frac{j-k}{l-k} = \frac{r(1+\sqrt{5})}{2r} = \varphi \; (= \frac{l-k}{j-l} = \frac{2r}{r(\sqrt{5}-1)})$
These same relationships apply, by similar triangles, to the corresponding parts of the inflection line.
Mission accomplished!

Lengths



Some history

This was discovered in 2004 by Lin McMullin when working on determining the relationship between the enclosed areas, a problem set by John F Mahoney.

He proudly describes using his Texas Instruments Voyage 200 with its built-in CAS (Computer Algebra System) to determine the above intersections; in terms of k and l, the other two are $i = \varphi k - \frac{1}{\varphi} l$ and $j = \varphi l - \frac{1}{\varphi} k$, which, I believe rightly, he found even more interesting than the areas.

Lin McMullin

The North Carolina Association
Of
Advanced Placement Mathematics Teachers
Newsletter

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Volume 13 **Winter 2005** **Issue No. 1**

How I Found the Golden Ratio on my CAS

By Lin McMullin

Our friend John Mahoney has an area problem that goes like this. Draw the line through the two points of inflection of a fourth-degree polynomial. The line intersects the graph of the polynomial in four places making three closed regions. From left to right the areas of these regions are in the ratio of 1:2:1. (See *The Mathematics Teacher* November 2002.) I was trying to develop an applet in Winplot to demonstrate this, but I never got it done. Something better came along. Here's what happened.

www.ncaapmt.org/calculus/newsletters/Winter2005/NCAAPMTv13no1.pdf
Lin McMullin also has a blog, https://teachingcalculus.com/

John F Mahoney

John Mahoney

John F. Mahoney

BS Mathematics, 1969
Bucknell University

MA Mathematics, 1972
Temple University

High School Mathematics Teacher,
Benjamin Banneker Academic High School, Washington DC

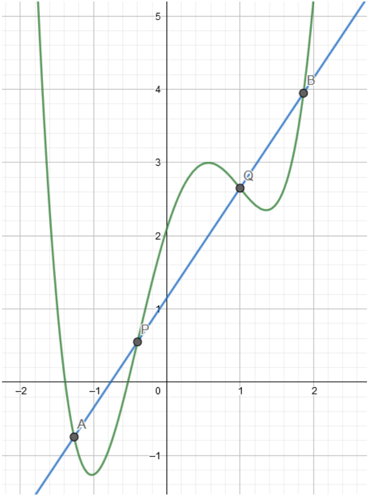
As I was growing up, my favorite classes were in mathematics. So, when I enrolled at Bucknell University in 1965 I decided, somewhat automatically, to major in mathematics. To the likely chagrin of my professors, I devoted more time and energy to anti-war activities than to studying. After graduating, I enrolled in graduate school at Temple University. I was awarded a teaching assistantship and taught one or two courses to freshmen each semester. Many of my students were Vietnam War veterans and women returning to college after raising families. Thus, I was often one of the youngest people in the classroom. At this time I truly became enchanted with teaching mathematics. My students were often extremely anxious about math when they entered my classes; but, all I had to do to turn their views around was to teach humanely. My teaching was anchored in my students' own experiences, which I found to work very well. After completing the course work for my PhD and facing the daunting prospect of learning a second foreign language, I decided to quit graduate school and start teaching high school mathematics.

Because of my contact with Quakers in the peace movement, I accepted a job teaching at Moorestown Friends School in Moorestown, NJ for $7,200 a year. Teaching 9 - 12th graders was harder than teaching college freshmen (but more rewarding). The students were inquisitive, asking questions and wanting to know why certain mathematical statements were true. I remember one student asking me why sine was the y-coordinate and cosine the x-coordinate on a unit circle. Right then I realized that I had just accepted this fact, without understanding it. Teaching high school made me think deeply about the mathematics that I had learned far too quickly. I introduced AP Calculus to the school and really learned the subject well as I was teaching it.

https://www.maa.org/careers/career-profiles/academia-teaching/john-mahoney

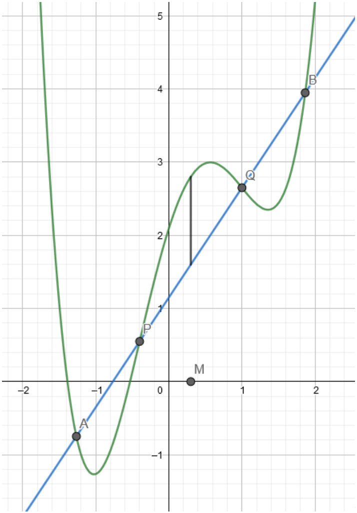
Areas

Let's consider also the areas.



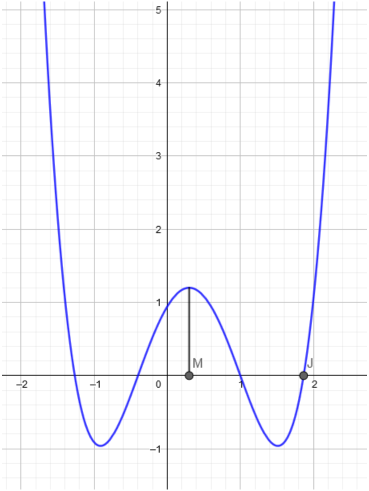
Areas

We can split the middle one at $x = m$.
We have four regions of different shapes.



Areas

We need the integral $\int (f(x) - g(x))dx$



Areas

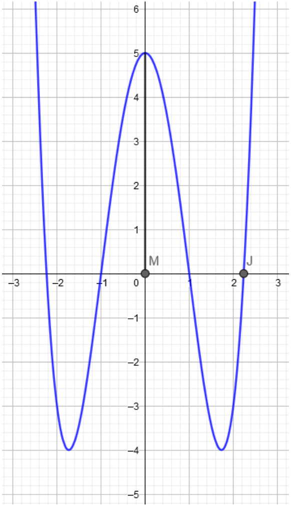
The integrand is symmetric in the line $x = m$, so the two regions below the line are congruent, as are the two halves of the region above the line.

Also,

$\int_m^{m+r\sqrt{5}}(f(x) - g(x))dx = \int_m^{m+r\sqrt{5}}((x - m)^4 - 6r^2(x - m)^2 + 5r^4)dx =$
$\{\text{set } ru = x - m\}$
$= r^5 \int_0^{\sqrt{5}}(u^4 - 6u^2 + 5)du$

Areas

The new integrand

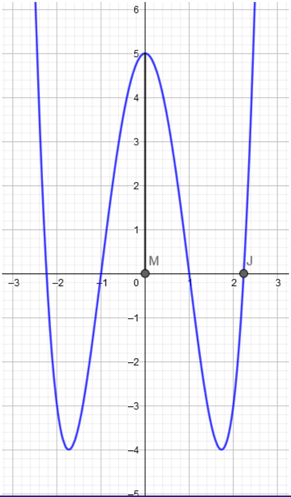


Areas

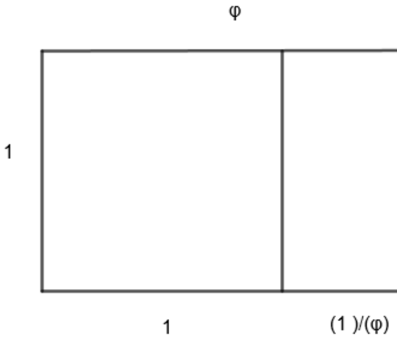
$\int_m^{m+r\sqrt{5}}(f(x)-g(x))dx =$
$= r^5\int_0^{\sqrt{5}}(u^4-6u^2+5)du = r^5[\frac{1}{5}u^5-2u^3+5u]_0^{\sqrt{5}} =$
$= \frac{r^5}{5}[u^5-10u^3+25u]_0^{\sqrt{5}} = \frac{r^5}{5}[u(u^2-5)^2]_0^{\sqrt{5}} = 0$
so half the "above" area is equal to the "below" area.
(This area is $\frac{r^5}{5}\cdot 1\cdot(1^2-5)^2 = \frac{16}{5}r^5 = \frac{(l-k)^5}{10}$.)
I was quite surprised to find that a primitive function of $(u^2-1)(u^2-5)$ is $\frac{1}{5}u(u^2-5)^2$!

Advanced level

A more sophisticated approach to both the length and the area problem is to notice that any quartic is affinely equivalent to this last one, where the relationships are almost obvious.



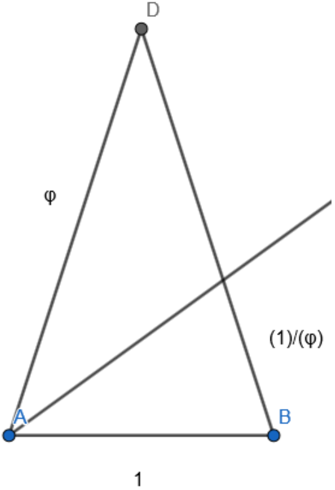
Golden Rectangle



$$\varphi = \frac{\sqrt{5}+1}{2} = \frac{2}{\sqrt{5}-1}$$

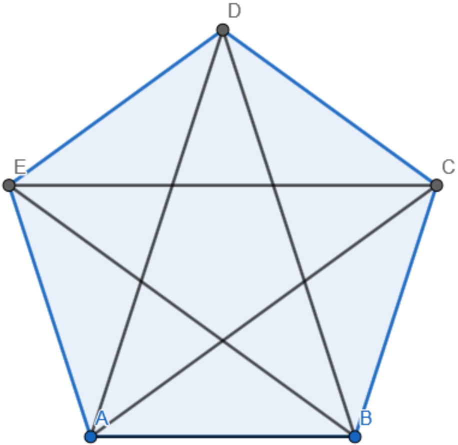
Golden Triangle

The angle at D is 36° and the angle at A is bisected.



Pentagon, pentagram

φ is everywhere.



Reference

NOTES ON QUARTIC CURVES

H. T. R. AUDE, Colgate University

1. Introduction. In these notes certain properties of the Cartesian graph of the quartic function

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

are considered. Some of these, though well known, are linked to other relationships which have turned up in classwork. Thus certain characteristics are pointed out which may assist in the sketching, and add to the understanding of quartic curves.

For convenience, a translation of axes will be applied to remove the term in x^3. Also, a change in the scale of the y-coördinates will, without loss of generality, allow the quartic to be taken in the form

$$(1) \qquad y = f(x) = x^4 + px^2 + qx + s.$$

The American Mathematical Monthly
Vol. 56, No. 3, March 1949, pp. 165-170

As far as I know, this is the first place where the golden section in quartics is pointed out.

NOTES

Quartic Polynomials and the Golden Ratio

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Suppose we have a pentagram in the xy-plane, oriented as in FIGURE 1a, and want to find a quartic polynomial whose graph passes through the three vertices indicated. Out of infinitely many possibilities, there is exactly one quartic polynomial that attains its minimum value at both of the two lower vertices. This graph—shaped like a smooth W with its local maximum at the upper vertex—is shown in FIGURE 1b. Now, how does the graph continue? Will it touch the pentagram again on its way up to infinity? As it turns out, the graph passes through two more vertices, as shown in FIGURE 1c. Furthermore, the two points where the graph crosses the interior of a pentagram edge lie exactly below two other vertices, as shown in FIGURE 1d.

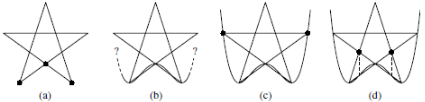

Figure 1 A pentagram and a quartic polynomial

Mathematics Magazine
Vol. 82, No. 3, June 2009, pp. 197-201

Reference

93.27 Fourth degree polynomials and the golden ratio

A fascinating property of fourth degree polynomials with two real points of inflection F and G has been described by McMullin in [1] and [2]. Let E and H be the two other points of intersection between the inflection secant and the graph, see Figure 1. Then

(R1) $EF = GH$ and (R2) $\dfrac{FG}{GH} = \dfrac{\sqrt{5}+1}{2}$.

This means that G divides FH into the golden section.

Moreover,

(A1) Area (P) = Area (P'), and

(A2) $2 \times$ Area (P) = Area (C).

The results (R1) and (R2) were stated and proved by Aude in 1949, see [3]. He also mentions that the area properties were proved by a student in 1948. My aim is to use affine transformations to prove the claims. This has two advantages.

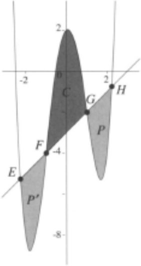

FIGURE 1

R. A. RINVOLD
Hedmark University Collge, Norway

The Mathematical Gazette
Vol. 93, No. 527, July 2009, pp. 292-295

My source

I heard this problem from Swedish high school teacher Bengt Åhlander, Uddevalla, who is also involved in Texas Instruments' T3 (Teachers Teaching with Technology). He had previously given me another problem by John F Mahoney:

Let $f(x)$ be a third-degree polynomial with three zeroes. If a and b are two of those zeroes, let $m = \frac{a+b}{2}$ and show that the tangent to $y = f(x)$ at $(m, f(m))$ cuts the graph at the third zero.

I have of course solved this problem, which actually seems quite well known, but only this spring did I realize the proper way to look at it!

My graphs

This was my first experience with GeoGebra, so my file is working material rather than a finished product.
You find it at https://www.geogebra.org/m/cupjmb9k
where the sliders a and b are the x-values of the inflection points which are called k and l in the talk (I changed them since GeoGebra gave the names K and L to the points on the x-axis). You also find the variable q which is the quotient $\frac{|PQ|}{|QB|}$ which is the golden section so doesn't change when you vary a or b; you may think that I just put a constant in there, but slide both a and b to 0 and see what happens!
You may also enjoy seeing that for large (positive or negative, after all we have symmetry) values of c, we only have one critical point instead of the three we had in the talk, but of course we still have the two inflection points.