Mathematical colorings

A class at the ETH Math Youth Academy

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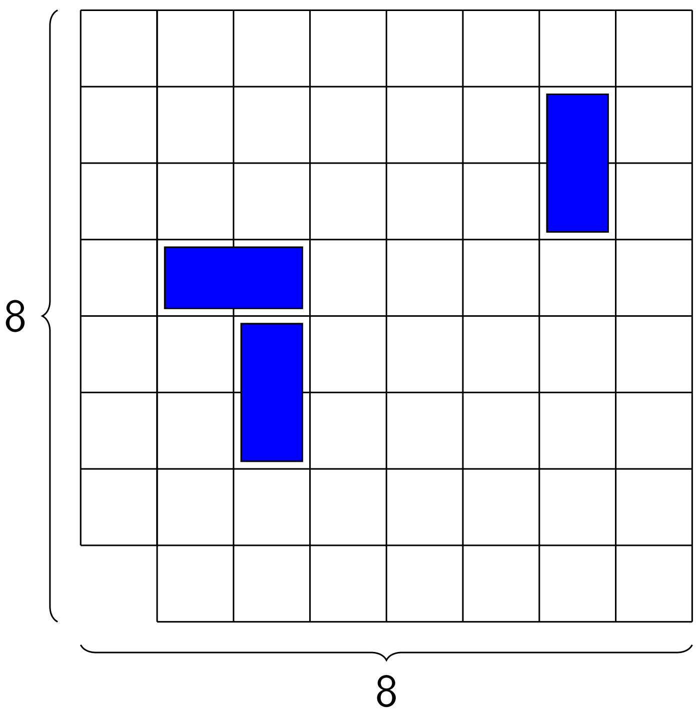
Domino tiling — a)



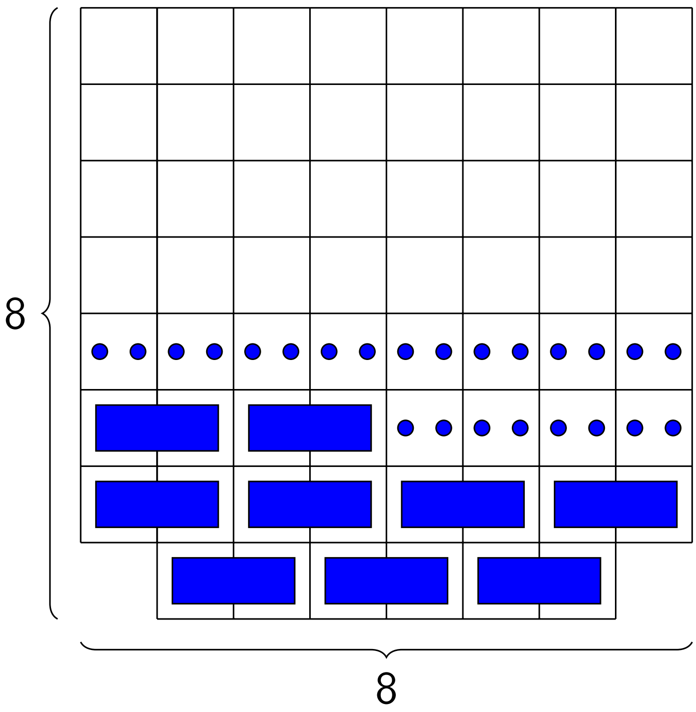
a) Is it possible to *tile* this board by 2×1 domino pieces — i.e., to cover it

- without overlaps, and
- without leaving the board?

— **No,** because there is an *odd* number of squares (63). ✓

If you can tile a board $\Longrightarrow$ the number of squares must be even.

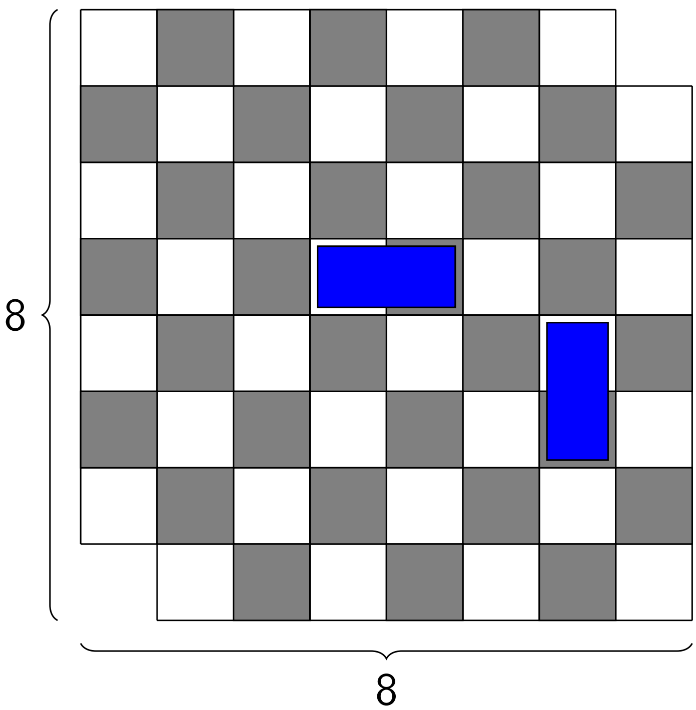
Domino tiling — b)



b) Is it possible to tile this board by 2×1 domino pieces?
— Student: **Yes,** because there is now an *even* number of squares (62).
— ...
Student: — **Yes,** because we can tile as follows. ✓

you can tile the board $\underset{?}{\Longleftarrow}$ the number of squares is even

Proof – c)

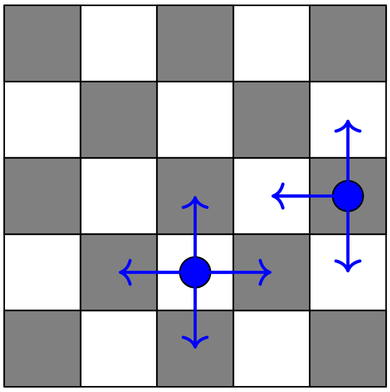


c) Is it possible to tile this board by 2×1 domino pieces?

No. Consider the chess coloring of the board. Each domino takes one black and one white square. However, there are 32 white and 30 black squares. ✓

you can tile the board $\underset{\not\Longleftarrow}{\Longrightarrow}$ the number of squares is even

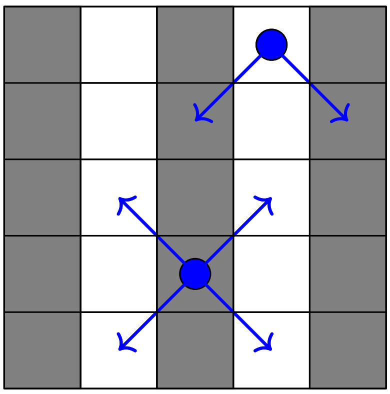
Bugs – a)



a) There is a bug at each square of a 5×5 grid. At a given instant, each bug craws horizontally or vertically to a neighboring square. Prove that some square will remain empty.

Proof. Color the board in a chess patern.
Each bug changes the color of its square.
There are 13 black and 12 white squares.
$\implies$ some black square will remain empty. ✓

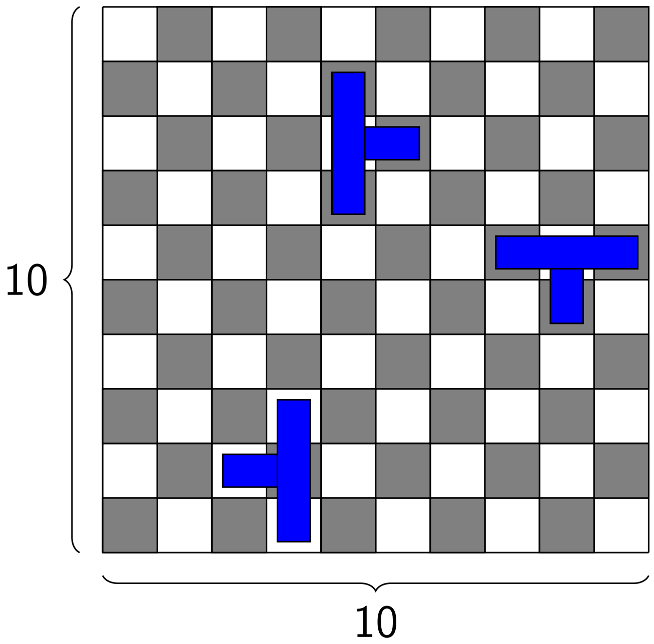
Bugs – b)



a) There is a bug at each square of a 5×5 grid. At a given instant, each bug craws *diagonally* to a neighboring square. Prove that at least 5 squares will remain empty.

Proof. Color the board as shown.
Each bug changes the color of its square.
There are 15 black and 10 white squares.
$\implies$ at least 5 black squares will remain empty. ✓

T–tetraminos



Is it possible to tile a 10×10 board by T-tetraminos?

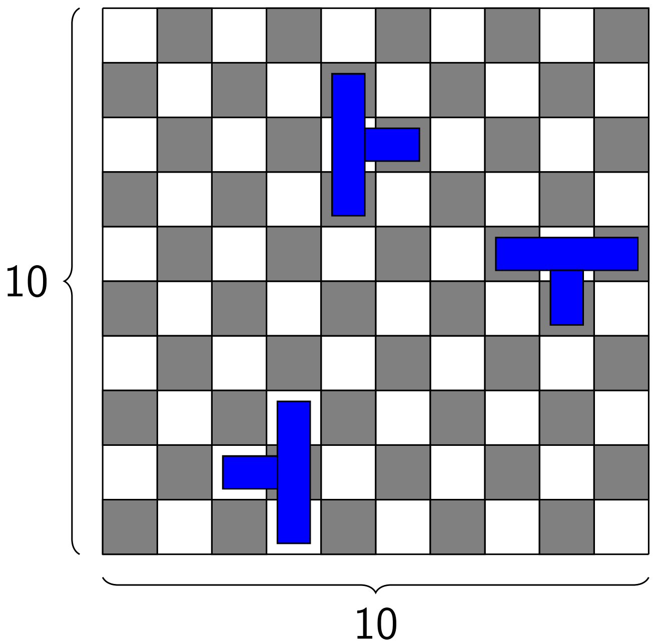
No. Color the board in a chess pattern. Each T–tetramino takes either

a) 3 black and 1 white, or
b) 3 white and 1 black

squares.

Argument 1. Each T–tetramino covers an odd number of black squares. If a tiling exists, there would be $\frac{100}{4} = 25$ tetraminos. Altogether, they would cover an *odd* number of black squares. However, there are 50 black squares.

T–tetraminos



Is it possible to tile a 10×10 board by T-tetraminos?

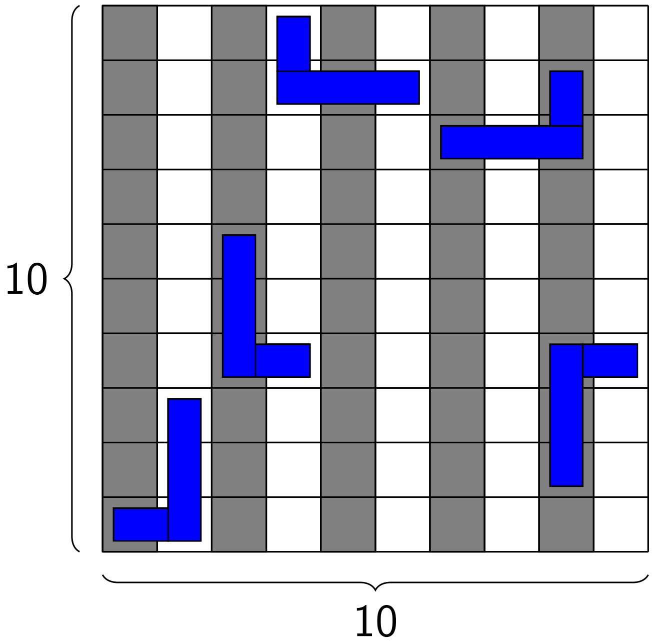
No. Color the board in a chess pattern. Each T–tetramino takes either

a) 3 black and 1 white, or
b) 3 white and 1 black

squares.

Argument 2. If there are more T–tetraminos of type a), there would be more black squares total. So, there must be as many T–tetraminos of type a) as of type b). However, the total number of T–tetraminos is 25 — odd.

L–tetraminos



Is it possible to tile a 10×10 board by L–tetraminos?

No. Color the board as shown. Each L–tetramino takes either

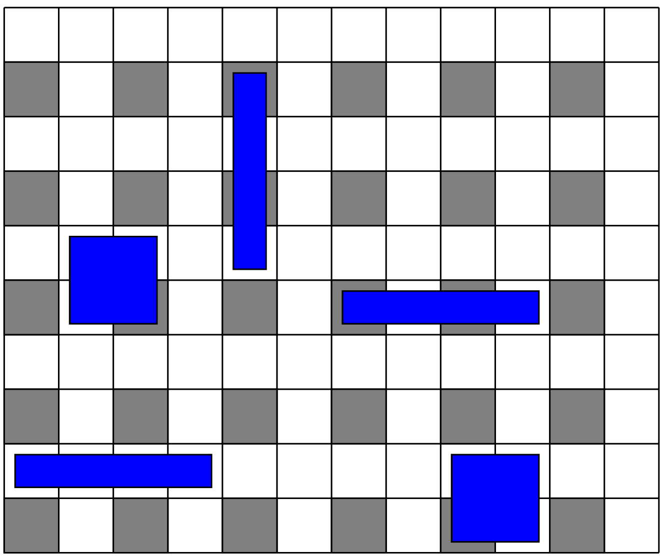
a) 3 black and 1 white, or

b) 3 white and 1 black

squares.

Finish as either argument from the previous problem. ✓

A bathroom floor



A rectangular bathroom floor was tiled by tiles of two kinds:

- 2×2, and
- 1×4.

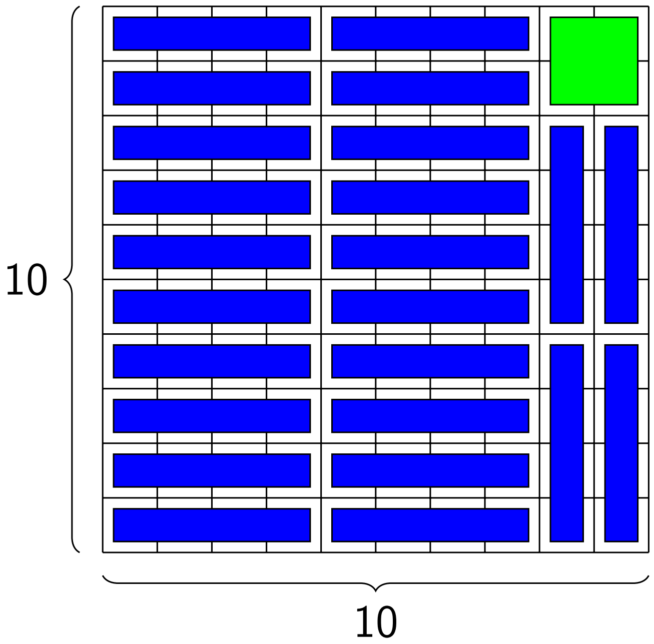
Before gluing the pieces, one of the 2×2 tiles got lost. A spare 4×1 tile is available. Is it still possible to tile the floor?

No. Color the board as shown.

- Each 2×2 tile covers exactly 1 black square. odd!
- Each 4×1 tile covers 0 or 2 black squares. even!

$\Longrightarrow$ the total number of 2×2 tiles in *any* tiling must be (in this picture) even!

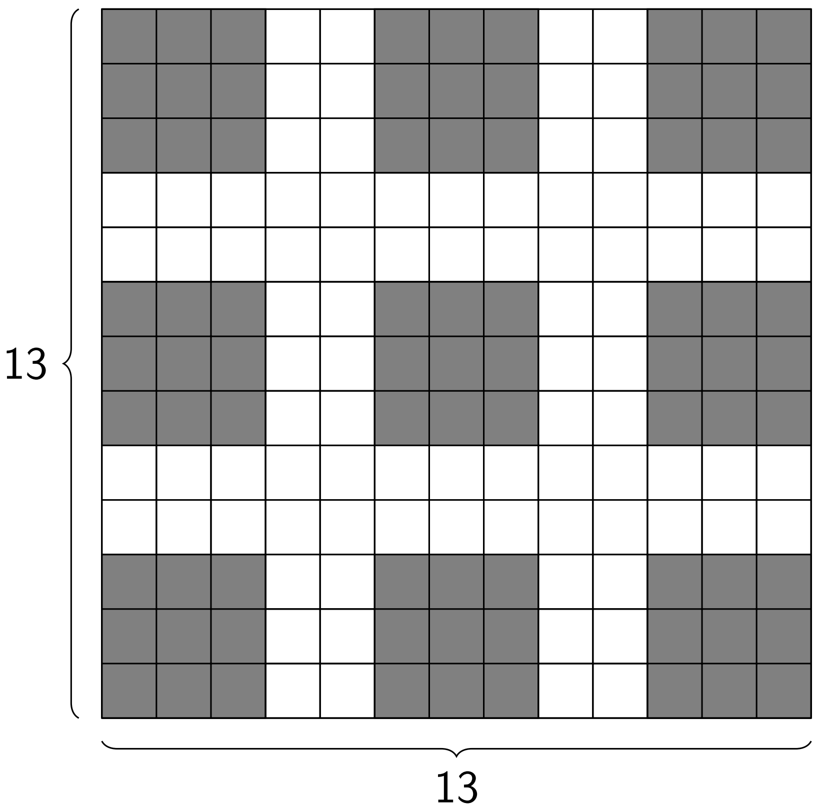
4×1 tetraminos



Is it possible to tile a 10×10 board by 4×1 tetraminos?

No. By the previous problem, the 2×2 piece cannot be replaced by a 4×1 tetramino. ✓

Cutting out



A board 13×13 is given. A total of 8 squares of size 3×3 have been cut out. Is it always possible to cut out yet another 3×3 square?

Yes. Color as shown. Each of the 8 removed 3×3 squares hits only *one* of the 9 black 3×3 squares.

$\Longrightarrow$ at least one of the black squares has remained unhurt. ✓

Public Talks

- Mathematical Induction, LG Rämibühl, 27.10.2015
- Mathematical Games, KS Baden, 07.04.2016
- (upcoming) Mathematical Colorings, RG Rämibühl, 30.09.2016
- ....