Kryptografie: Mit Mathematik gegen Hacker und die NSA

Ueli Maurer

Departement Informatik, ETH Zürich

27. Schweiz. Tag über Mathematik und Unterricht, KS Wil, 7. Sept. 2016

Ziele des Vortrags

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1. Mathematik mit zentral wichtigen Anwendungen

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2. Coole mathematische Probleme
   in Mittelschulreichweite

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1. Mathematik mit zentral wichtigen Anwendungen

2. Coole mathematische Probleme
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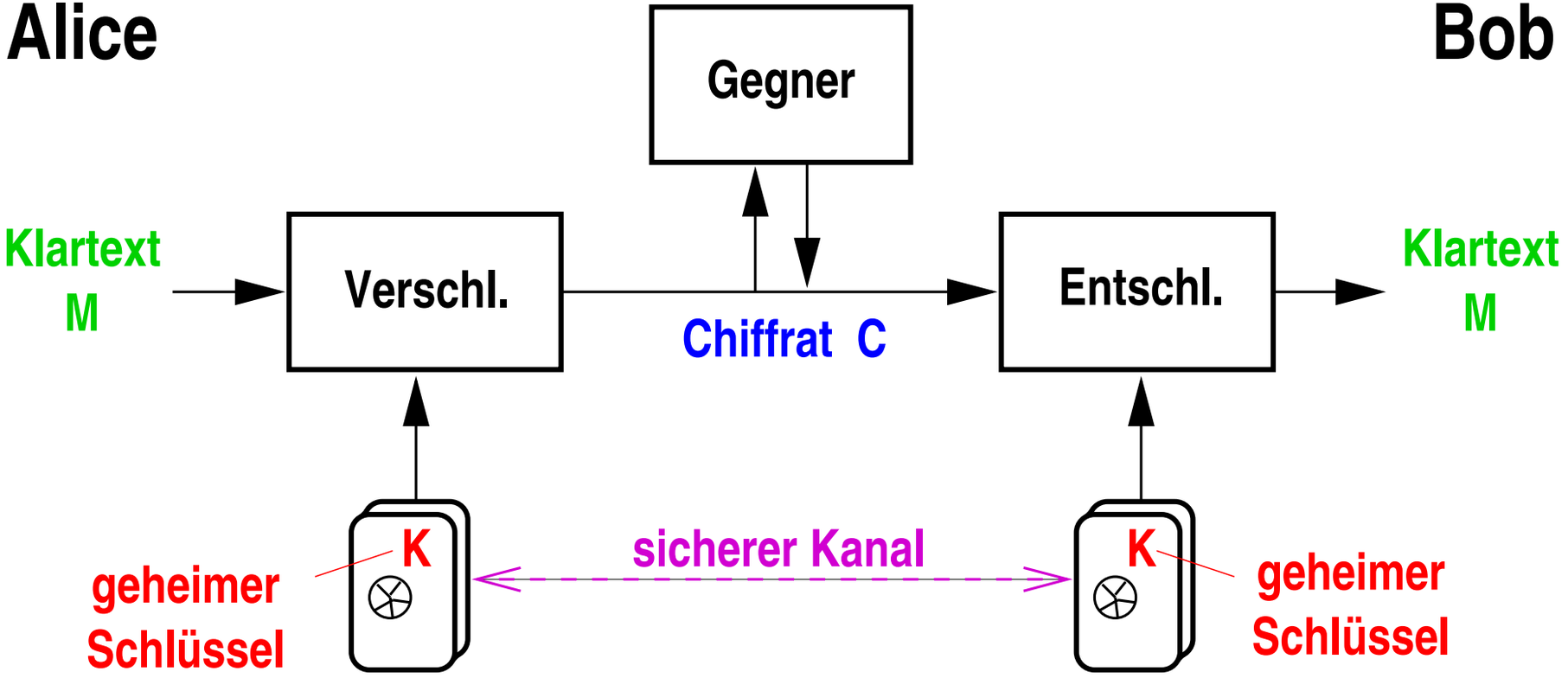
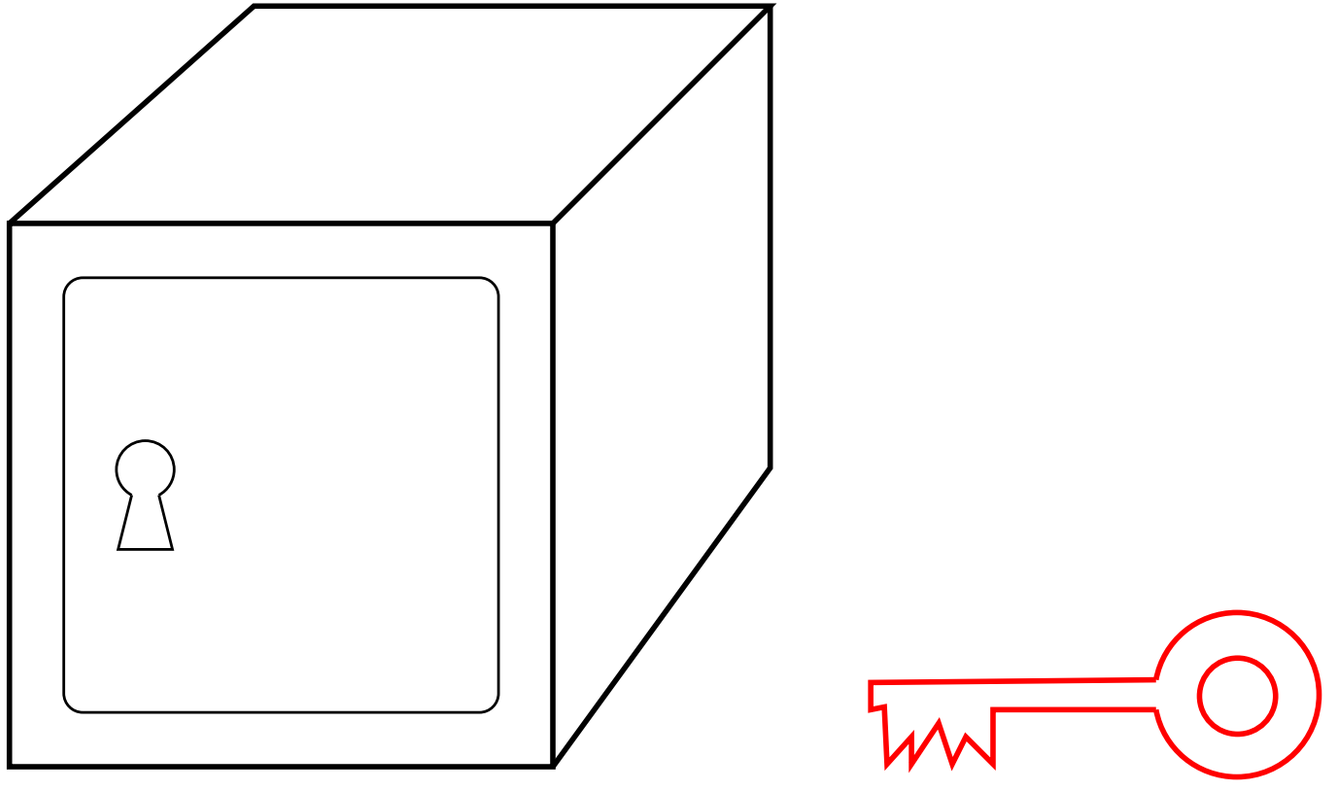
3. Werbung für das ETH-Informatikstudium

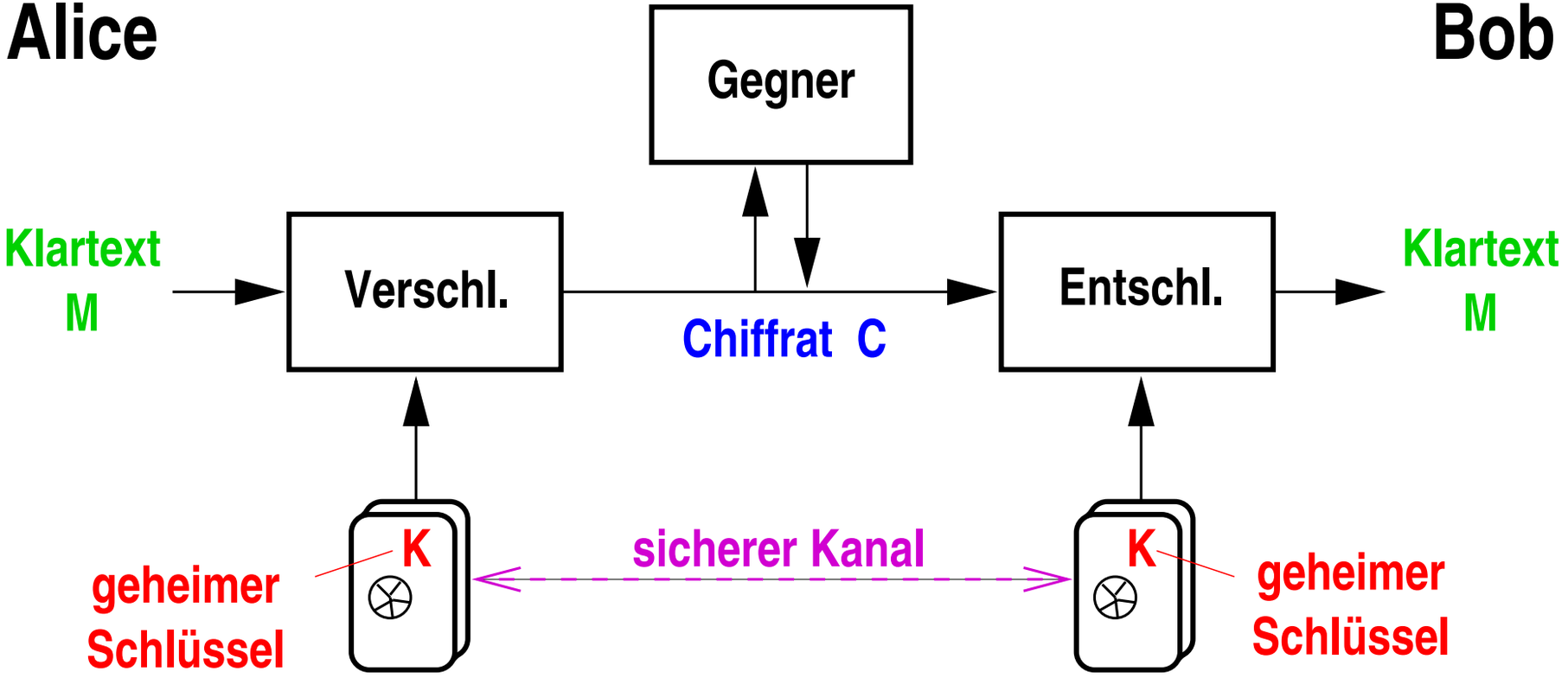
Verschlüsselungssystem

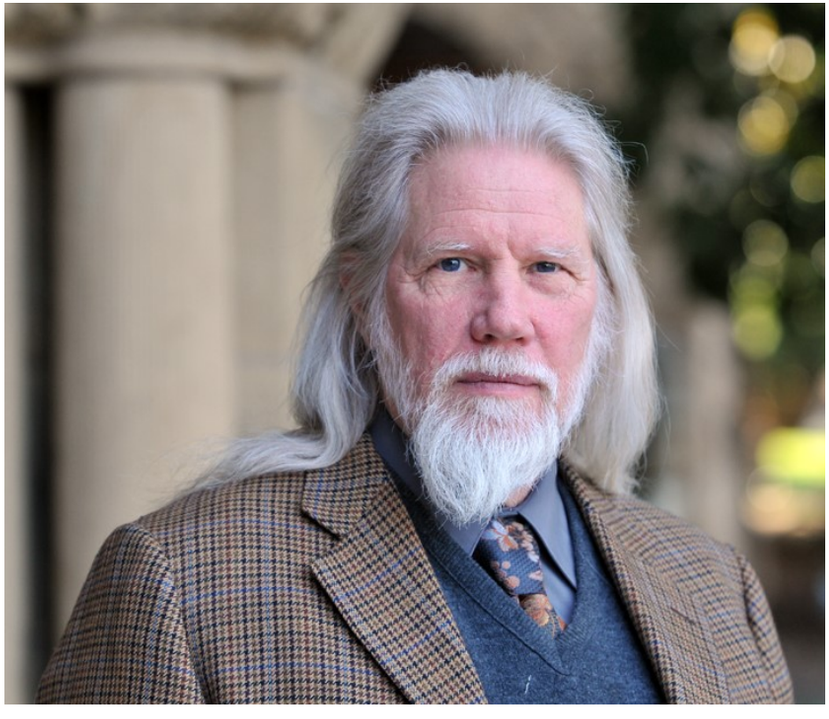


Verschlüsselung: mechanisches Analogon



Verschlüsselungssystem





Whitfield Diffie

Martin Hellman

Faktorisierung von Zahlen

$$323 = 17 \cdot 19$$

Faktorisierung von Zahlen

1098227363564981025449827

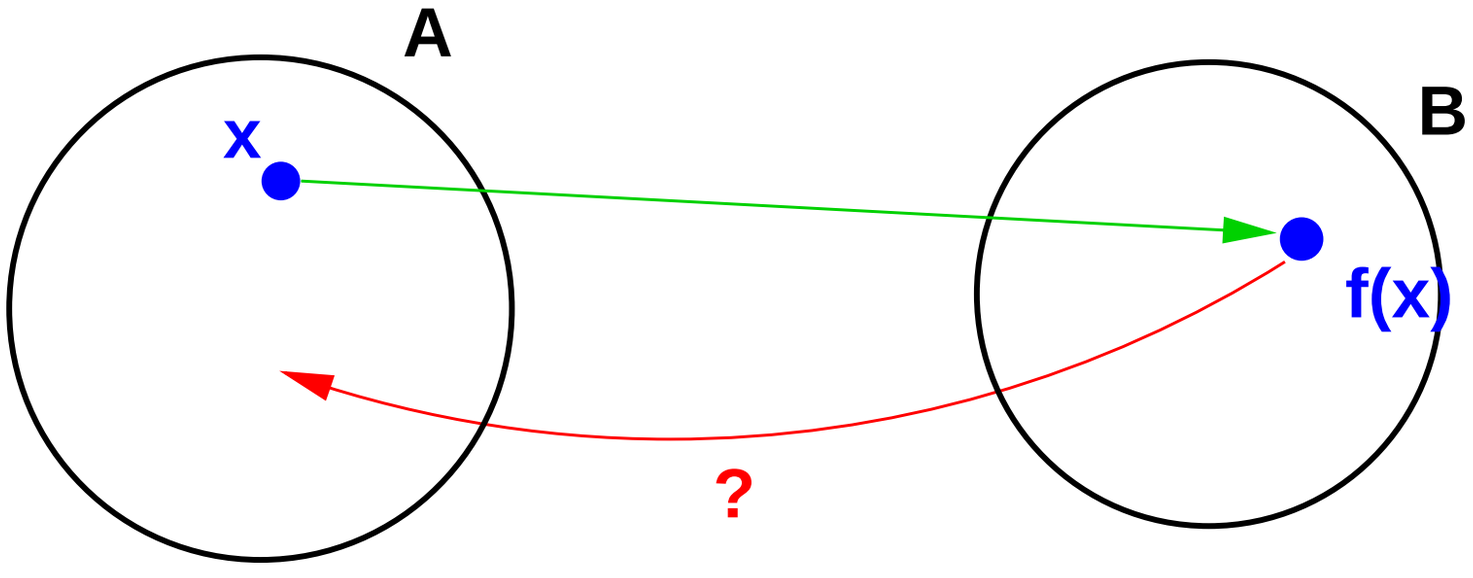
364098583475109184536287 6

341743634934764453302938 2

746711723976589866033564 3

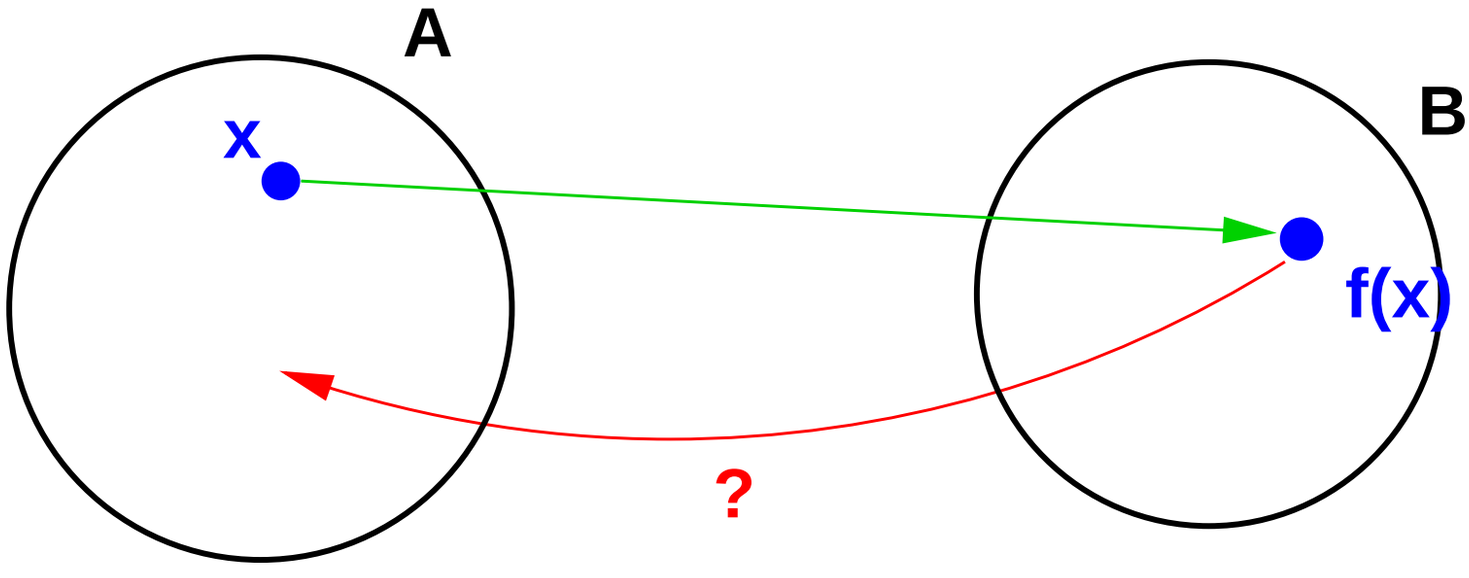
893364751029800354637562 3 = ? · ?

Einwegfunktion f



f(x) ist einfach berechenbar für jedes x.

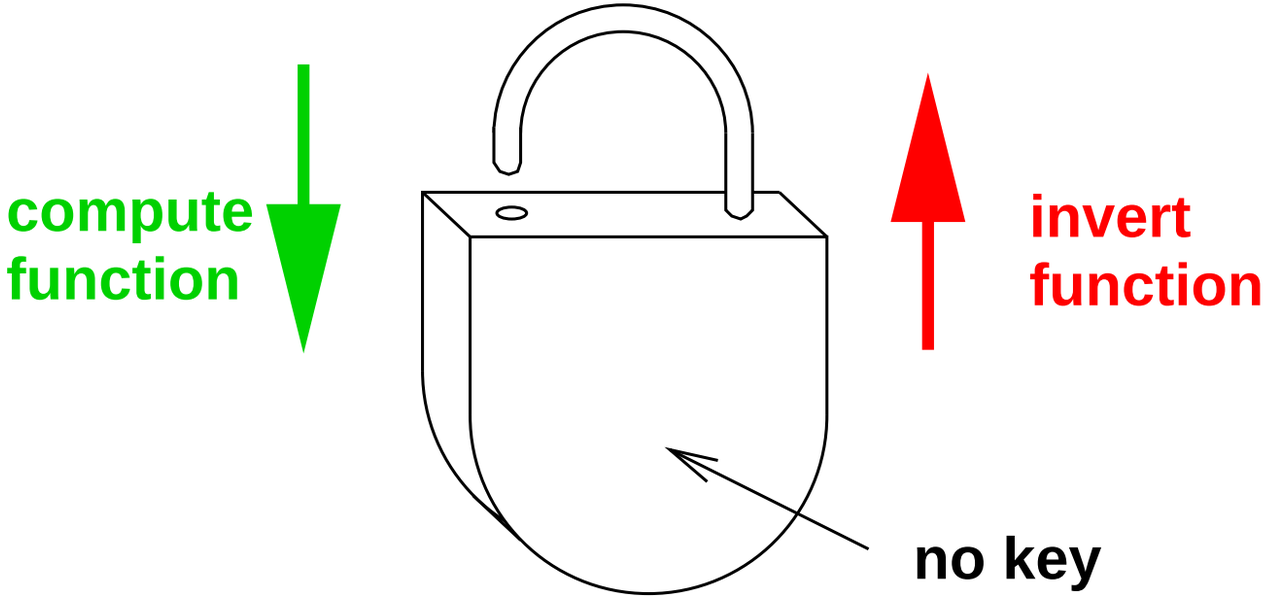
Einwegfunktion f



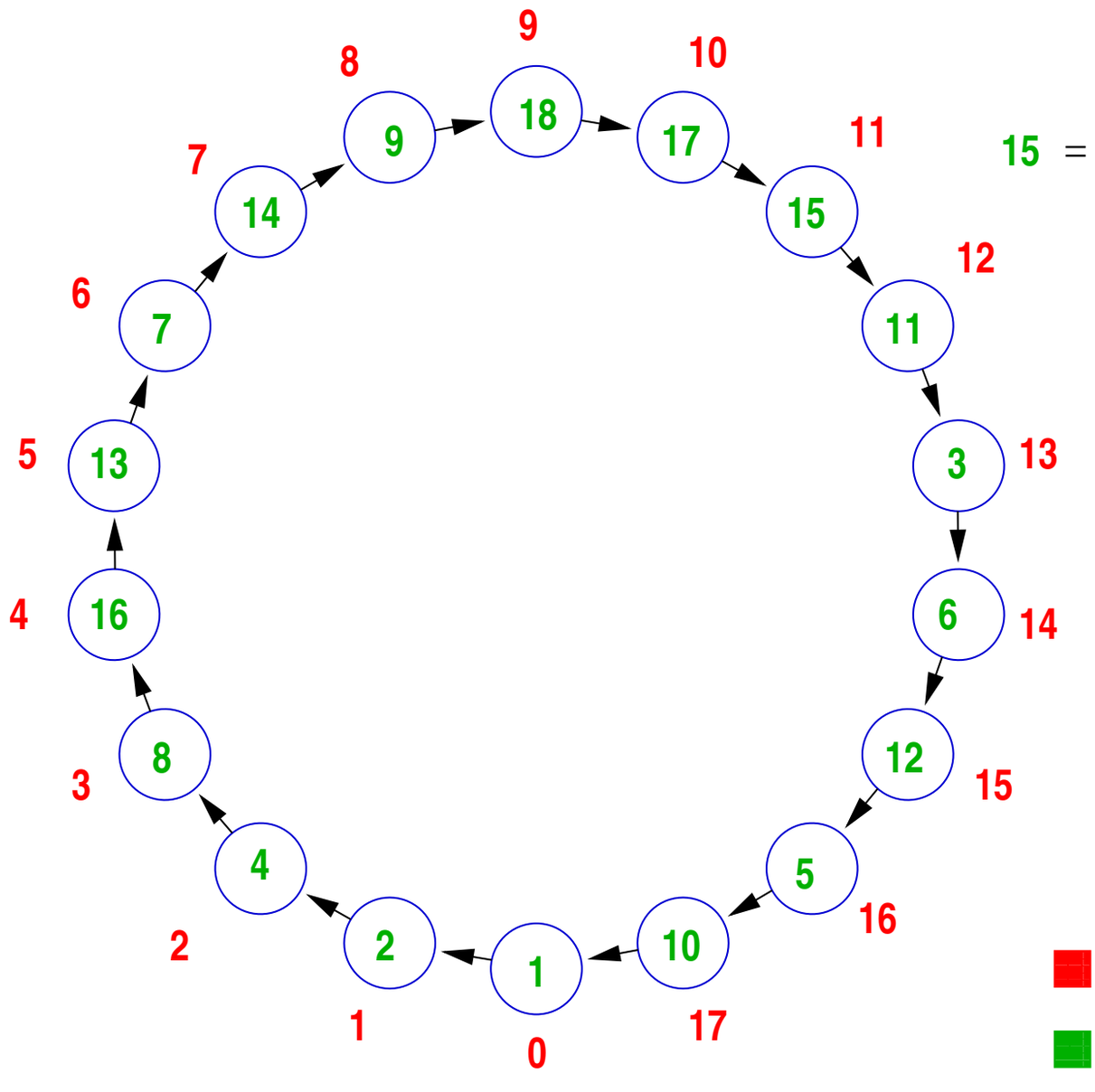
f(x) ist einfach berechenbar für jedes x.

Für zufälliges y ist es berechenmässig zu schwierig, ein x mit f(x) = y zu finden.

Einwegfunktion: Mechanisches Analogon



Potenzierung modulo 19

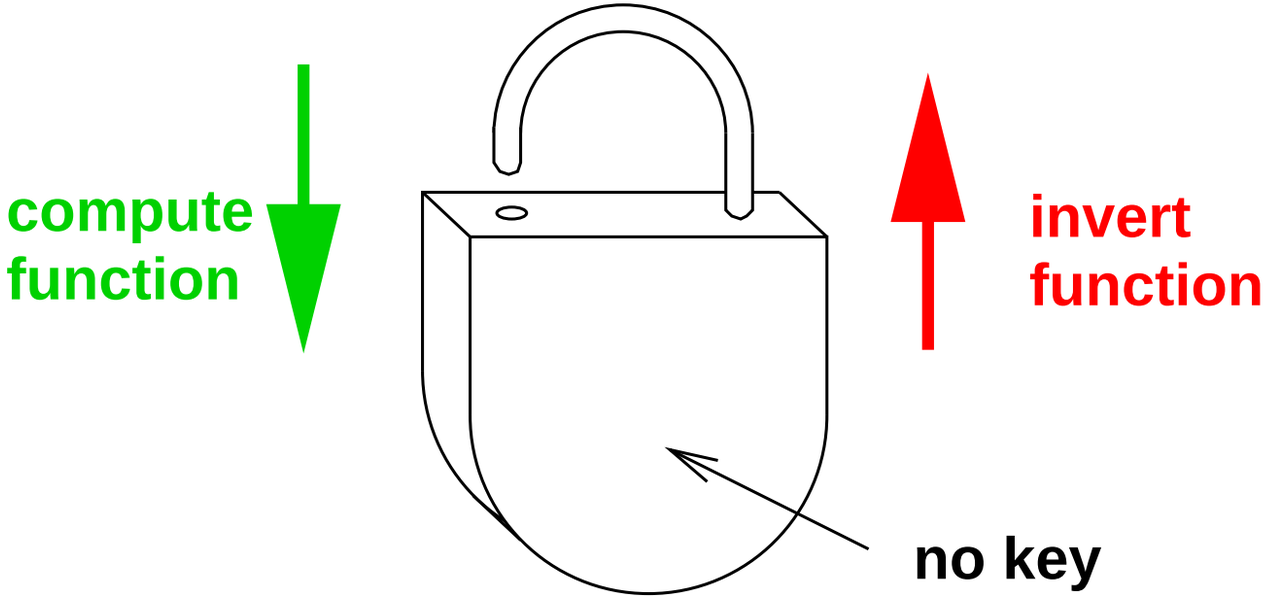


$15 = 2^{11} \pmod{19}$

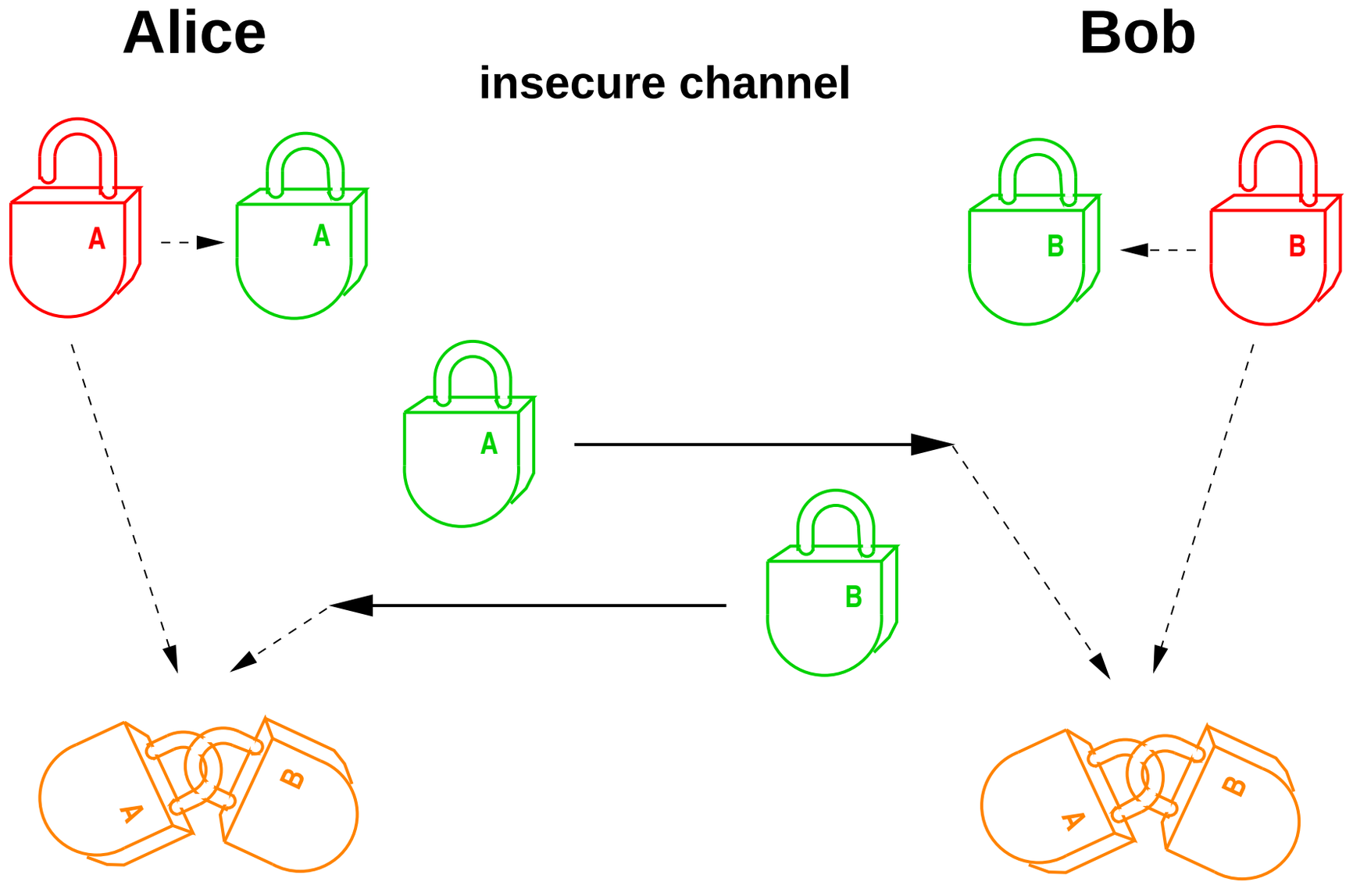
discrete logarithm

element of group

Einwegfunktion: Mechanisches Analogon



Diffie-Hellman Protokoll:



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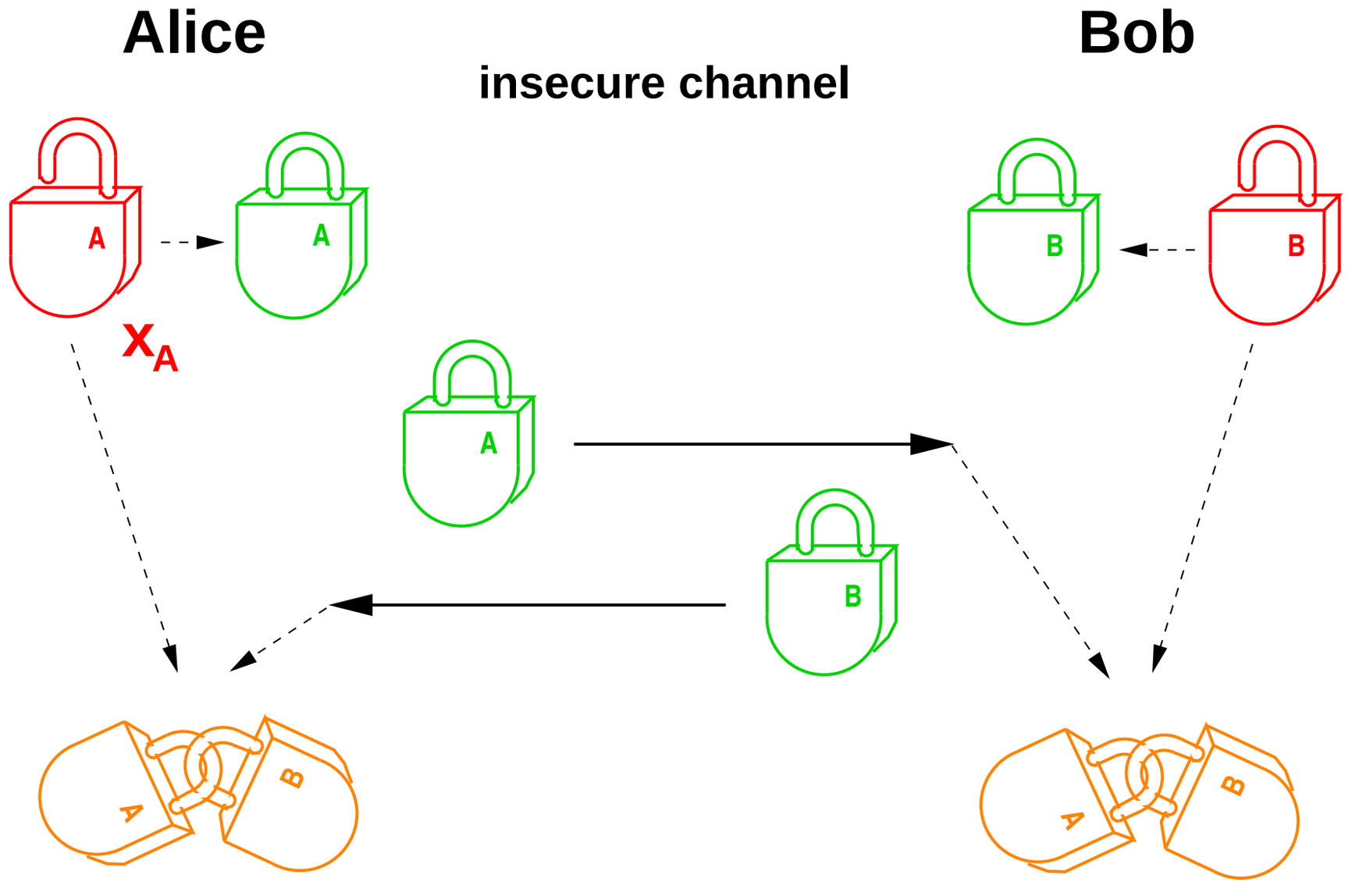

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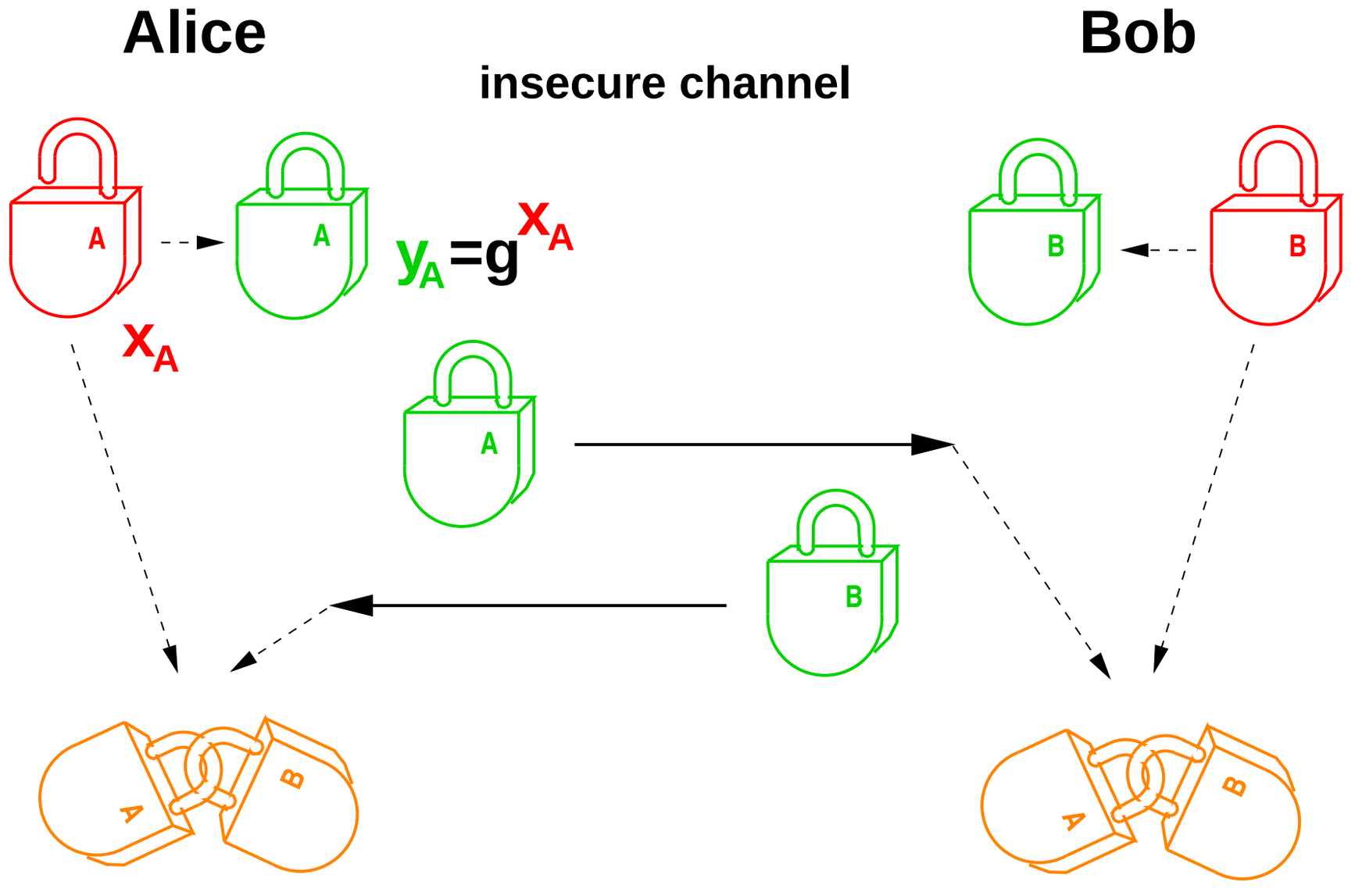

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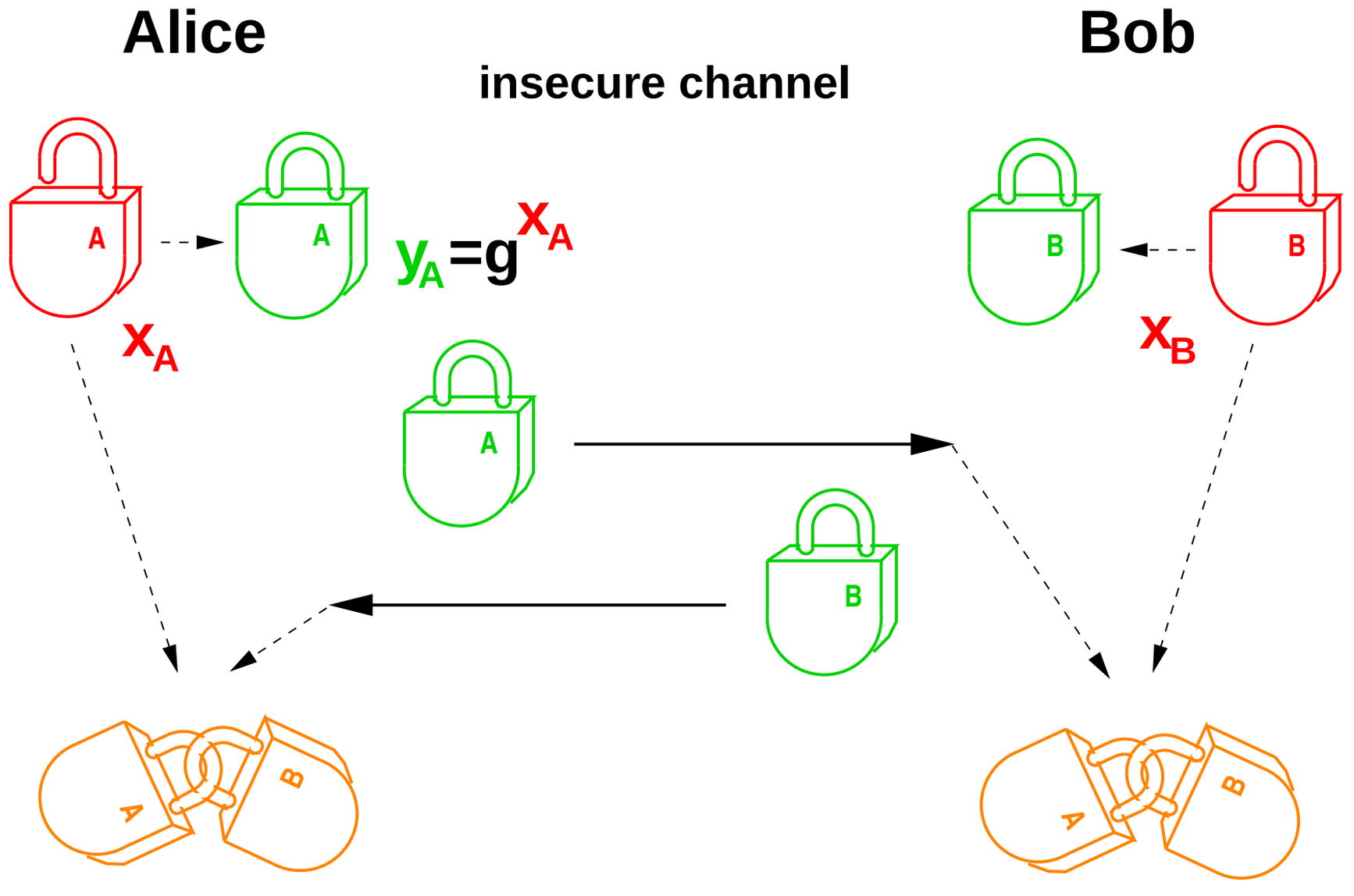

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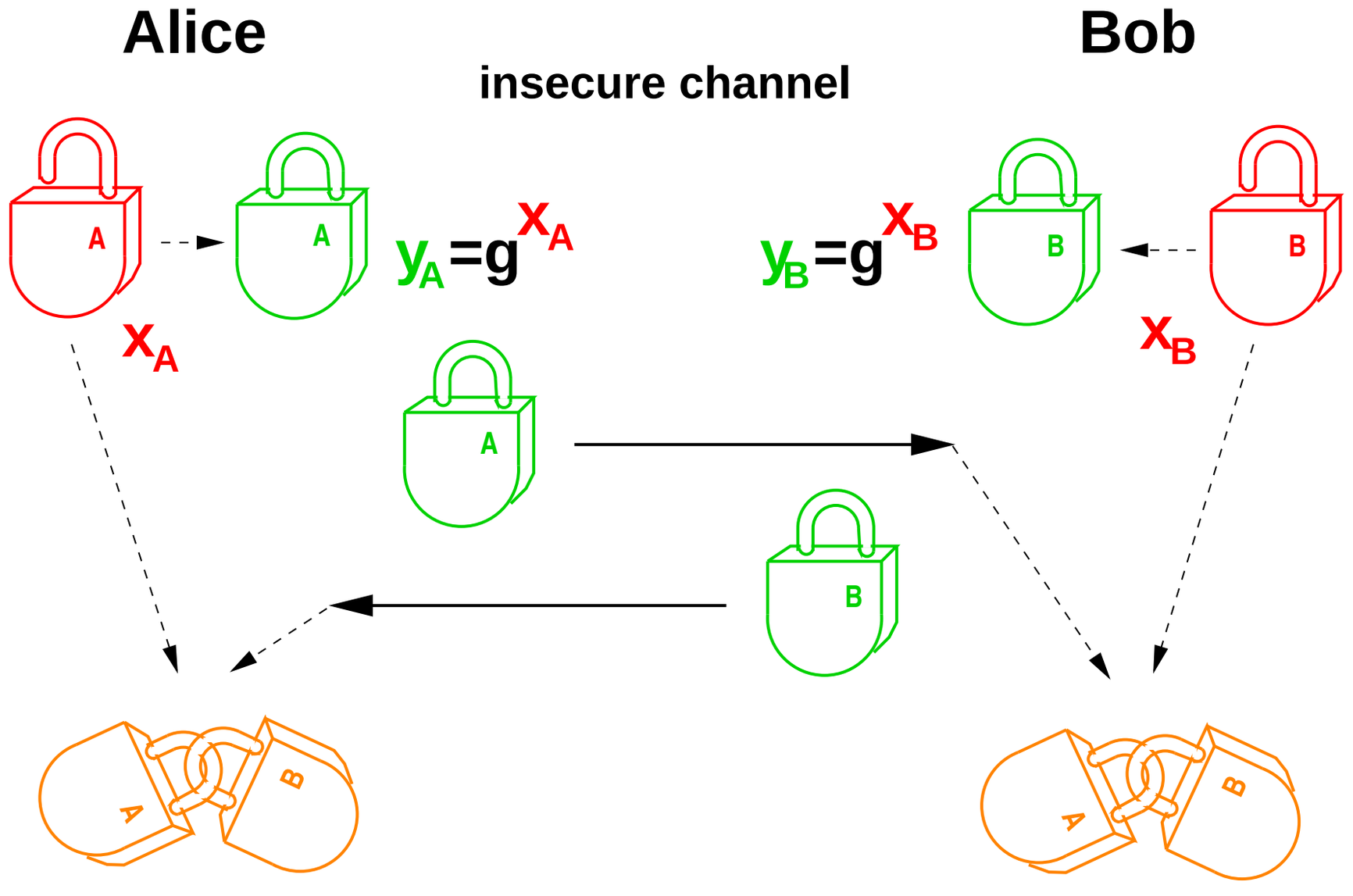

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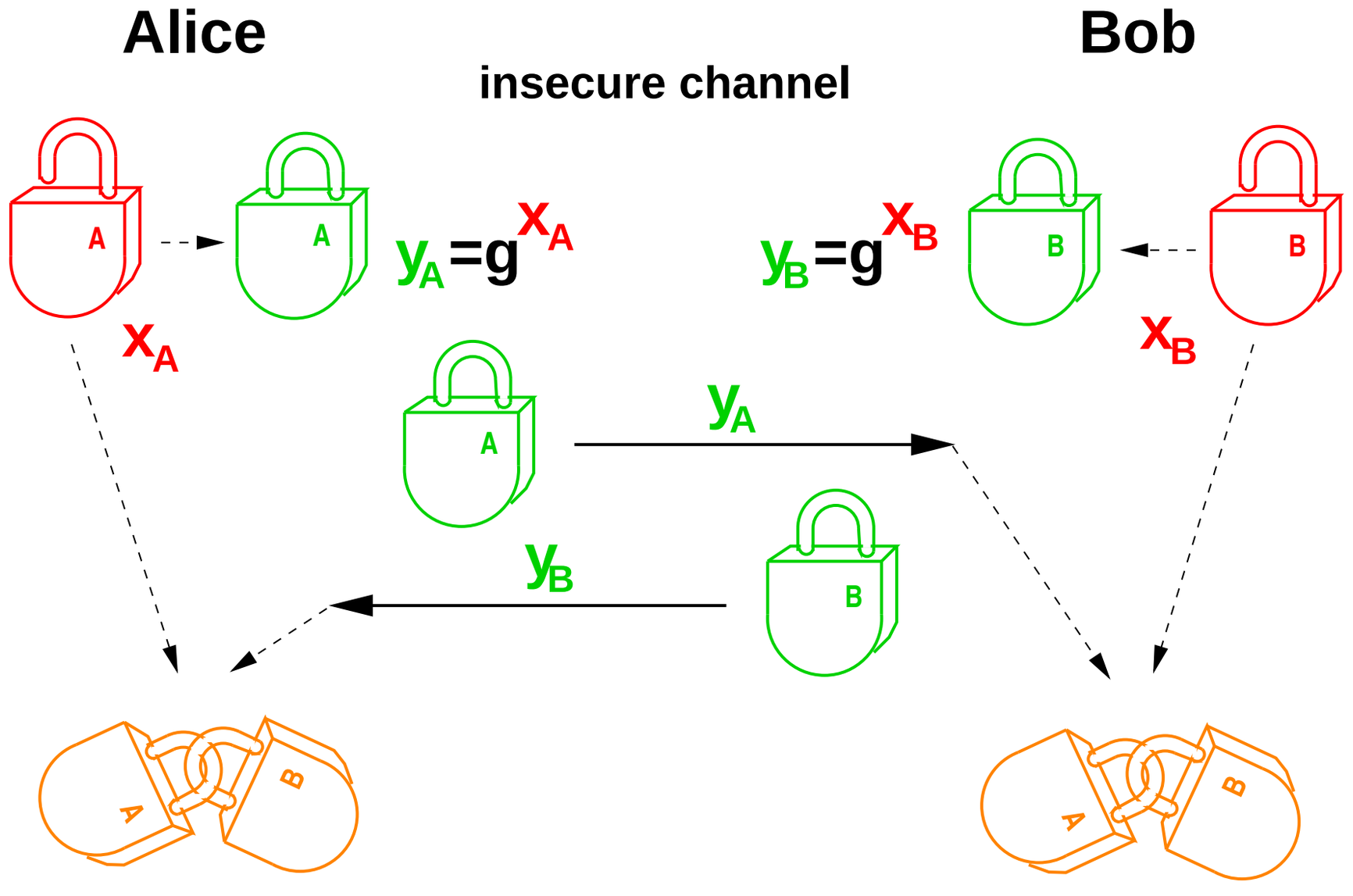

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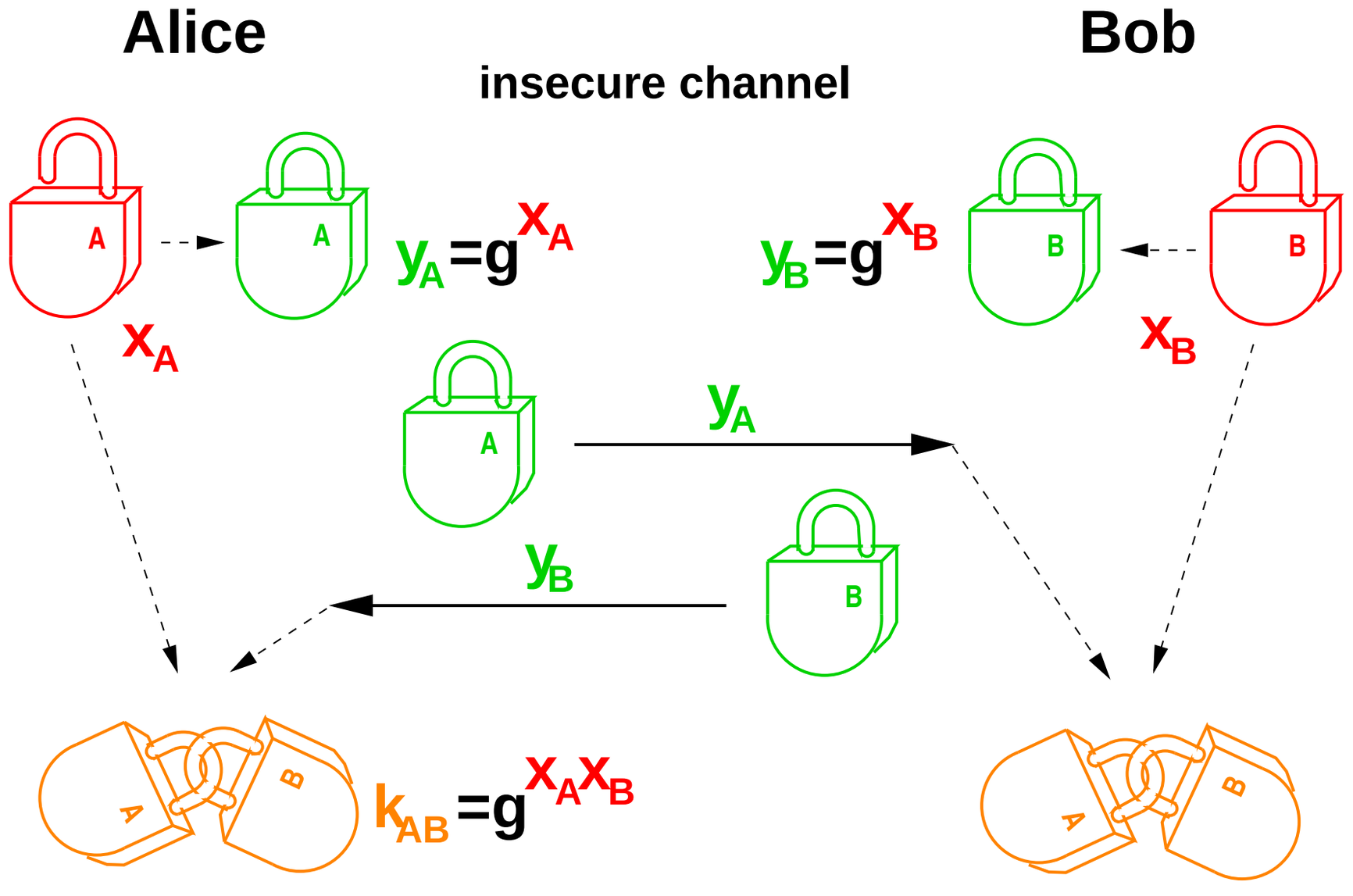

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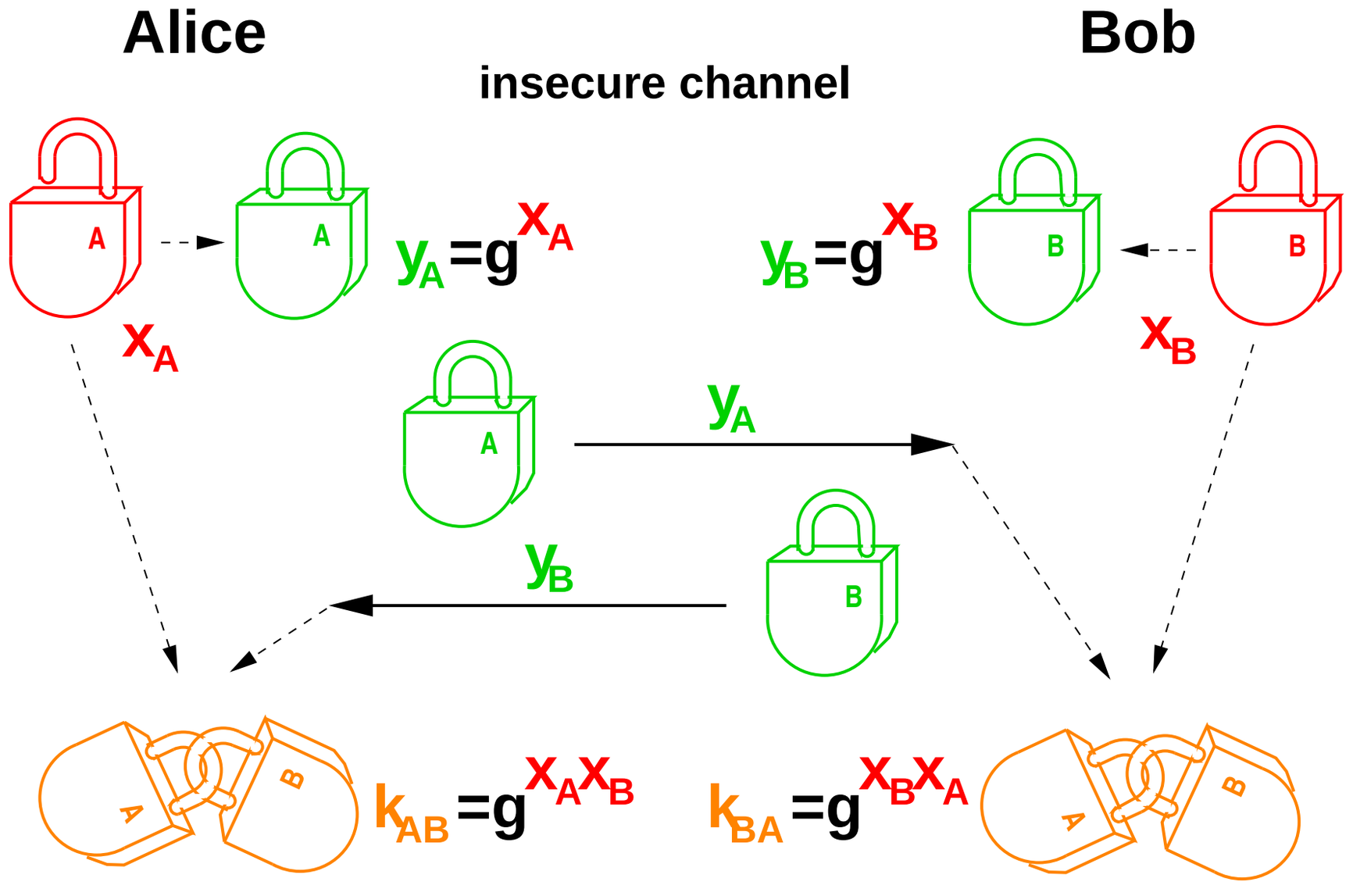


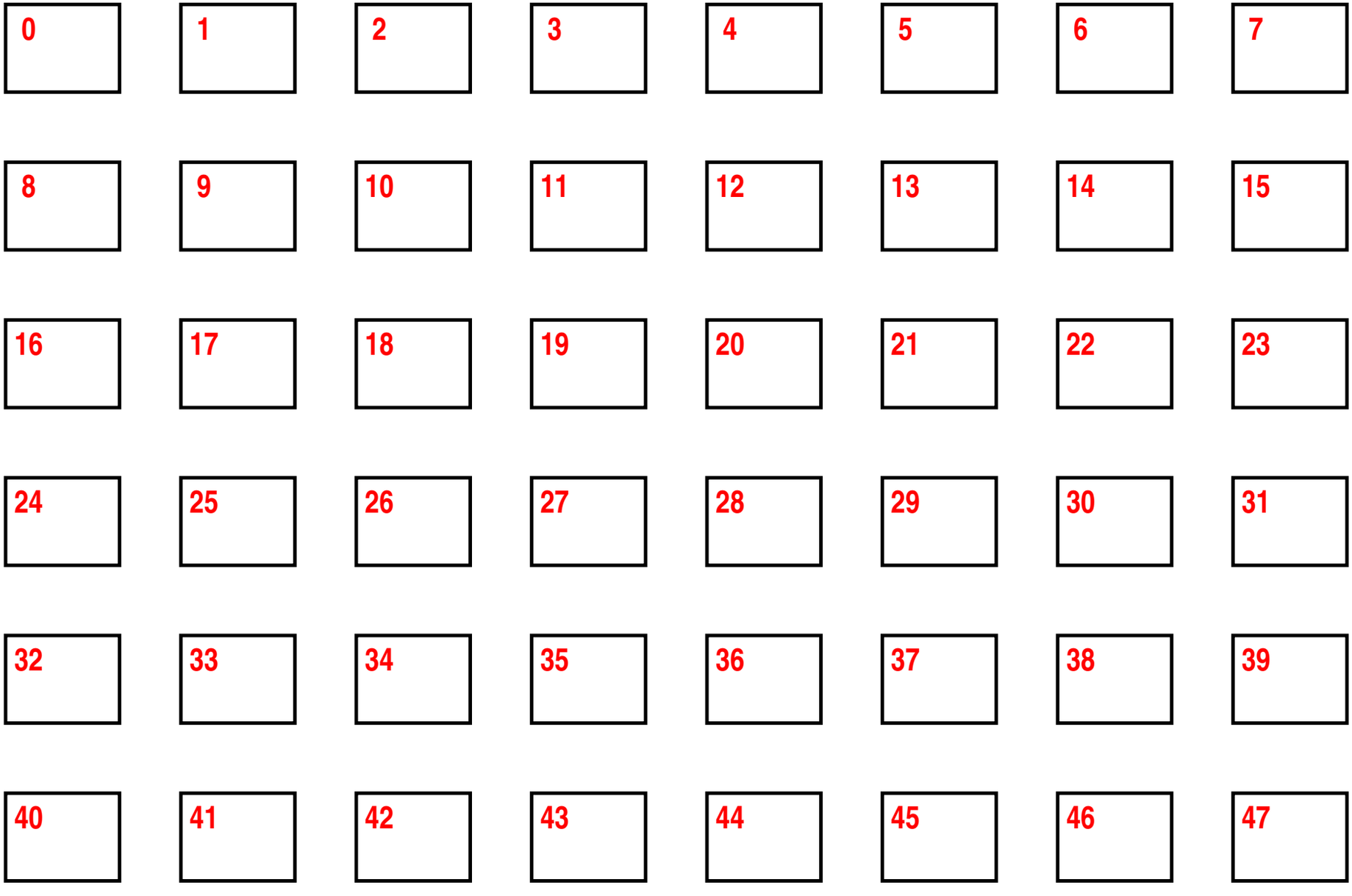
Discrete logarithm (DL) problem

Cyclic group G of order n with generator g:

$$\mathrm{G} = \langle \mathbf{g} \rangle = \{\mathbf{g}^i : 0 \leq i < n\}$$

DL problem: Given $\mathrm{a} \in \mathrm{G}$, find x such that $\mathrm{a} = \mathbf{g}^{\mathbf{x}}$.

$\mathbf{x} \longrightarrow \mathbf{g}^{\mathbf{x}}$ is for many groups believed to be a OWF.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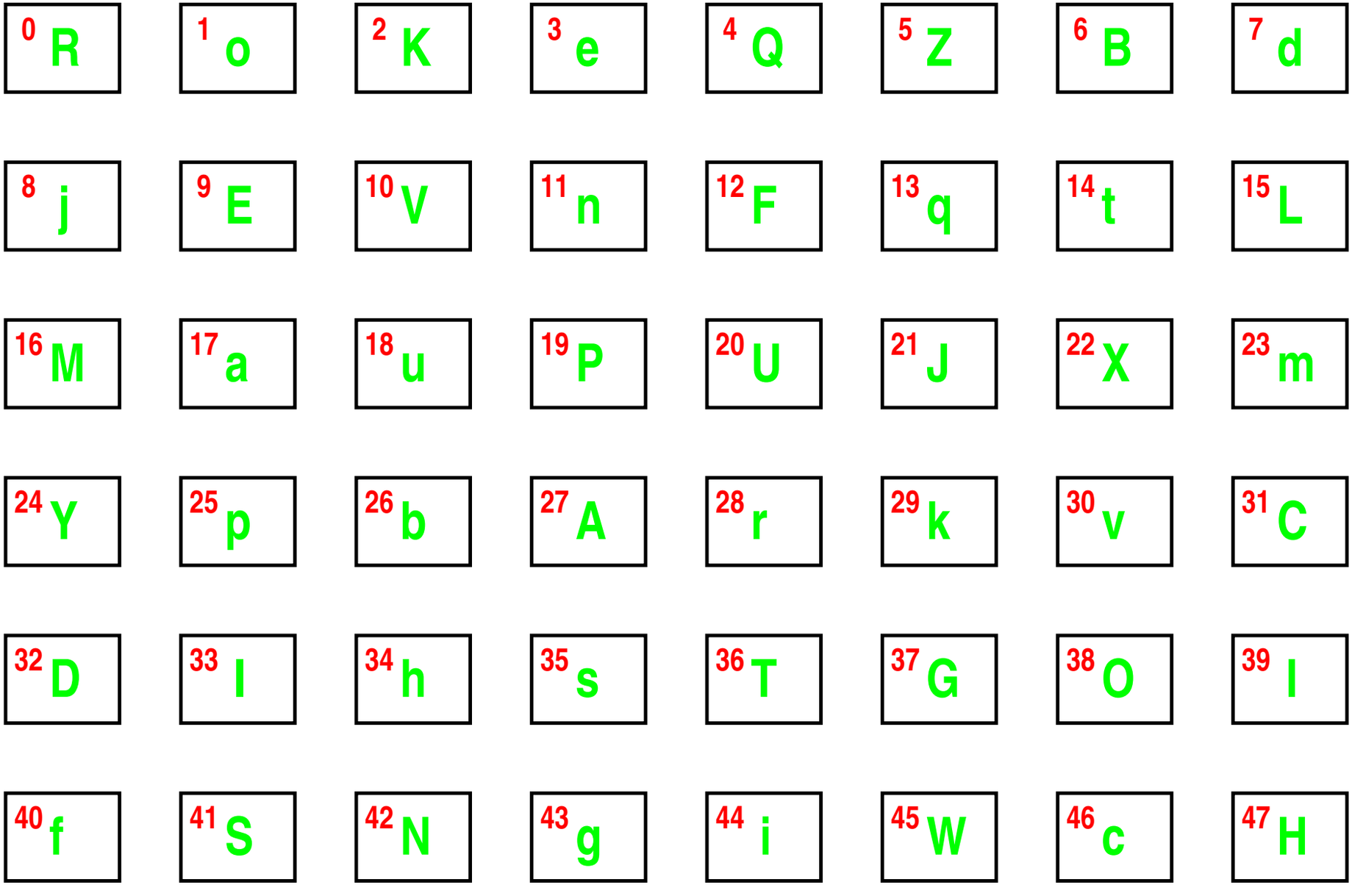
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18 u
19 P
20 U
21 J
22 X
23 m
24 Y
25 p
26 b
27 A
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29 k
30 v
31 C
32 D
33 I
34 h
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45 W
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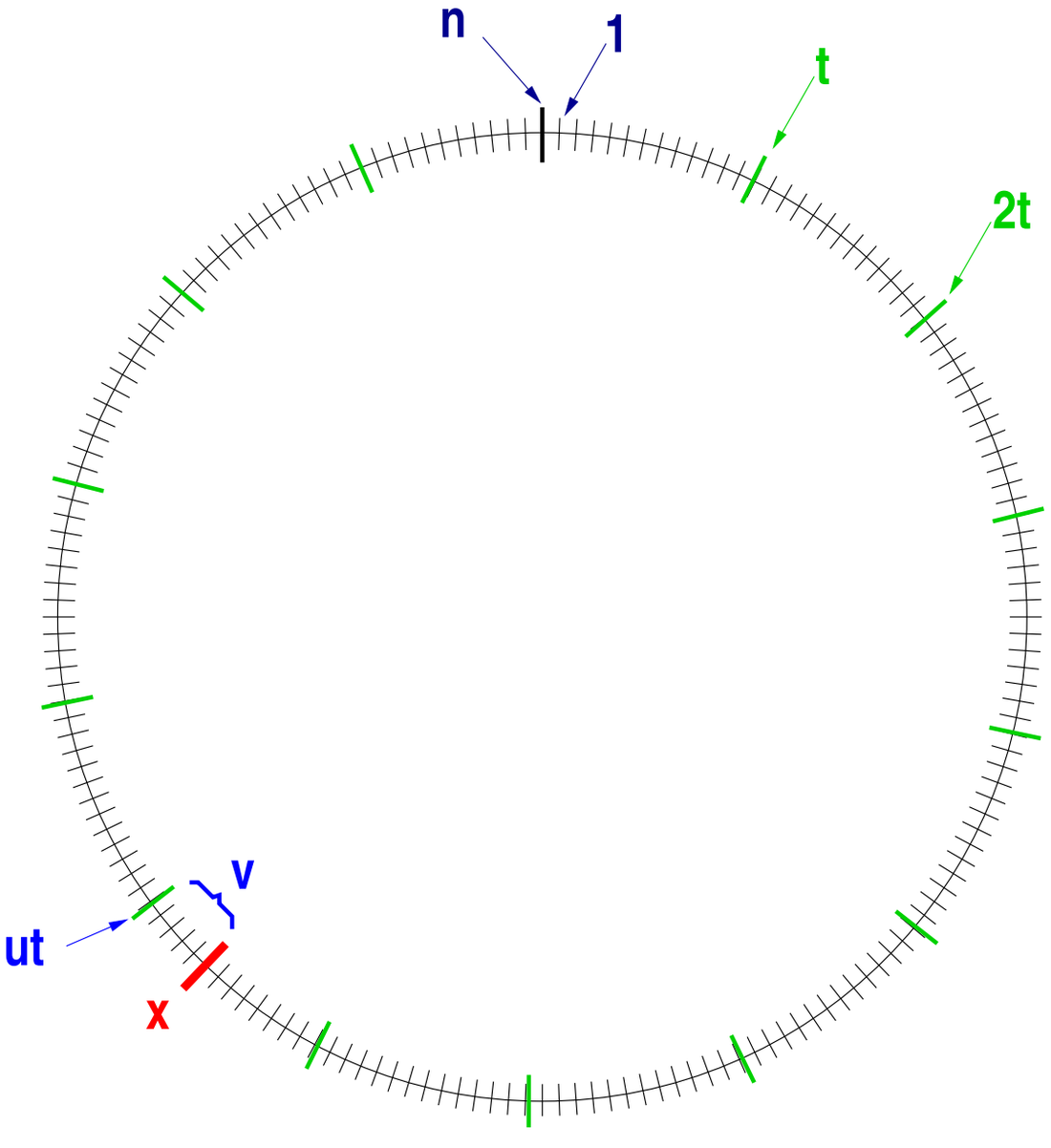
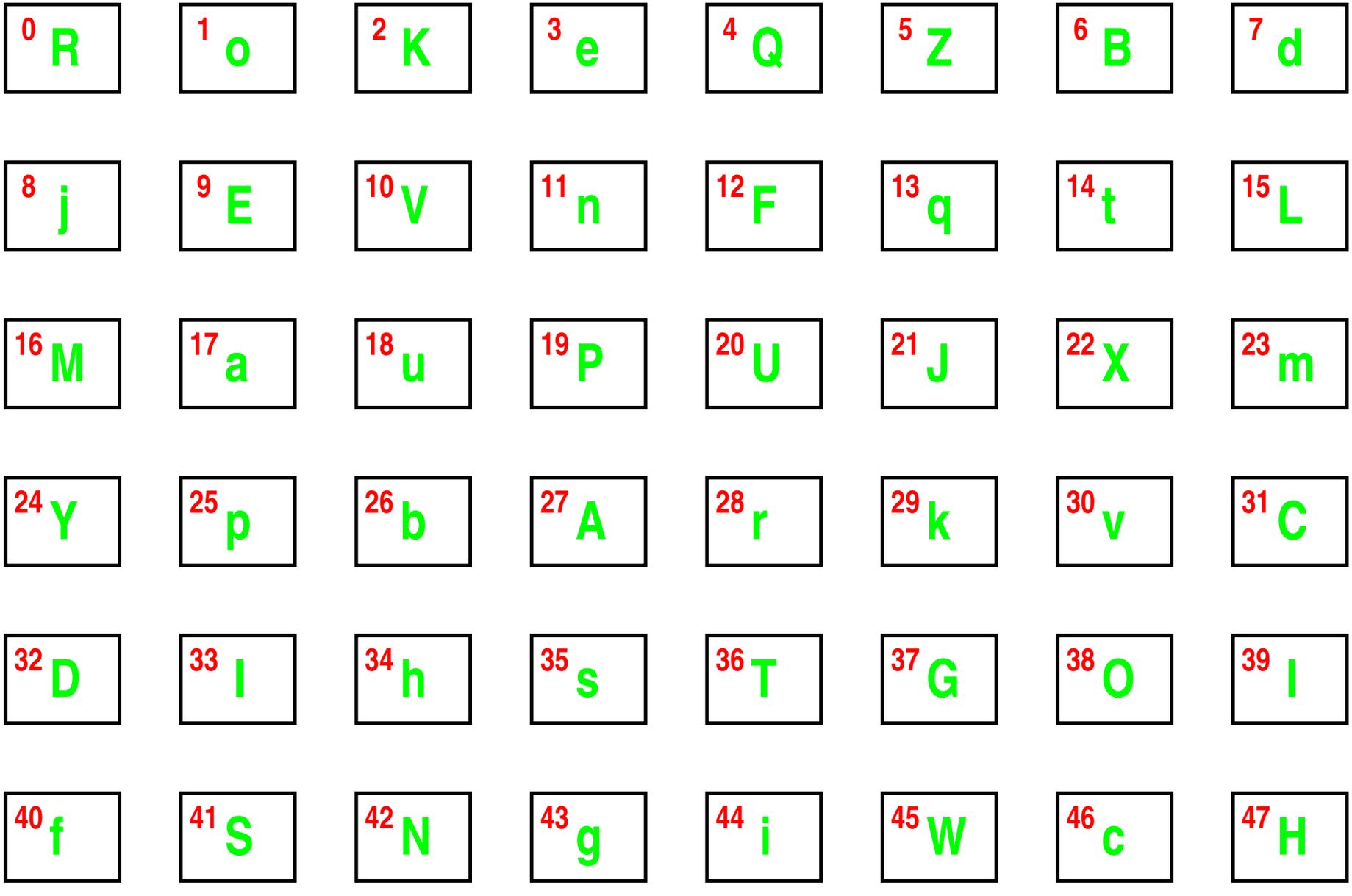
47 H

Baby-step giant-step DL algorithm (1)

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18 u
19 P
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21 J
22 X
23 m
24 Y
25 p
26 b
27 A
28 r
29 k
30 v
31 C
32 D
33 I
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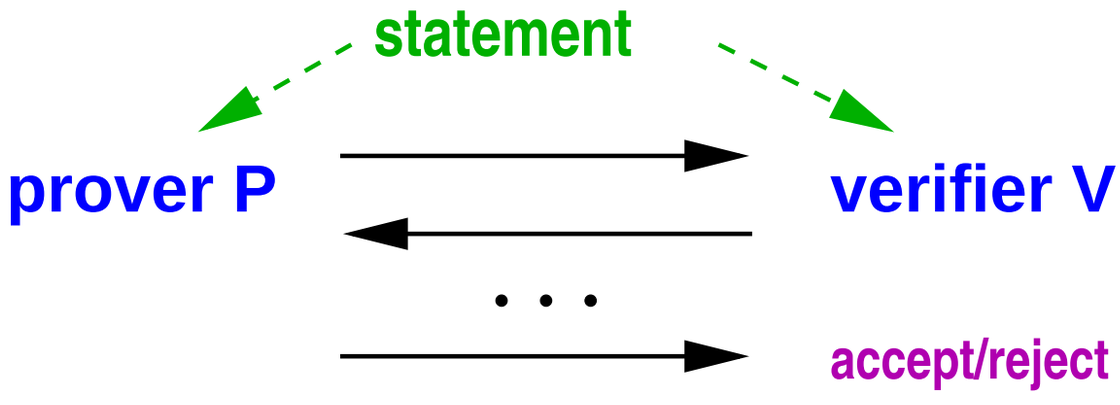
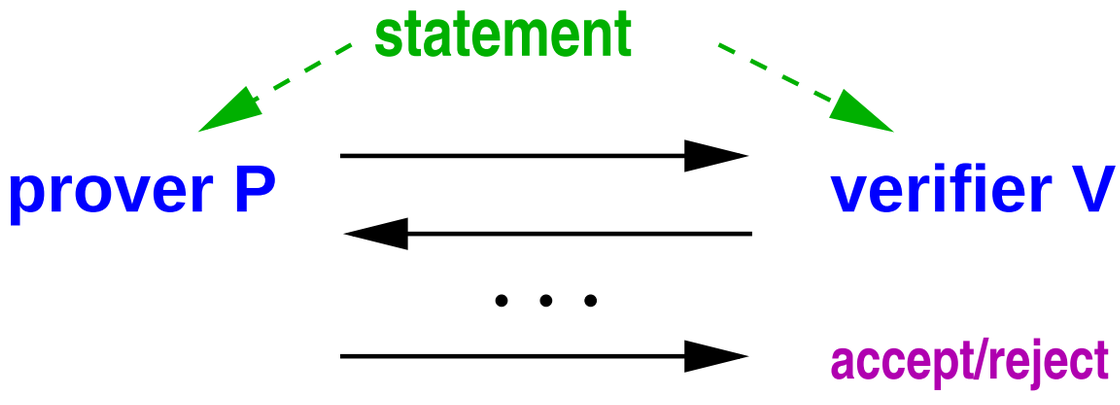
Interactive proofs

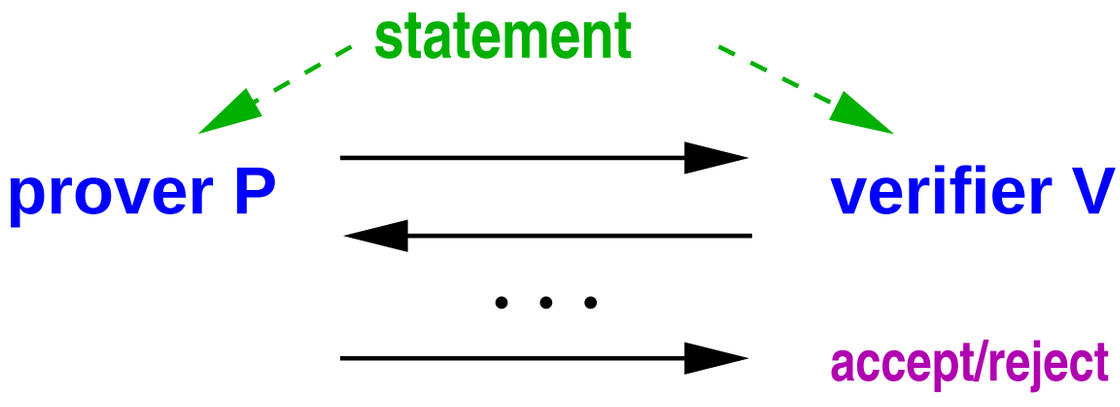


Interactive proofs



Motivations for interactive proofs:

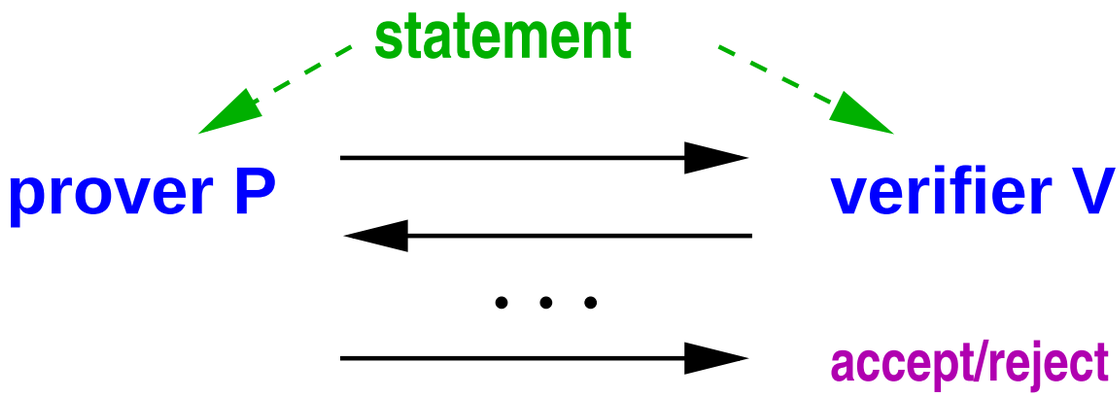
Interactive proofs



Motivations for interactive proofs:

- **Interactive proofs can be zero-knowledge.**

Interactive proofs



Motivations for interactive proofs:

- **Interactive proofs can be zero-knowledge.**
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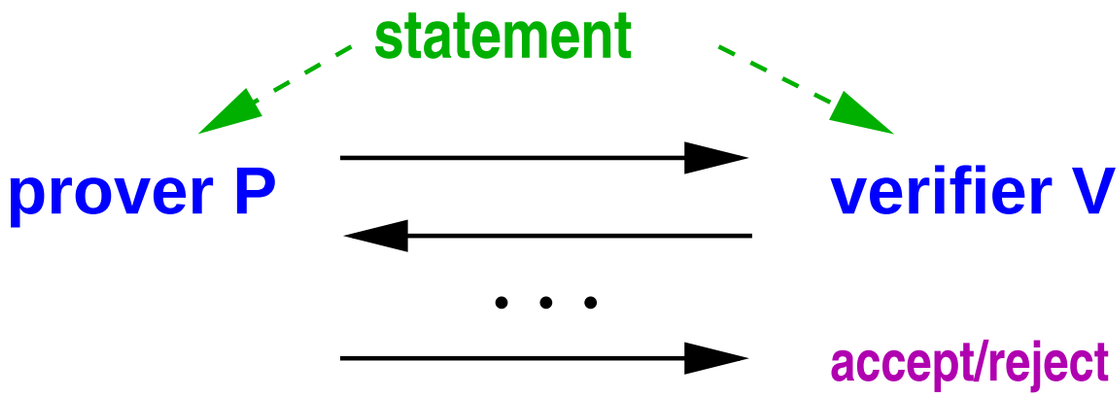
Interactive proofs



Motivations for interactive proofs:

- **Interactive proofs can be zero-knowledge.**
- **Interactive proofs are more powerful than static proofs**
- **Applications:**
  - **Digital signature schemes**
  - **entity authentication**
  - **secure multi-party computation**

Proving knowledge of a DL (Schnorr)

Prover Peggy | **Verifier Vic**

knows $x \in \mathbb{Z}_q$ | $z = h^x$

$k \in_R \mathbb{Z}_q$

$t = h^k$

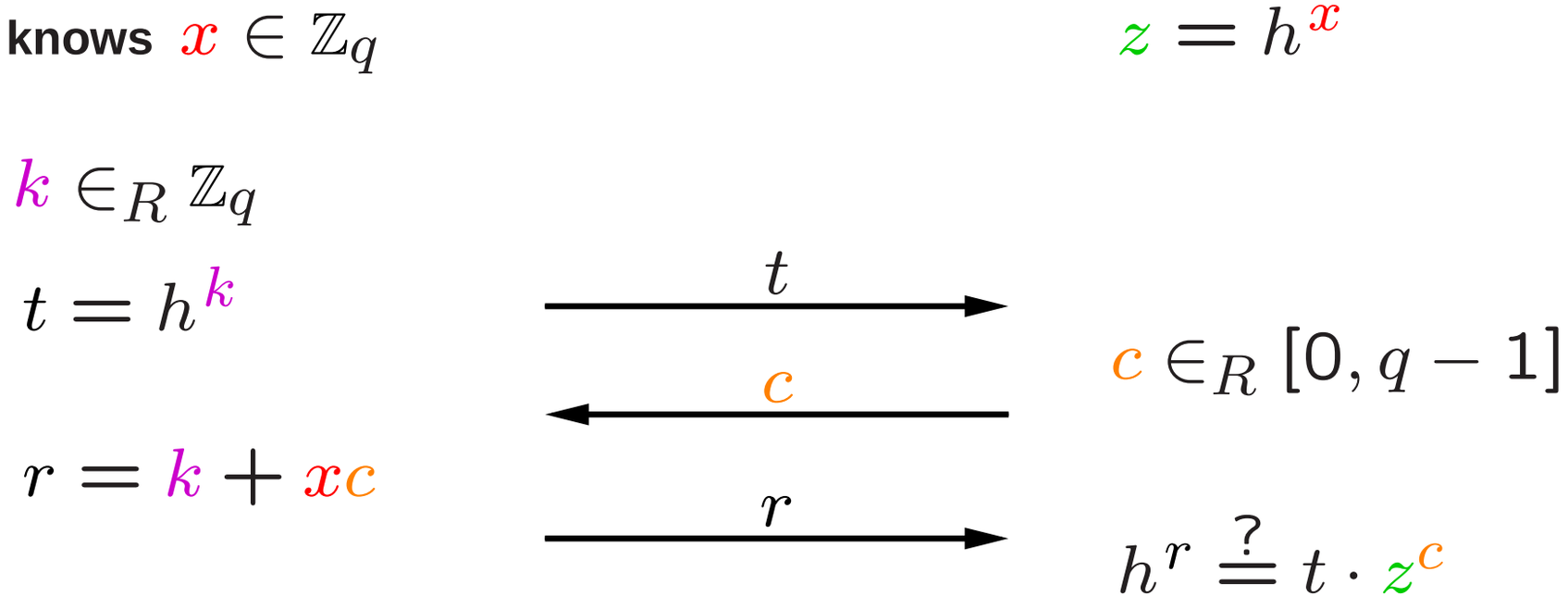
t →

← c ; $c \in_R [0, q-1]$

$r = k + xc$

r →

$h^r \stackrel{?}{=} t \cdot z^c$


knows $x \in \mathbb{Z}_q$
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t
$c \in_R [0, q-1]$
c
$r = k + xc$
r
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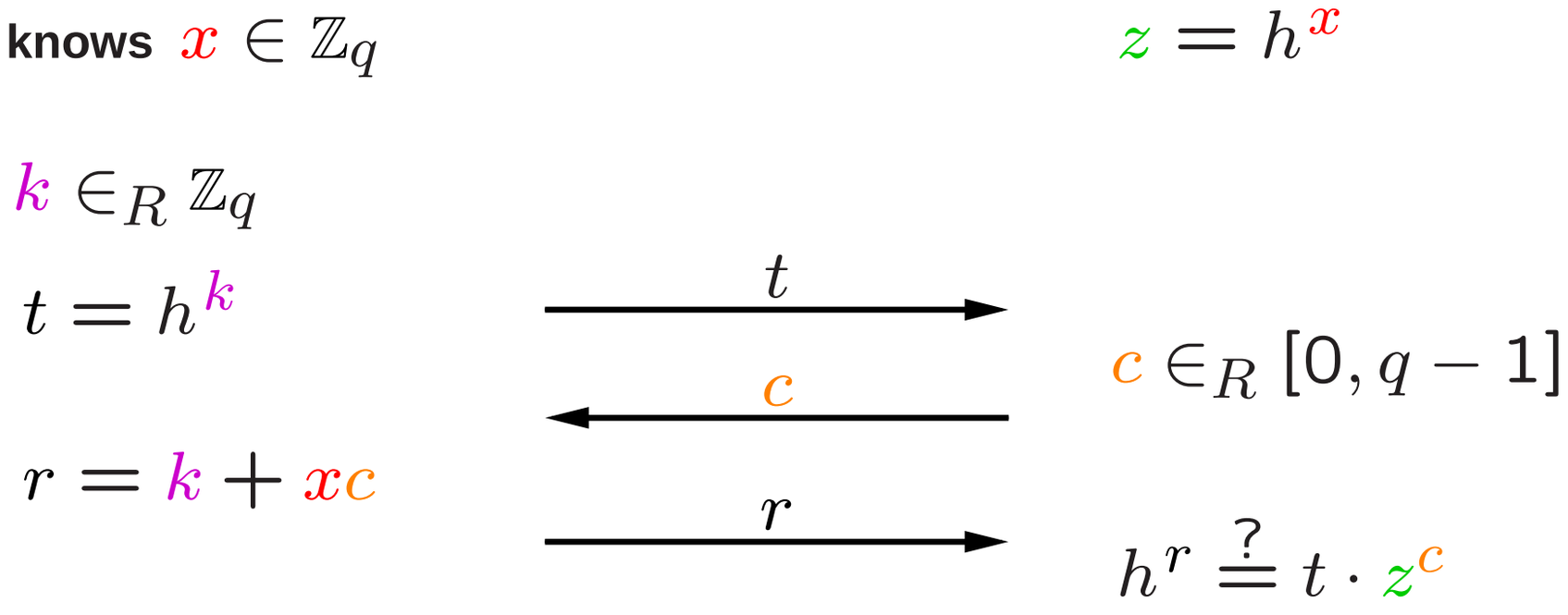
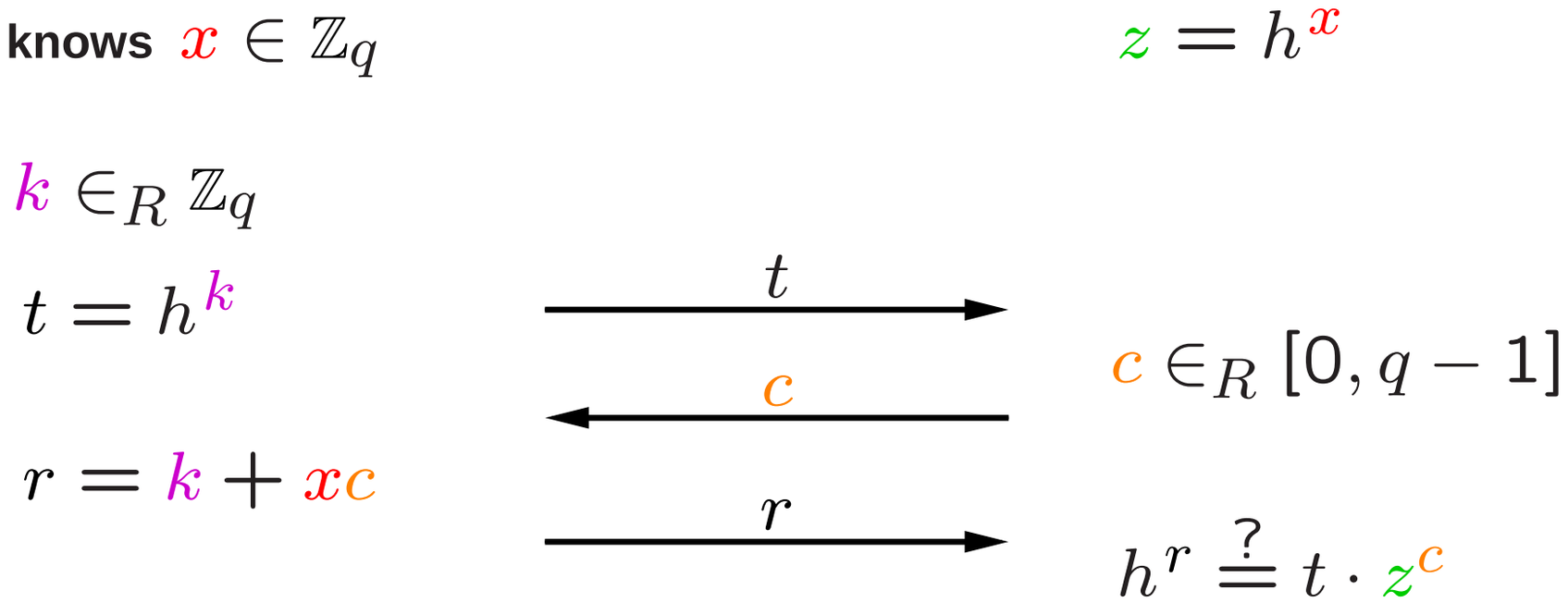


Fact: P can answer any 2 challenges $\Rightarrow$ P knows x.



Fact: P can answer any 2 challenges $\Rightarrow$ P knows x.

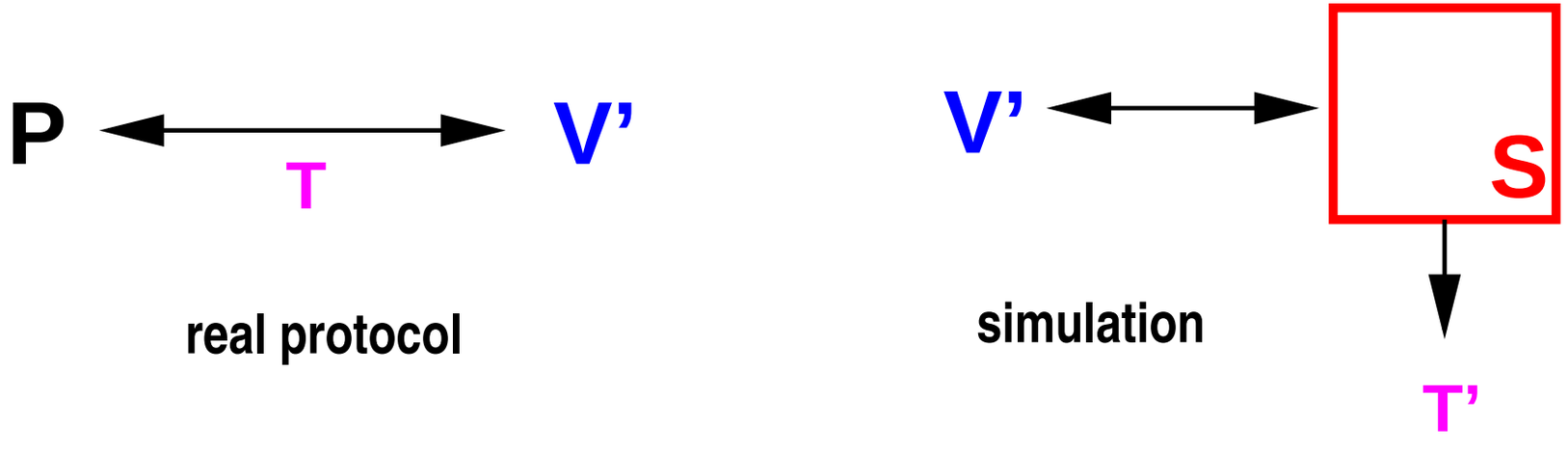
Proof:

$$h^r = t \cdot z^c, \quad h^{r'} = t \cdot z^{c'} \Rightarrow h^{r-r'} = z^{c-c'} = h^{x(c-c')}$$

$$\Rightarrow r - r' \equiv x(c - c') \pmod{q}$$

$$\Rightarrow x \equiv \frac{r - r'}{c - c'} \pmod{q}$$

Definition of zero-knowledge (ZK)



Definition: A protocol (P,V) is called (black-box) *zero-knowledge* if there exists an efficient simulator S with access to a (possibly) dishonest efficient verifier V' such that for every V' it outputs a simulated transcript T' which is indistinguishable from the real transcript T.

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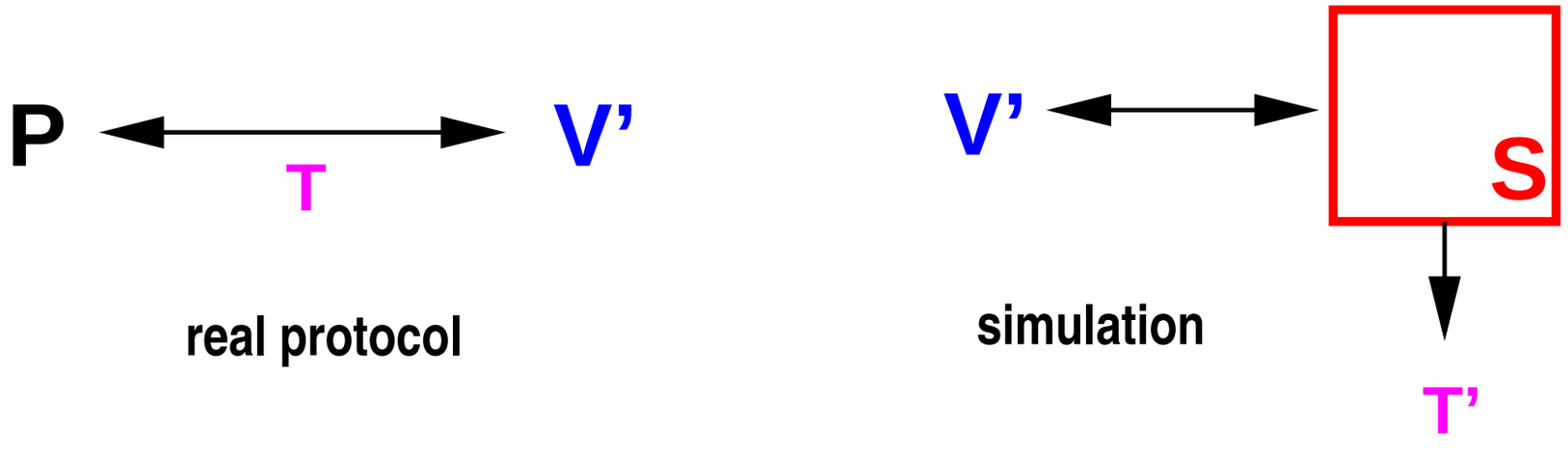

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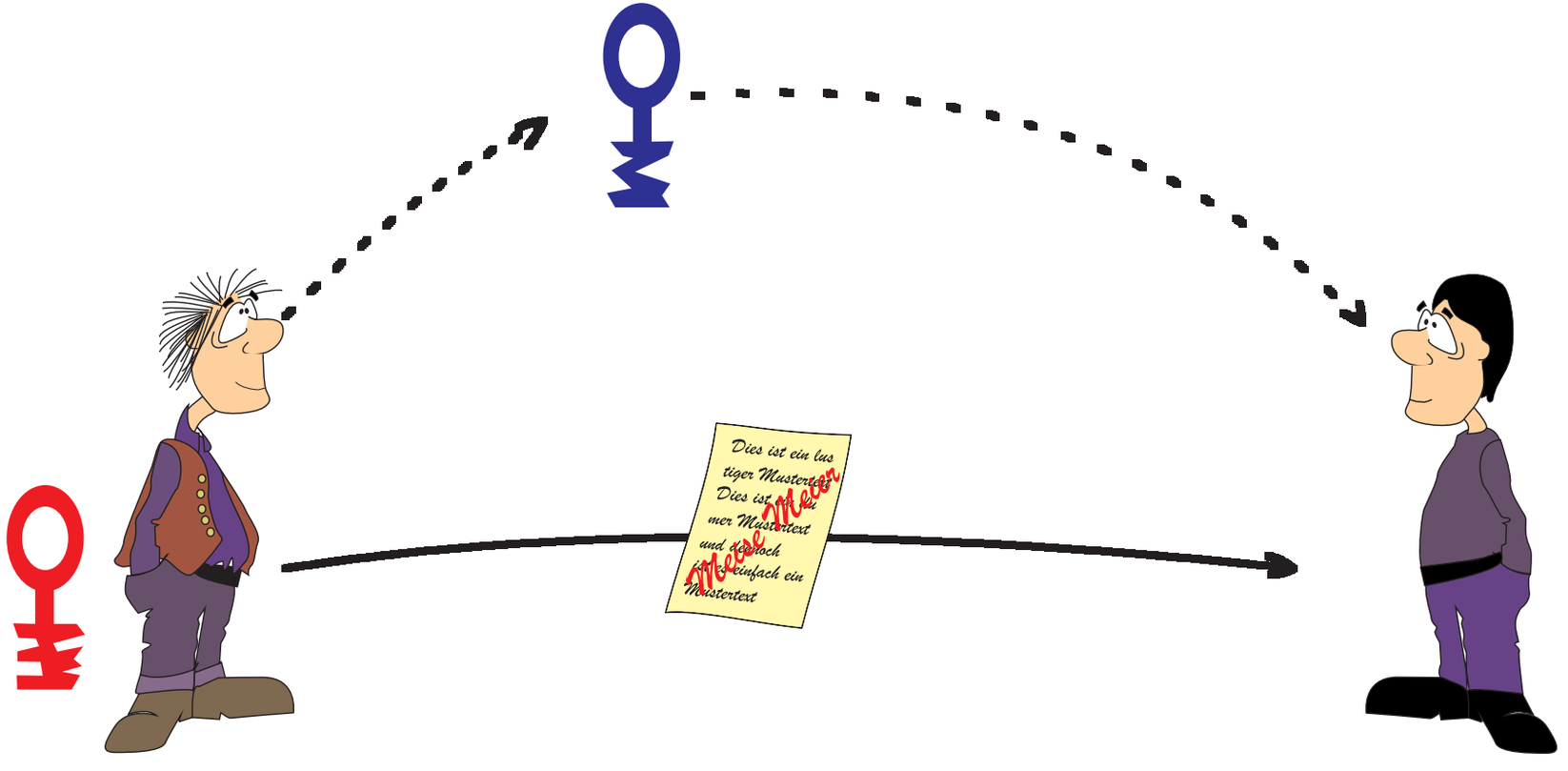
Types of ZK: perfect, statistical, computational.

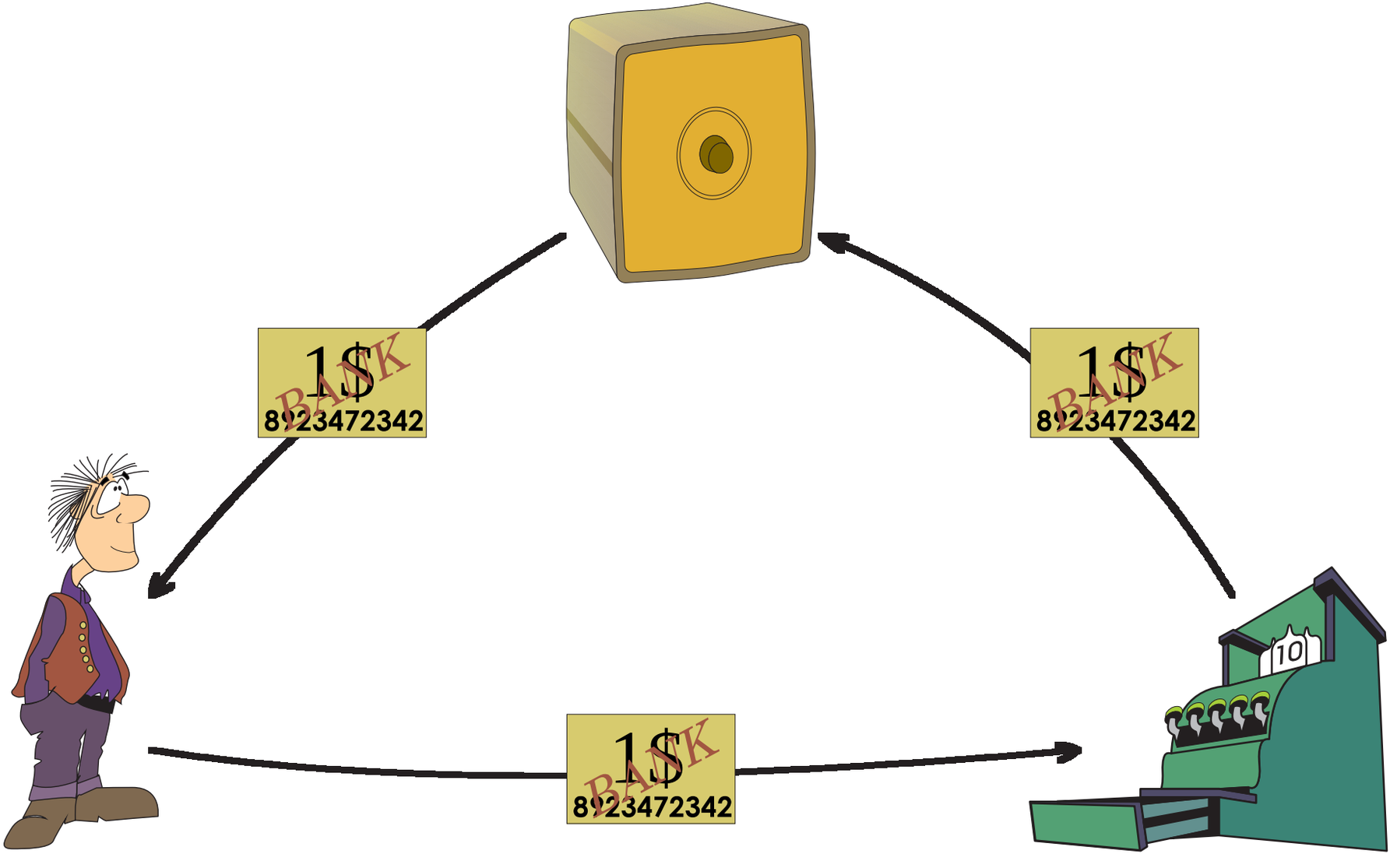
Digitale Signaturen



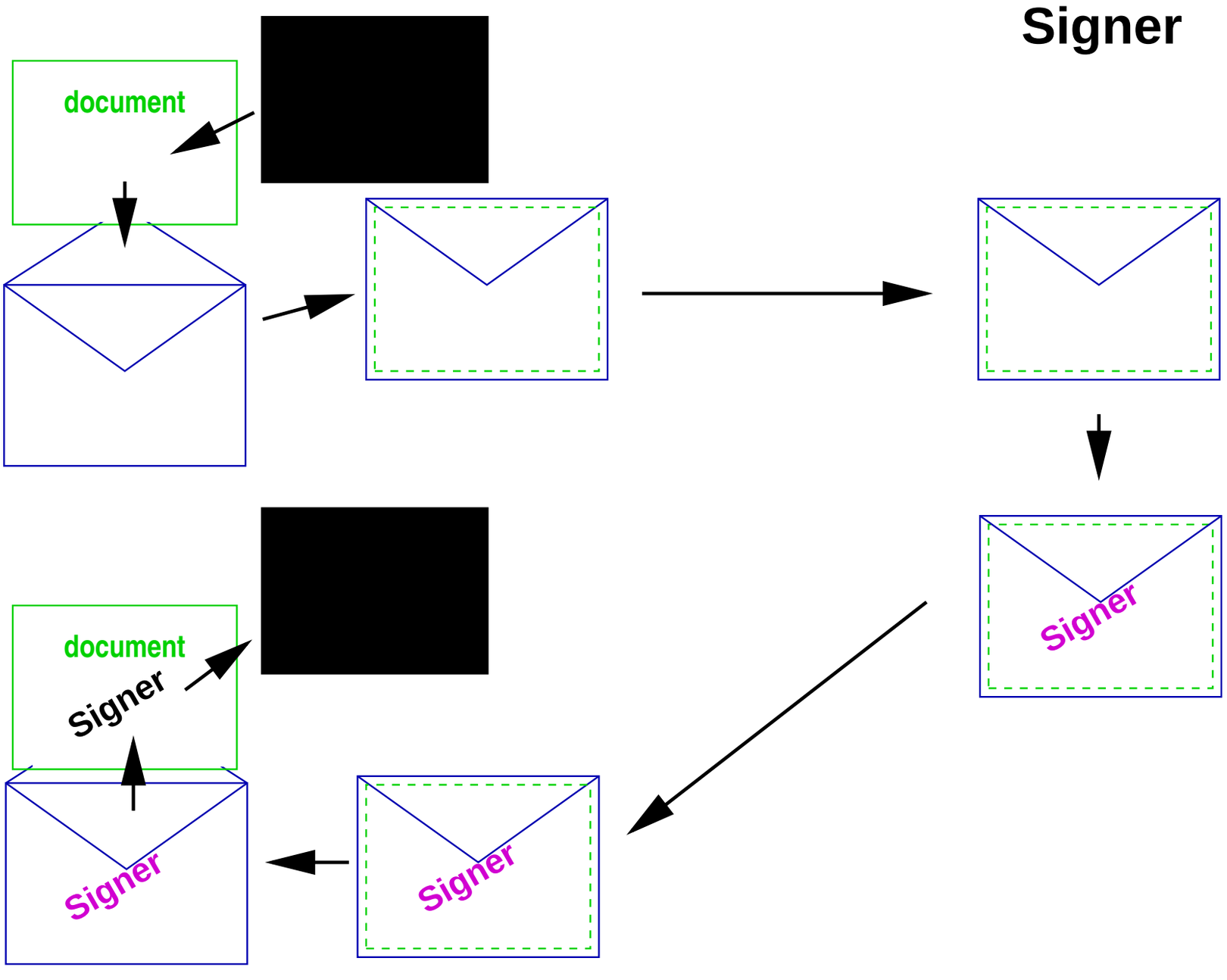
sign(,) =

verify(,) = { yes / no

Digitales Geld



Blinde digitale Signaturen


Signer
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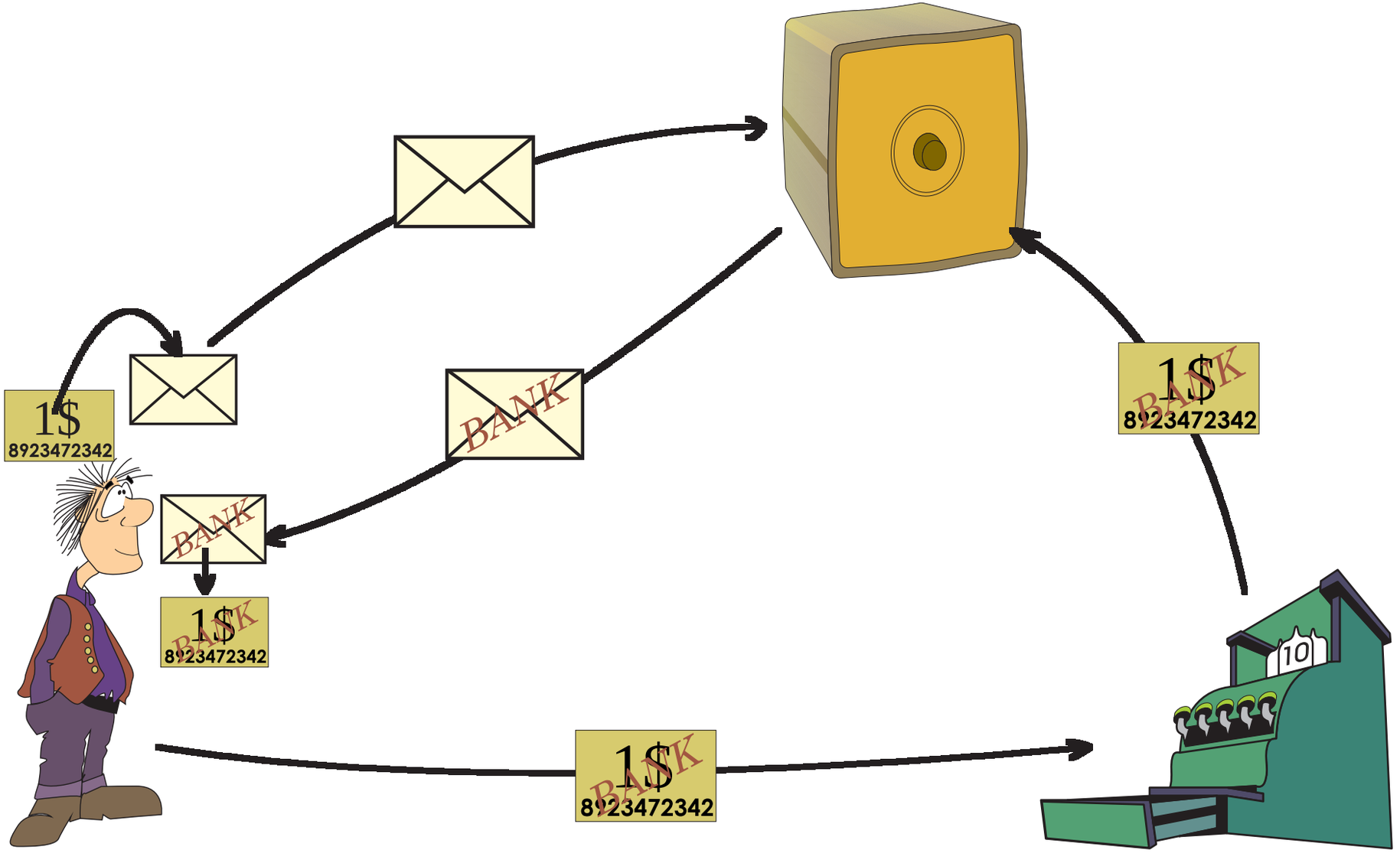
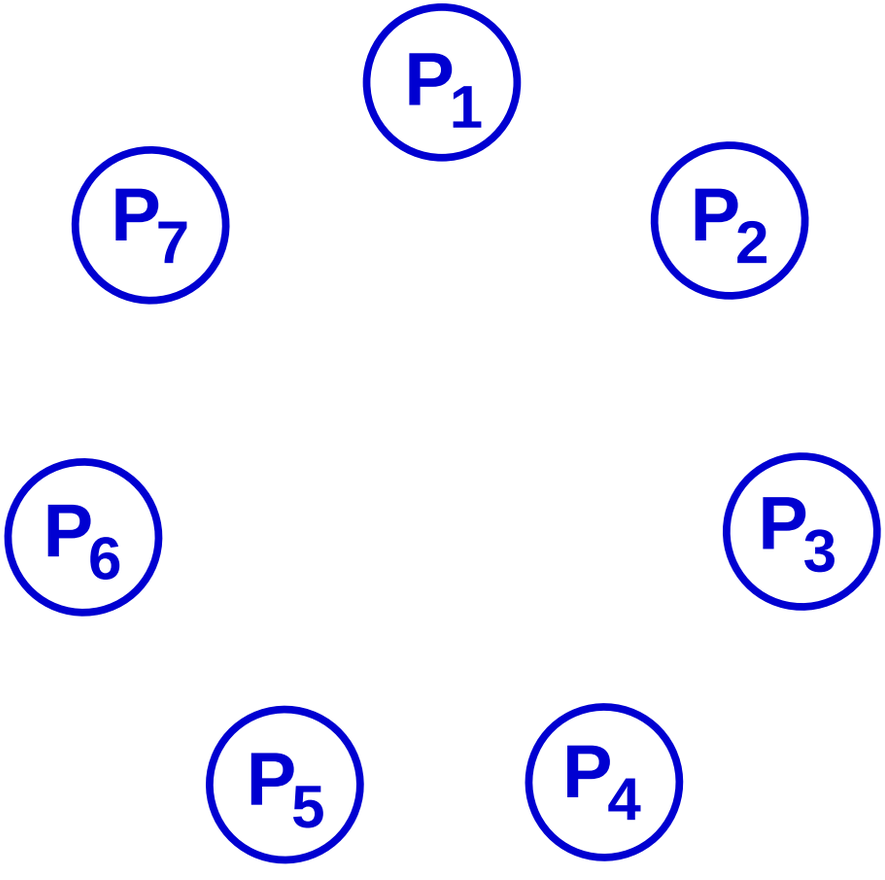
Anonymes digitales Geld (E-cash)



Broadcast / Byzantine agreement



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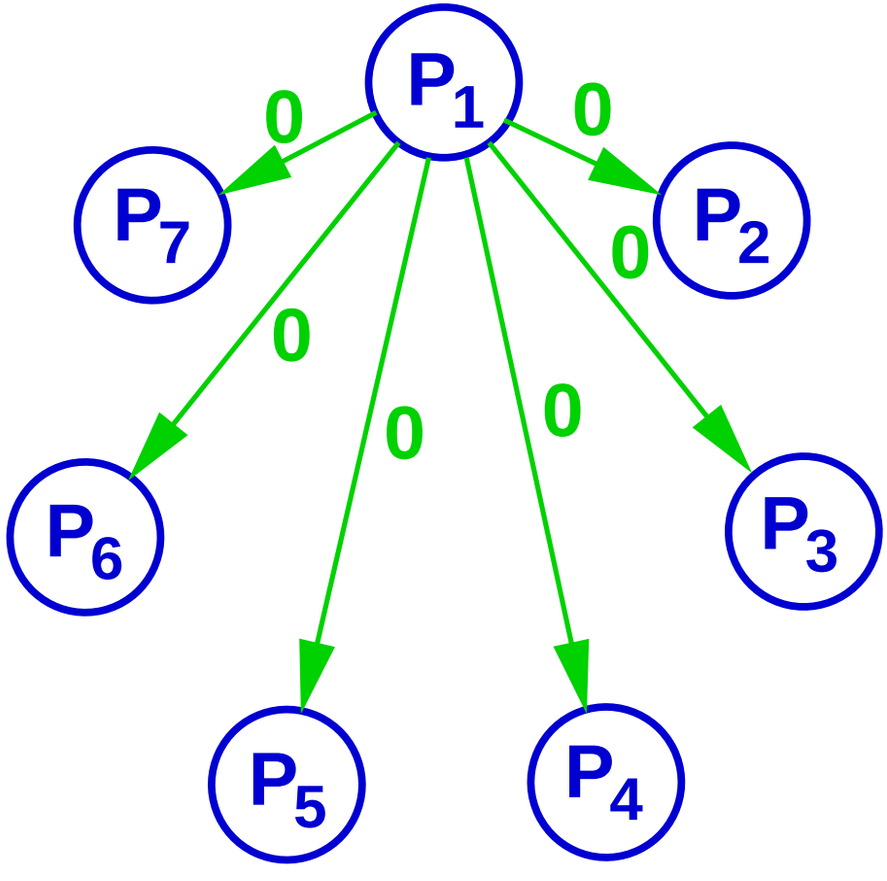

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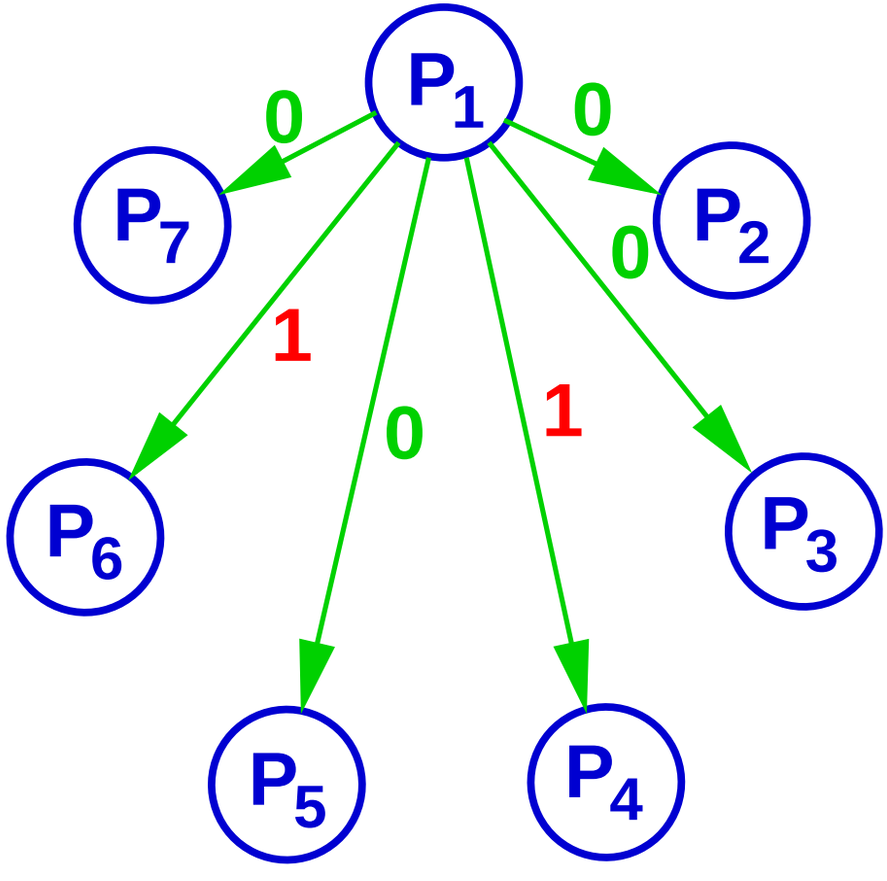

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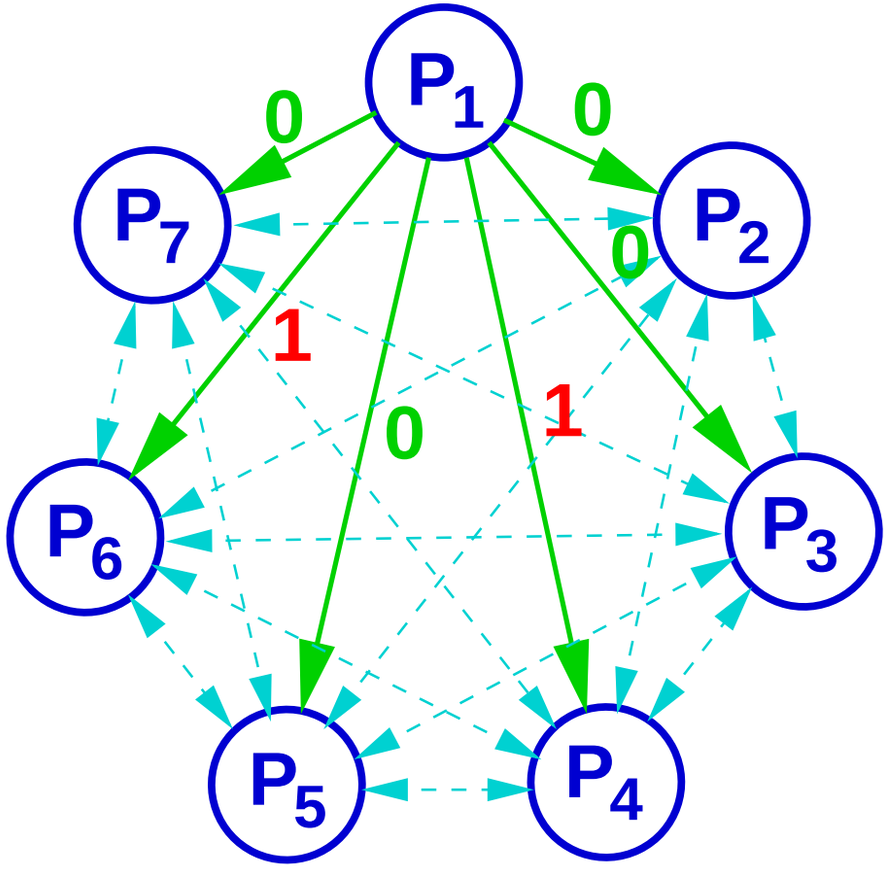

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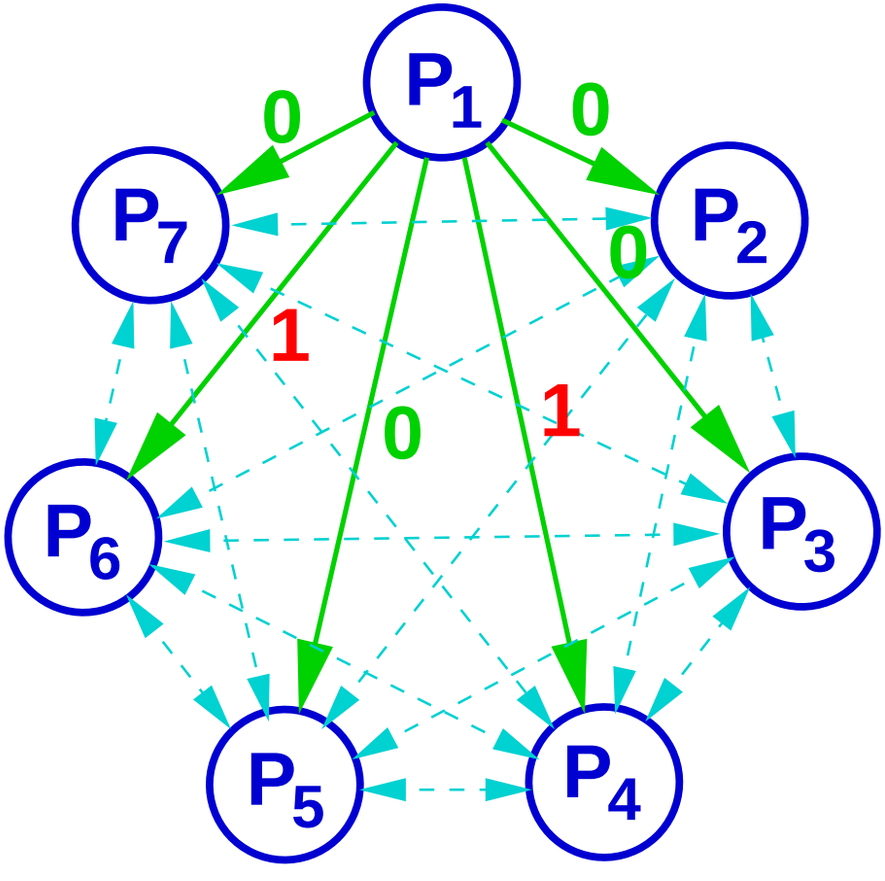
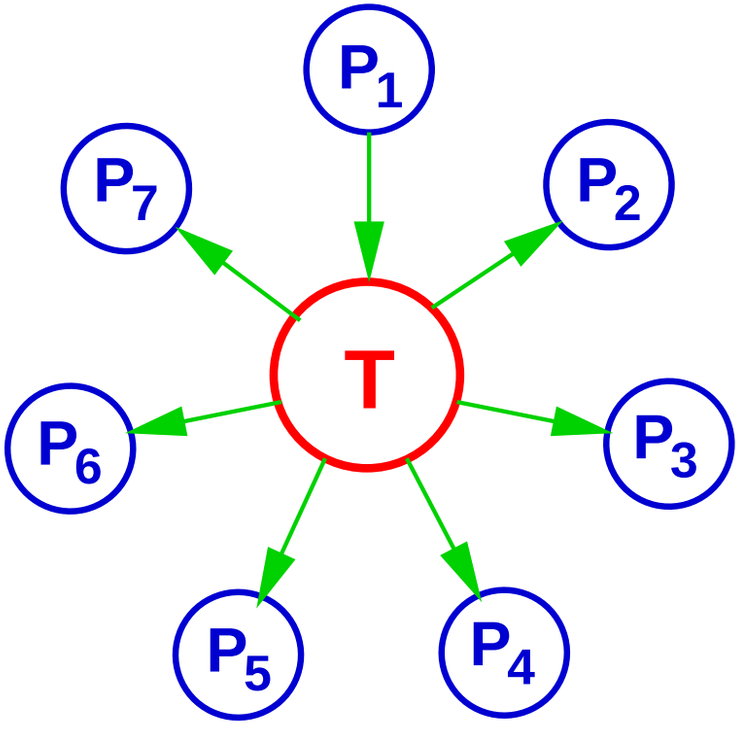
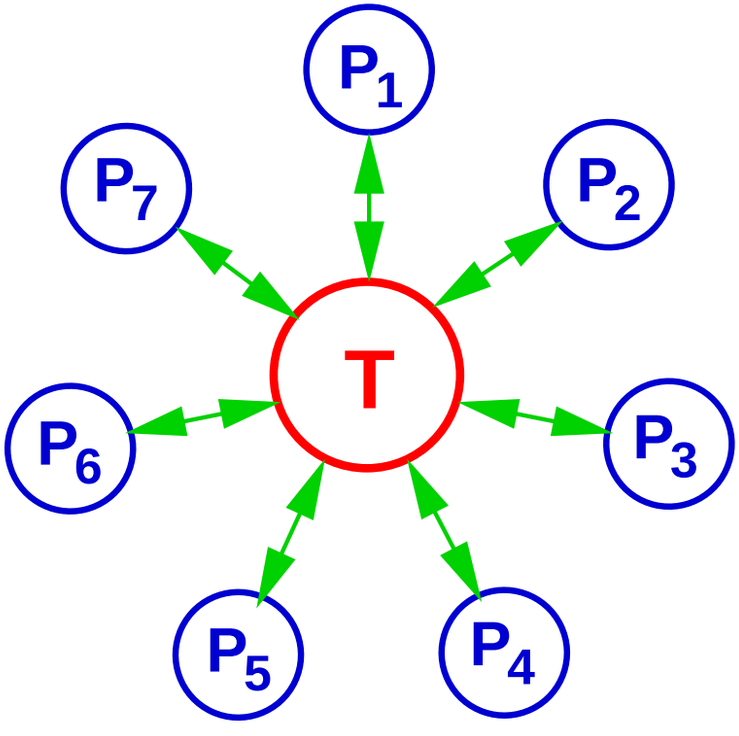


Theorem [LSP80]**:** **Among n players, broadcast is achievable if and only if t < n/3 players are corrupted.**

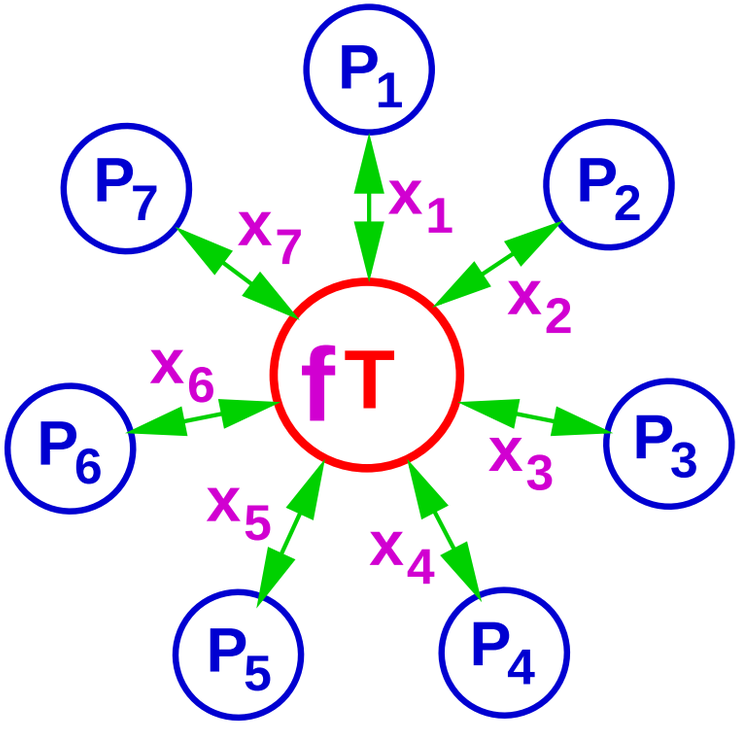
Generalization: Secure computation



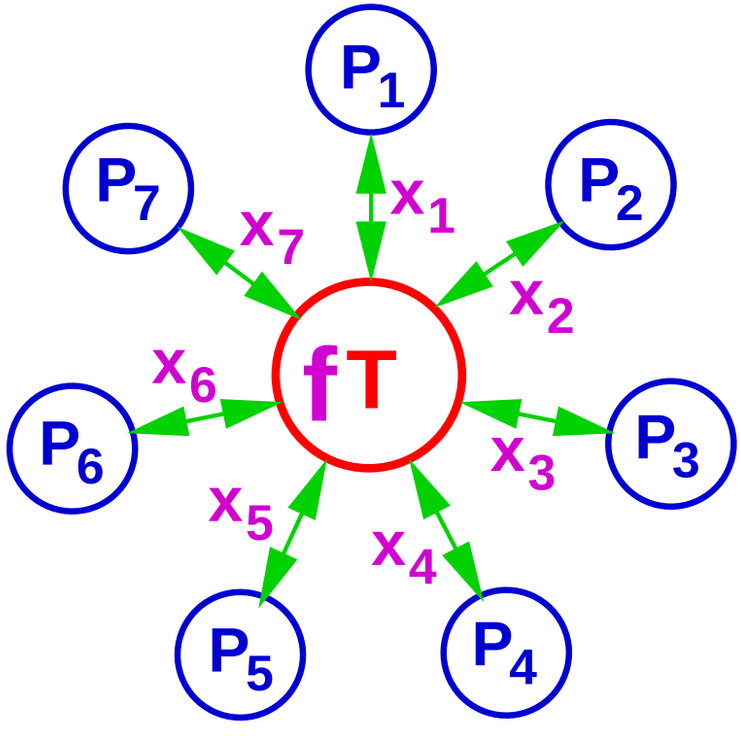
Generalization: Secure computation



Generalization: Secure computation

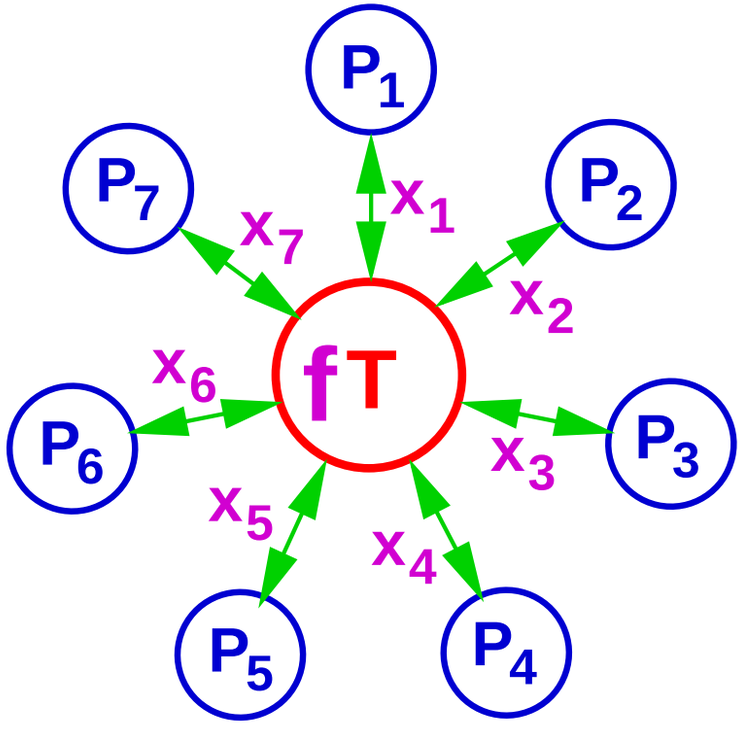


Generalization: Secure computation



T computes a function $f(x_1, \ldots, x_7)$ of the inputs.

Generalization: Secure computation



T computes a function $f(x_1, \ldots, x_7)$ of the inputs.

New operations of T:
- **receive secret input**
- **keep secret state**
- **perform operations on state**

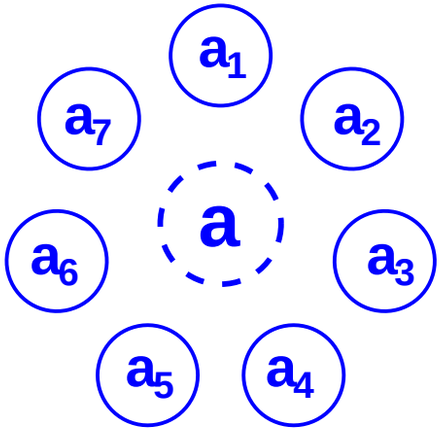
Some applications

- The millionaires' problem
- Preventing software piracy
- On-line auctions
- E-voting
- Secure aggregation of databases

Secret sharing

(t,n)-secret sharing: **Share a value a among n players such that**

- **any t players have no information about a,**
- **any t+1 players can reconstruct a.**



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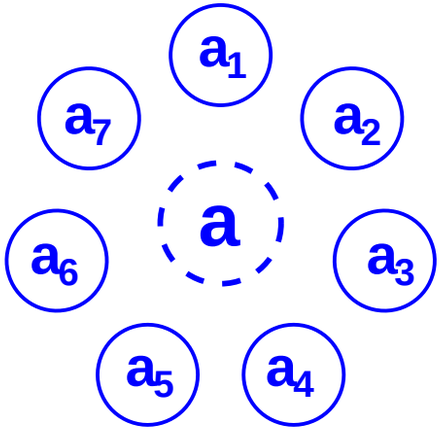
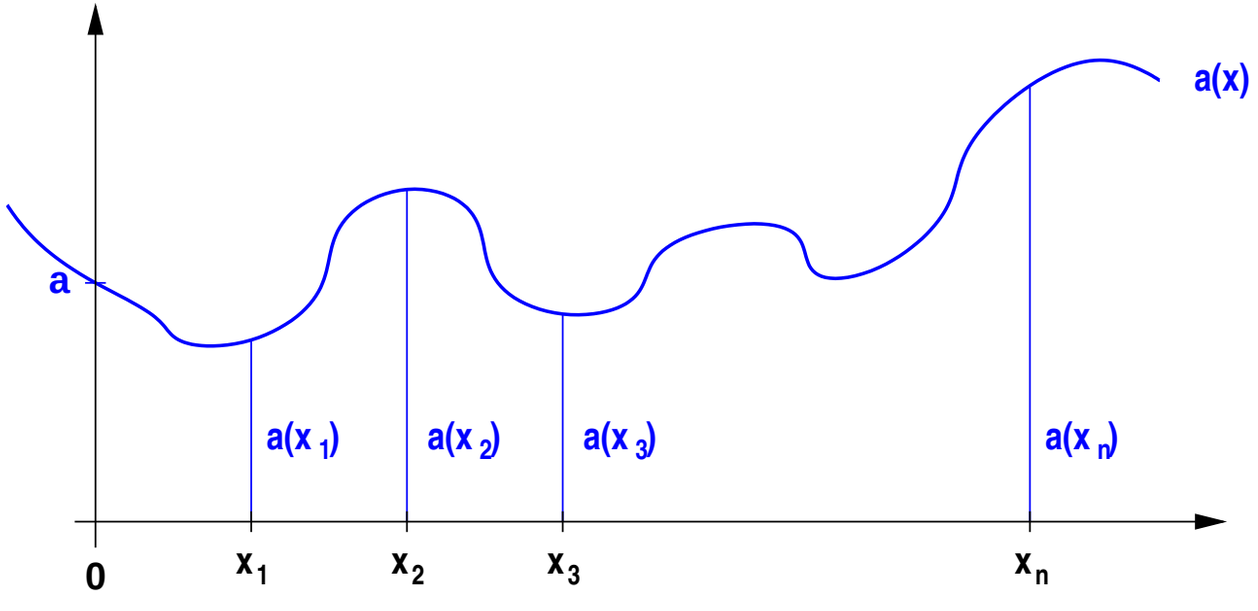
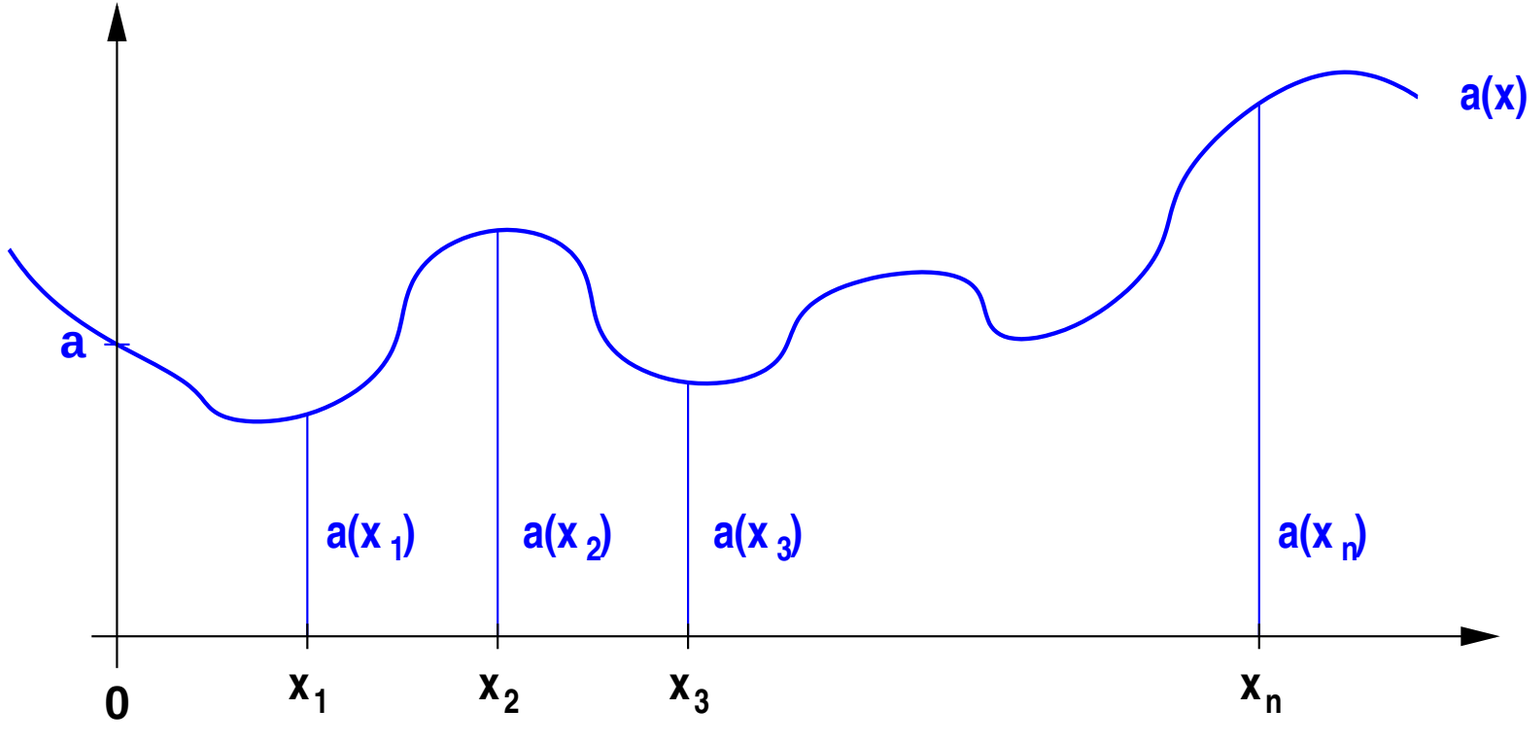




Every player is assigned a fixed value x_i from a finite field.

To share the value a, choose a random polynomial a(x) of degree t such that a(0)=a. The share of the i-th player is $a(x_i)$.

Addition of shared values



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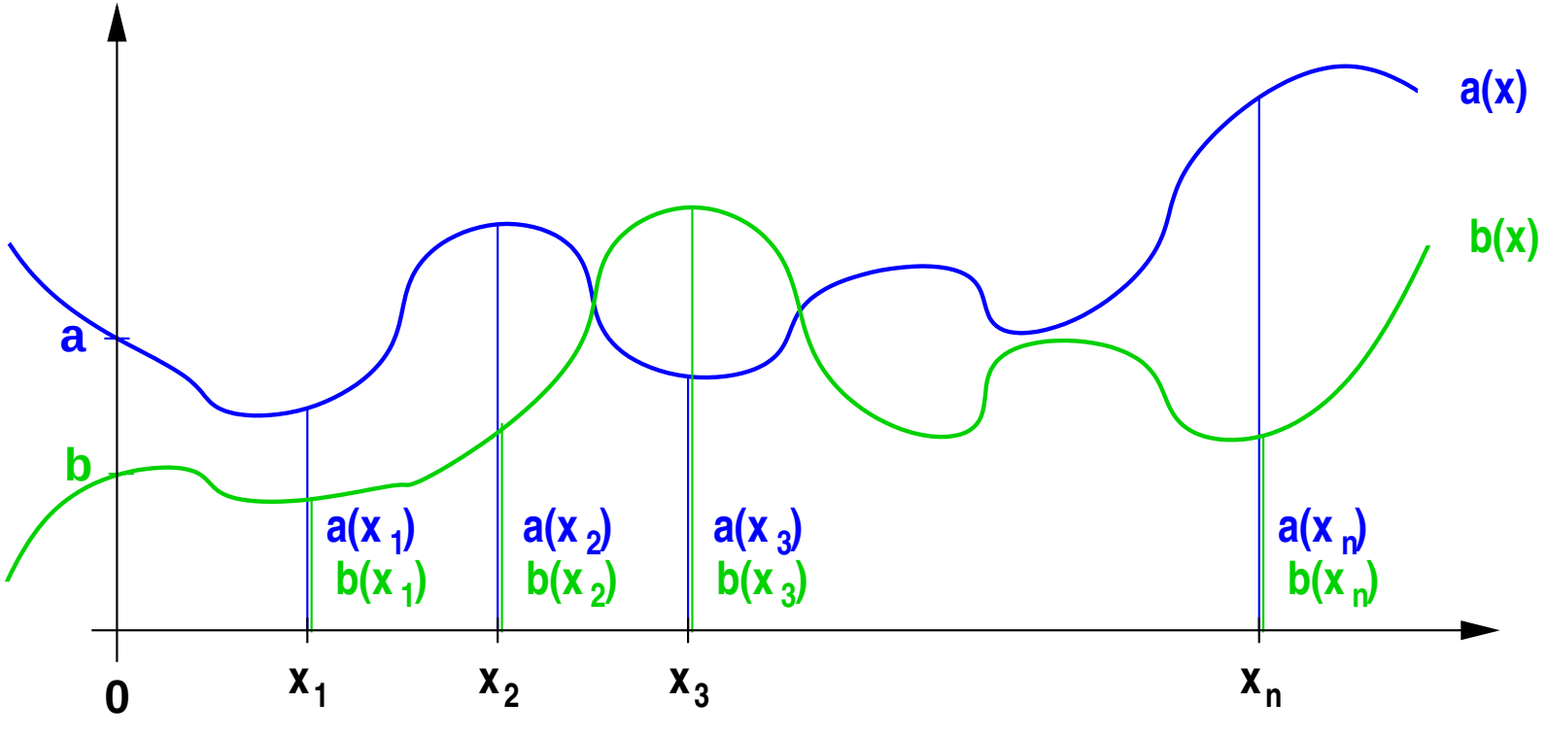

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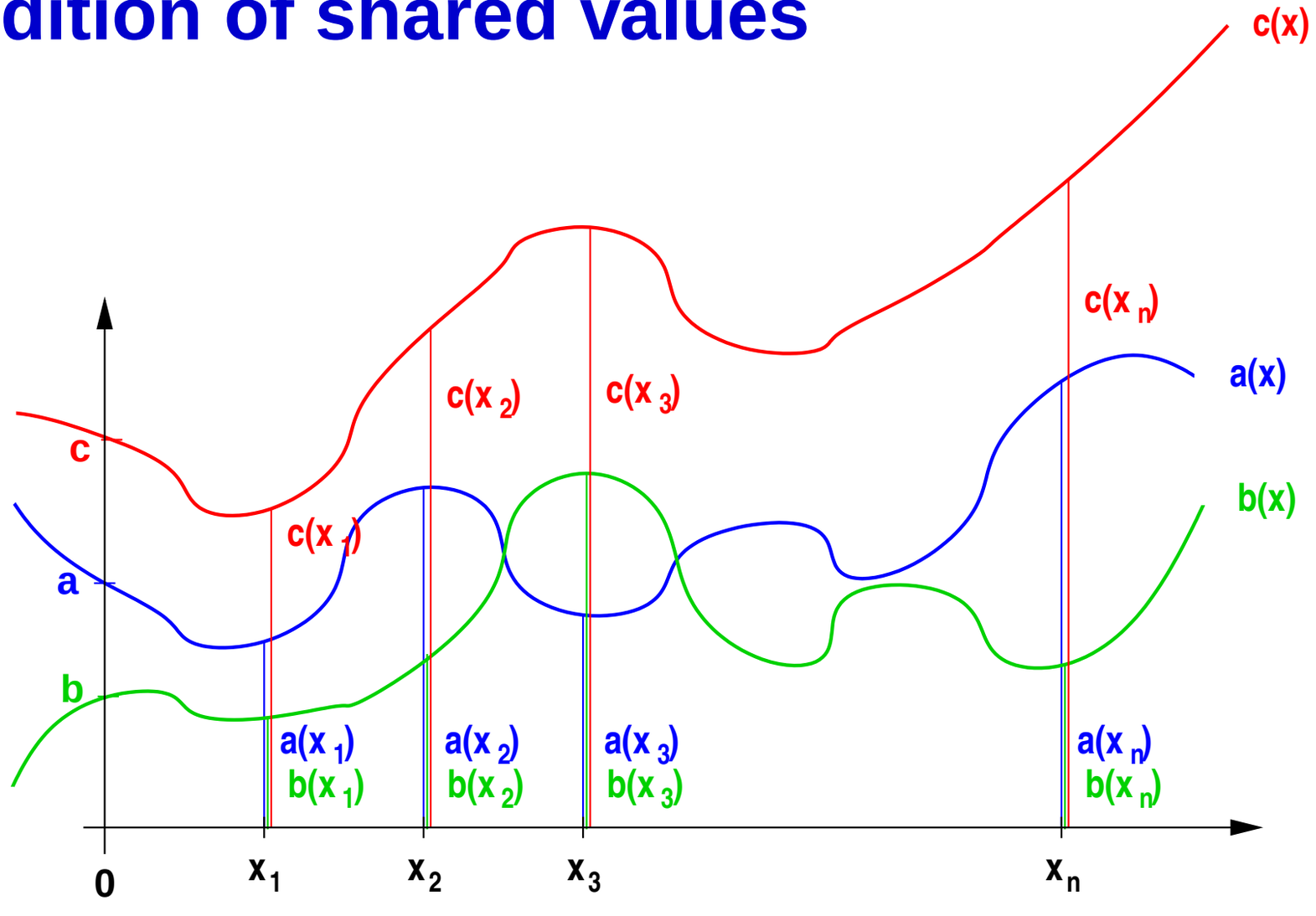

$c(x_i) = a(x_i) + b(x_i) \Rightarrow c(x) = a(x) + b(x) \Rightarrow c(0) = a(0) + b(0)$

Complexity of DL and factoring algorithms