Wallpaper Stamps

Step-by-Step Symmetry

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ETH Zürich

TMU, Wettingen, 13. Sept. 2017

Have you ever been to Lisbon?

Restauradores

Rossio

Belém

Chiado

Stepping on Mathematics

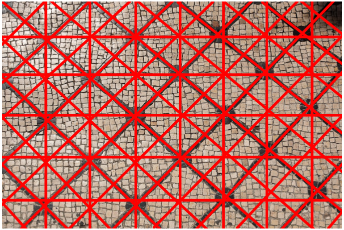
?

?

?

?

Simplest symmetry features



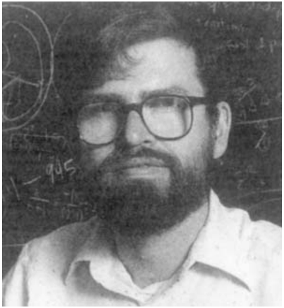
The fishnet / grid pattern is preserved under certain mirror reflections.

There are three types of mirror crossings:

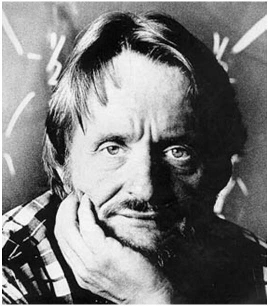
- **4 mirrors cross at the center of each white square;**
- **4 mirrors cross at each black node;**
- **2 mirrors cross at the center of each black edge.**

A new look into the symmetries of things

Murray MacBeath



Bill Thurston



John H. Conway

Conway is the most avid prophet of the new geometric perspective on symmetry and devised the terminology and signature notation.

The four fundamental **features of symmetry**

kaleidoscopes, **gyrations**, **miracles** & **wonders**

gyrations and wonders are true to orientation $\longrightarrow$ blue

kaleidoscopes and miracles reflect $\longrightarrow$ red

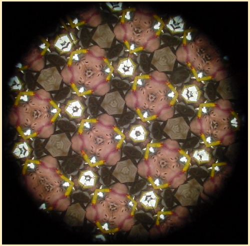
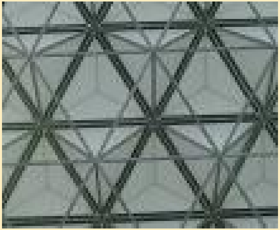
and their symbols: $\ast$, digits or ∞, X and O

each symmetry type corresponds to a symbolic signature such as $\ast$ 4 4 2 or 2 2 $\ast$ or 4 4 2 or X X

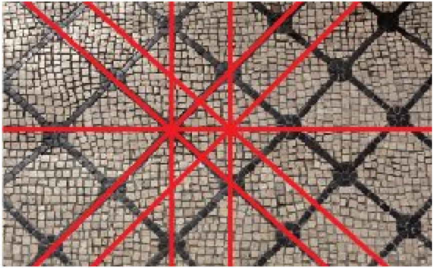
KALEIDOSCOPES

$\ast$ a b c symbolizes mirrors crossing with angles $\dfrac{180^\circ}{a}, \dfrac{180^\circ}{b}, \dfrac{180^\circ}{c}$.

Examples $\ast$ 3 3 3


A closer look at the fishnet pattern



the three representative mirror crossings have angles

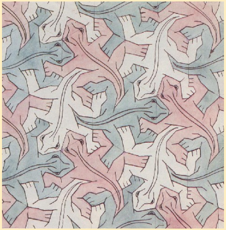
$$\frac{180^\circ}{\mathbf{4}}, \frac{180^\circ}{\mathbf{4}}, \frac{180^\circ}{\mathbf{2}}$$

$\Longrightarrow$ **pattern signature $\ast\ 4\ 4\ 2$.**

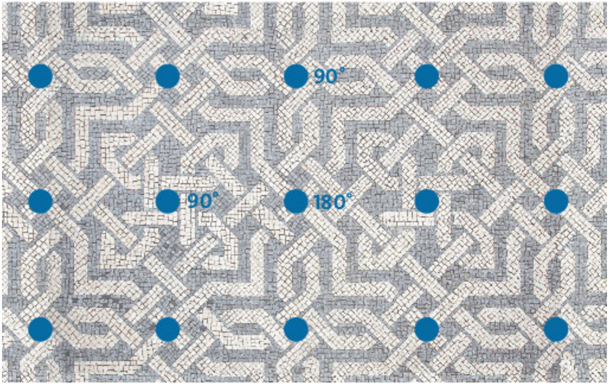
GYRATIONS

a b c symbolizes gyration points by angles $\frac{360°}{a}$, $\frac{360°}{b}$, $\frac{360°}{c}$, which do not lie on any mirror line.

Examples 3 3 3



A closer look at the *ropes* pattern



is preserved by rotation about three points by angles

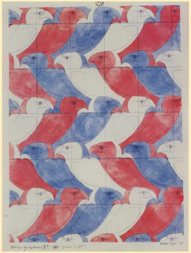
$$\frac{360^\circ}{\mathbf{4}}, \frac{360^\circ}{\mathbf{4}}, \frac{360^\circ}{\mathbf{2}}$$

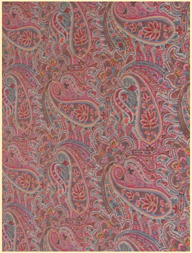
$\Longrightarrow$ **signature 4 4 2**

MIRACLES

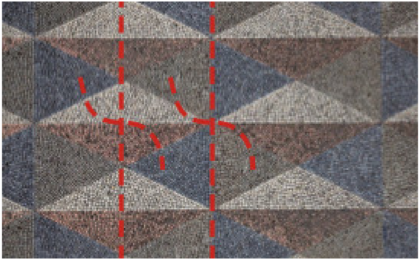
X represents a mirrorless crossing, i.e, a path from a point to a reflected copy of itself without ever touching a mirror line.

Examples **X X** and ∗ **X**





A closer look at the four-colour cobblestone pattern



has two miracles

$\Longrightarrow$ **signature XX**

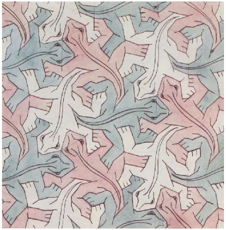
WONDERS

O (Oh! or zero) represents a pattern without kaleidoscopes, nor gyrations, nor miracles.

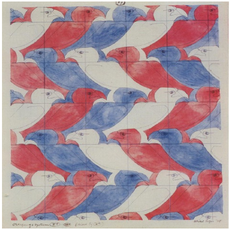
Examples O

The four symmetry features in Escher's work

Kaleidoscopes



Gyrations



Miracles

Wonders

How to find a signature of a pattern

1. **Mark one kaleidoscope corner of each type with a $*$ and the number of mirrors through it.**

2. **Mark one gyration point of each type with a $\bullet$ and its order.**

3. **Can you walk from some point to a reflected copy of itself without ever touching a mirror line?**
   If so, a miracle has occurred.
   Mark that path with a broken red line and a X.

4. **If you've found none of the above, then there is a wonder. Mark it with O.**

Friezes are analysed analogously

$\underbrace{2}_{\text{gyration}}$ $\underbrace{*\infty}_{\text{kaleidoscope}}$

gyration point by angle $\frac{360°}{2}$

infinite parallel mirrors cross at infinity

Back to Lisbon

4 4 2

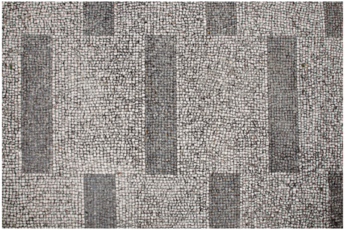
2 2 *

x x

* 4 4 2

2 2 2 2

3 * 3



2 * 2 2

* *

∗ 6 3 2

∗ X

∗ 2 2 2 2

∗ 4 4 2

And a few *friezes*

$2 * \infty$

$2\,2\,\infty$

$* \infty \infty$

$\infty\infty$

In general, we can find 17 + 7 types!

17 types of plane patterns

(signatures not involving ∞)

7 types of friezes

(signatures involving ∞)

*632	632	*442	442	*333	*22∞	22∞	2*∞
333	*2222	2222	4*2	3*3	2*22	*∞∞	∞∞
22*	**	*X	XX	22X	O	∞*	∞X

Menu: cost of the symmetry features

Symbol	Cost (Fr.)	Symbol	Cost (Fr.)
0	**2**	$*$ **or X**	**1**
2	$\frac{1}{2}$	**2**	$\frac{1}{4}$
3	$\frac{2}{3}$	**3**	$\frac{1}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
number n	$\frac{n-1}{n}$	**number n**	$\frac{n-1}{2n}$
∞	**1**	∞	$\frac{1}{2}$

For instance, 3 costs $\frac{2}{3}$ **and** $*$ **3 costs** $1 + \frac{1}{3} = \frac{4}{3}$**.**

The meal 2 2 ∞ **costs** $\frac{1}{2} + \frac{1}{2} + 1 =$ **Fr. 2.**

The meal * 4 4 2 also costs $1 + \frac{3}{8} + \frac{3}{8} + \frac{1}{4} =$ **Fr. 2.**

Magic Theorem

the types of patterns and friezes are exactly those with signatures (meals) of total cost 2

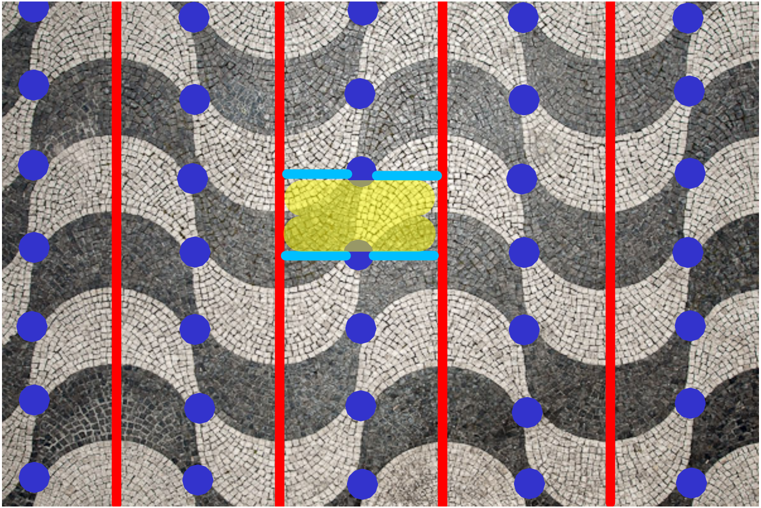
Summary:

- **There are 4 features of symmetry:**
  kaleidoscopes, gyrations, miracles and wonders.

- **The magic theorem implies that there are:**
  17 types of plane patterns and 7 types of friezes.

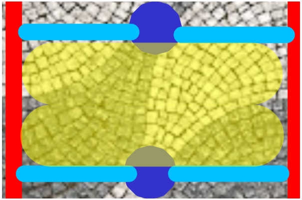
Where does this come from?!

A closer look at the *waves* pattern

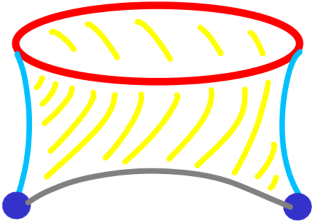


with its mirrors, gyrations and a chosen fundamental region

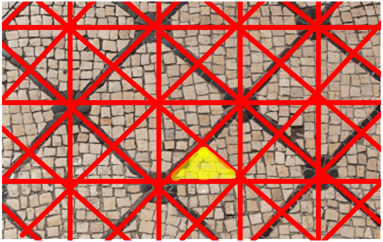
A closer look at the *waves* pattern

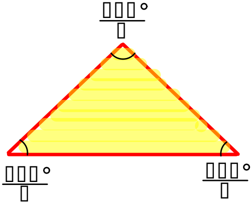


wallpaper stamp $\underbrace{2\quad 2}_{\text{gyrations}}\quad\underbrace{*}_{\text{mirror}}$



A closer look at the *fishnet* pattern





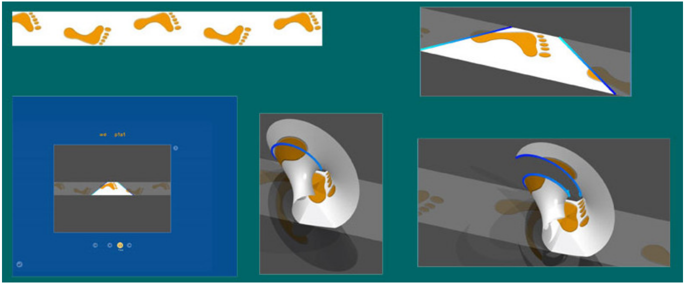
wallpaper stamp $\underbrace{*442}_{\textbf{kaleidoscope}}$

A closer look at the hopping and walking friezes

wallpaper stamp

$\Longrightarrow$

$\Longrightarrow$



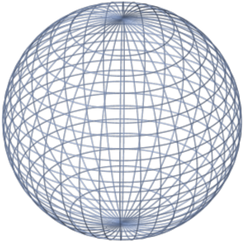
A wonderful Swiss characteristic

Euler Characteristic of a surface

$$\chi = (\#\textbf{vertices}) - (\#\textbf{edges}) + (\#\textbf{faces})$$

is independent of

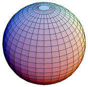
the chosen polygon subdivision!



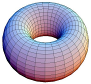
Euler Characteristic of a spherical surface

$$(\#\textbf{vertices}) - (\#\textbf{edges}) + (\#\textbf{faces}) = \mathbf{2}$$

Examples of surfaces $\mathrm{O}^m *^n$ or $*^n \mathrm{X}^p$

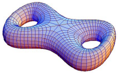


$\chi = 2$



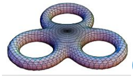
O

$\chi = 0$



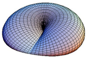
O^2

$\chi = -2$



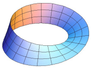
O^3

$\chi = -4$



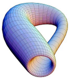
X

$\chi = 1$



$*\mathrm{X}$

$\chi = 0$



X^2

$\chi = 0$

...

Classification of surfaces $\quad O^m *^n \quad$ or $\quad *^n X^p$

Any (bounded and connected) surface can be obtained from a sphere by

- *** – punching holes (this introduces boundaries);**
- **O – adding handles and**
- **X – adding crosscaps, i.e. replacing disks by Möbius bands.**

These operations affect the Euler Characteristic χ:

- **punching a hole or adding a crosscap decrease χ by 1;**
- **adding a handle decreases χ by 2.**

Generalization to Orbifolds

A 2-dim **orbifold** *is a surface endowed with special points:*

either cone points of order n
or mirror points (along boundary lines)
or corner reflectors of order n (intersection of n mirror lines).

Orbifold Euler characteristic

$\chi = (\#\text{vertices}) - (\#\text{edges}) + (\#\text{faces})$

where now special vertices and edges are weighted:

- **an edge on a mirror contributes $-\frac{1}{2}$ (and not -1);**
- **if a vertex is a cone point of order n it contributes $\frac{1}{n}$;**
- **a vertex on a mirror, but that is not a corner reflector, contributes $\frac{1}{2}$;**
- **if a vertex is a corner refl. of order n it contributes $\frac{1}{2n}$;**
- **ordinary vertices and edges contribute 1 and -1, as before.**

Classification of orbifolds

Any (2-dim) orbifold can be obtained from a surface by

- **replacing some interior points by cone points and**
- **replacing some boundary points by corner reflectors.**

Introducing orbifold points affects the Euler Characteristic χ:

- **a cone point of order n decreases χ by $\frac{n-1}{n}$ and**
- **a corner reflector of order n decreases χ by $\frac{n-1}{2n}$.**

Summary of the proof of the *magic theorem*

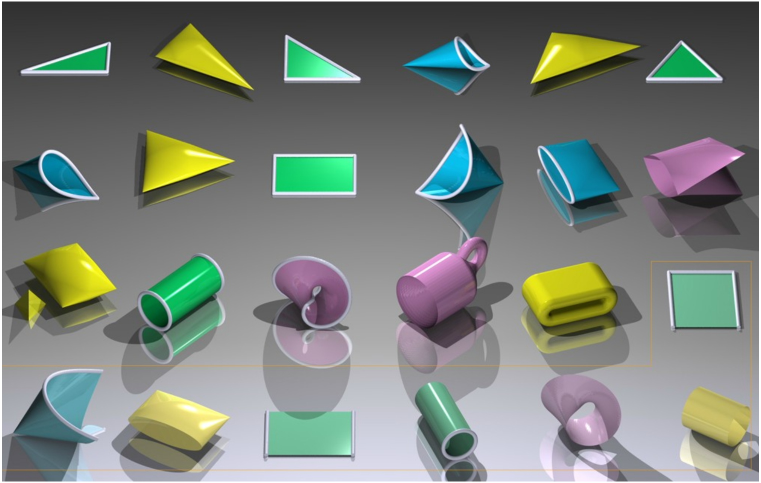
- **Any pattern can be folded-up into an orbifold – the stamp of that pattern – by taking all points of the same kind to a single point.**

- **The types of Euclidean plane patterns and friezes correspond one-to-one with the 2-dim. orbifolds having orbifold Euler characteristic zero.**

- **Every connected 2-dimensional orbifold can be obtained by surgery on a sphere: adding cone points, holes/mirror lines, corner reflectors, handles and cross-caps.**

- **The effect of adding such a feature to an orbifold has a certain cost on its Euler characteristic (see menu).**
  The total cost should be 2, so that $\chi = \mathbf{2} - \mathbf{cost} = \mathbf{0}$.

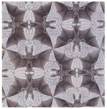
The 17+7 orbifolds with **zero** orbifold Euler characteristic



Symmetry-related activities


A

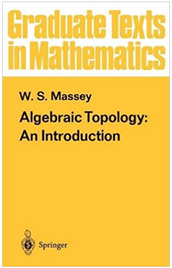

B


C


1


2

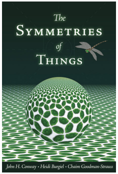

3



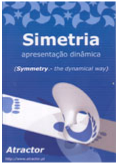
- **Match patterns.**
- **Memo game: pattern pairs. And other games.**
- **Classify patterns by writing their signature.**
- **Design patterns from orbifolds – see DVD.**
- **Check classif. of surfaces or other parts of the proof.**
- **Extend to patterns on sphere or on hyperbolic plane.**

Frieze fitness

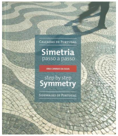
References:



The Symmetries of Things
book by Conway, Burgiel and Goodman-Strauss



Symmetry – the Dynamical Way
interactive software by `www.atractor.pt`



Symmetry Step by Step – Sidewalks of Portugal
book by Ana Cannas

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