Double counting

Snapshots from the ETH Math Youth Academy

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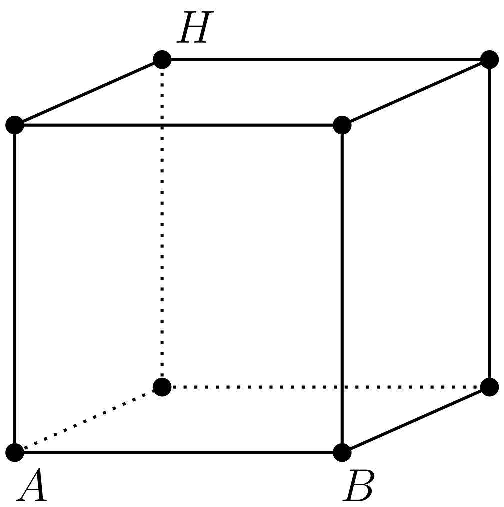
`https://www.math.ethz.ch/eth-math-youth-academy`

September 13, 2017



Can you put the numbers

$$1, 2, \ldots, 12$$

on the edges of a cube, so that the sum of the three numbers on the edges out of each vertex is the same?

Solution. Suppose that you can. Let

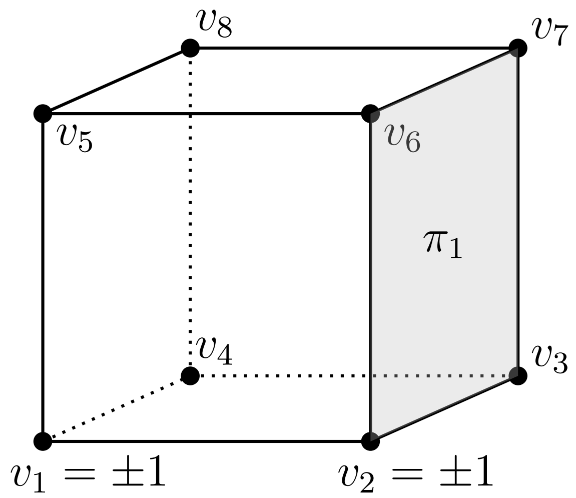
$$S_A = \text{the sum of the numbers on the } 3 \text{ edges out of } A$$

and similarly $S_B, \ldots, S_H$. Then

$$S_A + \cdots + S_H = (1 + 2 + \cdots + 12).2 = 12.13 = 156$$

not divisble by 8

$\Longrightarrow$ $S_A, \ldots, S_H$ cannot be all equal!



Sarah writes a number among ± 1 at each vertex of a cube.

At each face, she puts the product of the numbers written at its four vertices.

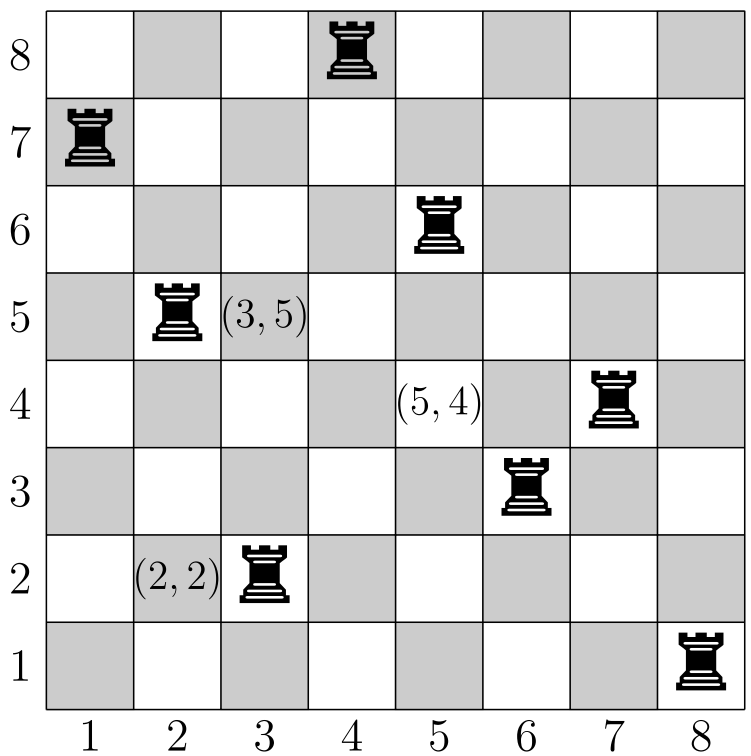
Altogether, she has written down $8 + 6 = 14$ numbers (each ± 1).

Is it possible that their sum is 0?

Solution. The product of the 14 numbers written down is

$$\begin{aligned} v_1 \ldots v_8 \pi_1 \ldots \pi_6 &= v_1^4 v_2^4 \ldots v_8^4 && \text{going vertex by vertex} \\ &= (v_1 \ldots v_8)^4 \\ &= 1 \end{aligned}$$

$\Longrightarrow$ we can't have 7 positive and 7 negative numbers on the left.



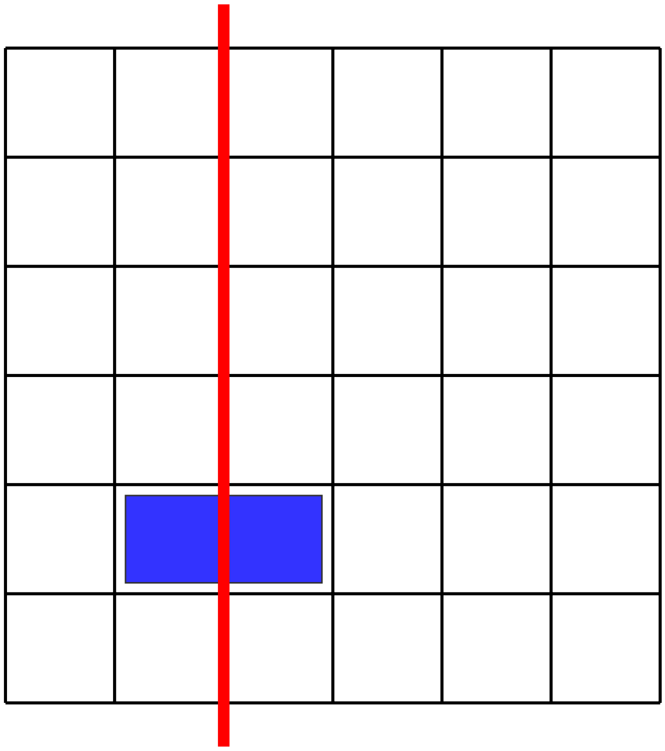
One has placed 8 rooks on a chessboard, no two attacking each other.

Prove that the number of rooks on *white* squares is *even*.

Proof. The square (x, y) is white $\iff x + y$ is *odd*.

Let $(x_1, y_1), \ldots, (x_8, y_8)$ be the positions of the rooks. Then

$$\begin{aligned} \underbrace{(x_1 + y_1) + \cdots + (x_8 + y_8)}_{\substack{\text{the number of odd terms} \\ \text{must be even}}} &= (x_1 + \cdots + x_8) + (y_1 + \cdots + y_8) \\ &= (1 + \cdots + 8) + (1 + \cdots + 8) \\ &= 72 \quad \text{even!} \end{aligned}$$



A 6×6 board is tiled by 2×1 domino pieces.

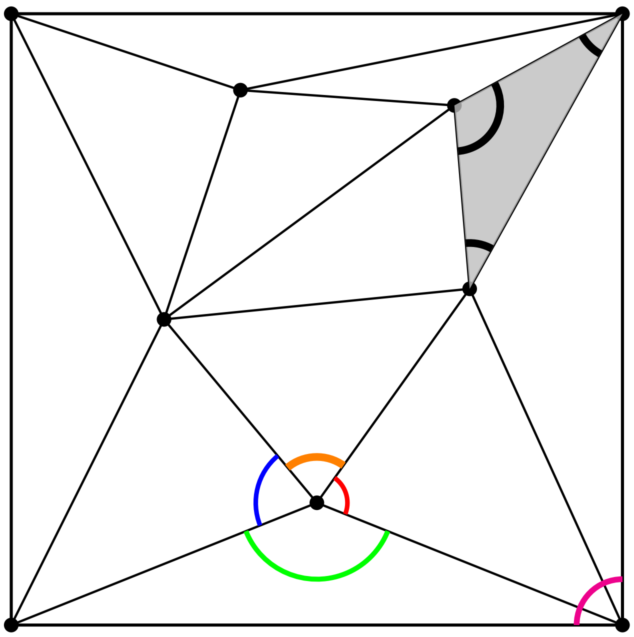
Prove that the board can be cut by a line that breaks *none* of the domino pieces.

Proof. Suppose that each of the 10 candidates for a line breaks at least one domino piece.

A line cannot break just a *single* domino piece.

$\implies$ each of these 10 lines breaks at least **two** domino pieces.

$\implies$ there are at least $10.2 = 20$ domino pieces. But, there are 18.



Example with 5 points inside.

Peter marks 20 points inside a square, as well as its 4 vertices.

Then he connects some pairs of marked points, so that

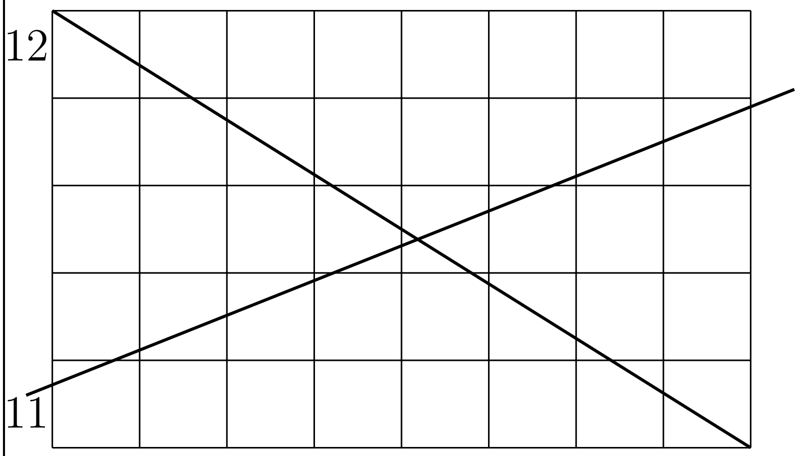
- no two segments intersect
- the square gets divided into triangles.

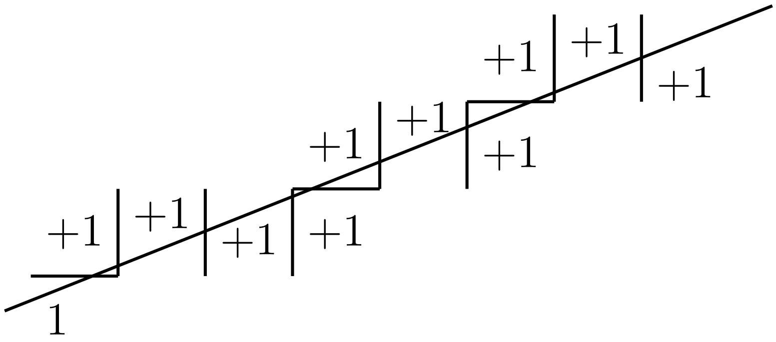
How many triangles does Peter obtain? Does this number depend on the way he chooses the 20 points or the specific subdivision?

Solution. Double–count *the sum of all angles of all triangles* in the subdivision.

$$\text{(number of triangles)}.180^\circ = 20.360^\circ + 4.90^\circ \qquad 20 \text{ points inside}$$

$$\implies \text{(number of triangles)} = 42.$$





Consider a grid 5×8.

What is the largest number of squares that a line can intersect (in their interiors)?

Claim: 12.

Proof. Consider a line and traverse it. Any time it intersects a gridline, it enters a new square. So, it can go through at most

$$1 + 4 + 7 = 12$$

squares.