Die Mathematik des Risikomanagements

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The solvency problem

- Liability holders of a financial institution are concerned that the institution may become **insolvent**, i.e.
- may fail to honor its future obligations.
- This is the case if the institution's **financial position** will be negative in some **future** state of the economy.
- By financial position we mean

$$\text{financial position} = \text{assets} - \text{liabilities}.$$

- How to operate in order to reduce the likelihood of insolvency?

A journey through citations (1)

- "when you cannot express it in numbers, your knowledge is of meagre and unsatisfactory kind"
  Lord Kelvin 1883

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- "...and if you can't measure it, measure it anyway"
  F. Knight upon reading the above quote of Lord Kelvin
- "if you can't measure it you can't manage it"
  Anonymous

The underlying mathematical description

- We consider a one-period economy with dates $t = 0$ and $t = T$.
- Let Ω be a (finite) state space equipped with a probability $\mathbb{P}$:
  - Ω represents the set of all future **scenarios** of the economy;
  - Ω is determined according to a scenario generation algorithm;
  - $\mathbb{P}$ is a **probability** measure assigning to every subset $A \subseteq \Omega$ its probability of occurrence $\mathbb{P}(A) \in [0, 1]$;
  - $\mathbb{P}$ is a frequency measure determined according to micro- and macro-economic analysis.
- We model assets and liabilities at time $t = T$ as random variables

$$A : \Omega \to \mathbb{R} \quad \text{and} \quad L : \Omega \to \mathbb{R}\,.$$

- The net financial position is also a random variable

$$X := A - L\,.$$

Financial positions

- Let $X := A - L$ be the financial position of a company. Then

$$X : \Omega \to \mathbb{R}$$

  is a random variable.

- The value $X(\omega) \in \mathbb{R}$ represents the **capital position** at time $t = T$ in case the scenario ω will occur.

- Three cases:

$$X(\omega) = A(\omega) - L(\omega) \begin{cases} > 0 & \text{(gain)} \\ = 0 & \text{(neither gain nor loss)} \\ < 0 & \text{(loss)} \end{cases}$$

- If $X(\omega) \geq 0$ for every $\omega \in \Omega$, the company is always solvent...

- but typically $\mathbb{P}(X < 0) > 0$.

The role of regulators

- To (among other issues) protect liability holders, financial institutions are subject to several **regulatory regimes**...
- and are required to hold **risk capital** as a buffer reserve against unexpected losses.
- Some regulatory frameworks:
  - Basel (now Basel III): banking system;
  - Solvency II: insurance companies within EU;
  - Swiss Solvency Test: insurance companies in CH.

How much risk capital?

- The key question is **how much** risk capital a financial institution should be required to hold to be deemed **adequately capitalized** by the regulator.
- The core of this lecture aims to describe the framework proposed in Artzner, Delbaen, Eber & Heath (1999).

Acceptable future positions

- The first step is to discriminate between "good" and "bad" financial positions...
- by introducing the concept of an acceptance set.
- A set $\mathscr{A}$ of random variables is called an **acceptance set** if

$$X \in \mathscr{A},\ Y \geq X \implies Y \in \mathscr{A}\,.$$

  This property is referred to as *monotonicity*.
- By $Y \geq X$ we mean $Y(\omega) \geq X(\omega)$ for all $\omega \in \Omega$.
- The acceptance set is **specified by the regulator**.

Testing for acceptability

- Let $\mathscr{A}$ be the acceptance set specified by the regulator.
- Testing whether a company is adequately capitalized or not reduces to establishing whether its financial position X belongs to $\mathscr{A}$ or not.
- Two situations:
  - if $X \in \mathscr{A}$, then the company is not required to hold risk capital;
  - if $X \notin \mathscr{A}$, then the company is forced to hold risk capital...
  - but how much?

Risk measures: Quantification of risk capital

- Risk capital is determined by using appropriate risk measures.
- Let $\mathscr{A}$ be the acceptance set specified by the regulator.
- Assume the financial position X of a company is such that $X \notin \mathscr{A}$.
- The amount of risk capital the company has to reserve is equal to

$$\rho_{\mathscr{A}}(X) := \inf\{m \in \mathbb{R}\,;\; X + m \in \mathscr{A}\}\,.$$

- We call $\rho_{\mathscr{A}}$ the **risk measure** associated to $\mathscr{A}$.
- The risk measure $\rho_{\mathscr{A}}$ gives a rule to compute risk capital according to the acceptance set $\mathscr{A}$:

$$X \quad \longmapsto \quad \rho_{\mathscr{A}}(X)\,.$$

Financial interpretation

- The risk measure associated to an acceptance set $\mathscr{A}$ is defined by
$$\rho_{\mathscr{A}}(X) := \inf\{m \in \mathbb{R};\ X + m \in \mathscr{A}\}\,.$$
- The quantity $\rho_{\mathscr{A}}(X)$ is an amount of capital, which we interpret as risk capital.
- More precisely, $\rho_{\mathscr{A}}(X)$ defines the minimal amount of capital that has to be added to X in order to transform X into an acceptable position (inf stands for the latin *infimum*).
- We might say that $\rho_{\mathscr{A}}(X)$ is the cost of making X acceptable.
- If $X \in \mathscr{A}$, then $\rho_{\mathscr{A}}(X) \leq 0$.
- If $X \notin \mathscr{A}$, then $\rho_{\mathscr{A}}(X) \geq 0$ and typically $\rho_{\mathscr{A}}(X) > 0$.

General properties of risk measures

- Let $\mathscr{A}$ be an acceptance set, and $\rho_{\mathscr{A}}$ the corresponding risk measure.
- The following properties hold:
  - $X \leq Y \Longrightarrow \rho_{\mathscr{A}}(X) \geq \rho_{\mathscr{A}}(Y)$ (monotonicity);
  - $\rho_{\mathscr{A}}(X + c) = \rho_{\mathscr{A}}(X) - c$ for all $c \in \mathbb{R}$ (cash additivity).
- *Proof*. If $X \leq Y$, then by the monotonicity of $\mathscr{A}$ we have

$$\{m \in \mathbb{R};\ X + m \in \mathscr{A}\} \subseteq \{m \in \mathbb{R};\ Y + m \in \mathscr{A}\}$$

and monotonicity of $\rho_{\mathscr{A}}$ follows by taking the inf on both sides. Now fix $c \in \mathbb{R}$. Then

$$\begin{aligned}
\rho_{\mathscr{A}}(X + c) &= \inf\{m \in \mathbb{R};\ X + c + m \in \mathscr{A}\} \\
&\overset{k=c+m}{=} \inf\{k - c \in \mathbb{R};\ X + k \in \mathscr{A}\} \\
&= \inf\{k \in \mathbb{R};\ X + k \in \mathscr{A}\} - c \\
&= \rho_{\mathscr{A}}(X) - c\,.
\end{aligned}$$

Example 1: Value-at-Risk

- Fix a level $\alpha \in (0,1)$, and define the acceptance set $\mathscr{A}_\alpha$ by

$$\mathscr{A}_{\mathrm{VaR}_\alpha} := \{X\,;\ \mathbb{P}(X<0) \le \alpha\}\,.$$

  Typically α is small, like $\alpha = 5\%$, $\alpha = 1\%$, $\alpha = 0.1\%$.

- Then $X \in \mathscr{A}_{\mathrm{VaR}_\alpha}$ is equivalent to X having a **default probability** capped by α.

Example 1: Value-at-Risk

- The corresponding risk measure $\rho_{\mathscr{A}_{\mathrm{VaR}_\alpha}}$ is called **Value-at-Risk**

$$\mathrm{VaR}_\alpha(X) := \rho_{\mathscr{A}_{\mathrm{VaR}_\alpha}}(X) = \inf\{m \in \mathbb{R}\,;\; \mathbb{P}(X + m < 0) \leq \alpha\}\,.$$

- Value-at-Risk is at the core of the Basel regimes and of the Solvency II regime.

Value-at-Risk for normal random variables

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$ be a normal random variable with mean μ and variance σ^2. For $Z \sim \mathcal{N}(0,1)$, set $\Phi(z) := \mathbb{P}(Z \leq z)$. Then

$$\mathrm{VaR}_\alpha(X) = -\mu - \sigma\,\Phi^{-1}(\alpha)\,.$$

- *Proof.* Since $X \sim \mathcal{N}(\mu, \sigma^2)$, we have $Z := \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. Hence we obtain

$$\begin{aligned}
\mathrm{VaR}_\alpha(X) &= \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \leq \alpha\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \mathbb{P}\left(Z < \frac{-m-\mu}{\sigma}\right) \leq \alpha\right\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \mathbb{P}\left(Z \leq \frac{-m-\mu}{\sigma}\right) \leq \alpha\right\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \Phi\left(\frac{-m-\mu}{\sigma}\right) \leq \alpha\right\} \\
&= \inf\left\{m \in \mathbb{R}\,;\ \frac{-m-\mu}{\sigma} \leq \Phi^{-1}(\alpha)\right\} \\
&= \inf\{m \in \mathbb{R}\,;\ m \geq -\mu - \sigma\,\Phi^{-1}(\alpha)\} \\
&= -\mu - \sigma\,\Phi^{-1}(\alpha)\,.
\end{aligned}$$

Value-at-Risk for discrete random variables

- How to compute $\mathrm{VaR}_\alpha(X)$ for some fixed level $\alpha \in (0,1)$ when X is a discrete random variable, for example

$$X = \begin{cases} 8 & [95\%] \\ -3 & [1\%] \\ 4 & [4\%] \end{cases} \qquad ?$$

- Consider a general discrete random variable X and reorder its outcomes in such a way that

$$X = \begin{cases} x_1 & [p_1] \\ x_2 & [p_2] \\ \cdots & \cdots \\ x_{n-1} & [p_{n-1}] \\ x_n & [p_n] \end{cases}$$

with

$$x_1 > x_2 > \ldots > x_n.$$

- For instance in the above case we would have:

$$X = \begin{cases} 8 & [95\%] \\ 4 & [4\%] \\ -3 & [1\%] \end{cases}$$

Value-at-Risk for discrete random variables: The algorithm

- Start with line n
  - If $p_n > \alpha$ then STOP and $\mathrm{VaR}_\alpha(X) = -x_n$
  - Else, proceed to line $n-1$
    - If $p_n + p_{n-1} > \alpha$, then STOP and $\mathrm{VaR}_\alpha(X) = -x_{n-1}$
    - Else, proceed to line $n-2$
    - If $p_n + p_{n-1} + p_{n-2} > \alpha$ then STOP and $\mathrm{VaR}_\alpha(X) = -x_{n-2}$
- $\cdots$ and so on

Summary:

- look at the probabilities defining X, starting from the lowest;
- as soon as you find a probability which is strictly greater than α, take the corresponding value of X, with the opposite sign.

Value-at-Risk for discrete random variables: Examples

- Consider again

$$X = \begin{cases} 8 & [p_1 = 95\%] \\ 4 & [p_2 = 4\%] \\ -3 & [p_3 = 1\%] \end{cases}$$

- We obtain based on the previous algorithm:
  - If $\alpha = 0.5\%$ then $p_3 > \alpha$ and $\mathrm{VaR}_\alpha(X) = 3$

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  - If $\alpha = 3\%$ then $p_3 < \alpha$ and $p_3 + p_2 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -4$

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  - If $\alpha = 4\%$ then $p_3 < \alpha$ and $p_3 + p_2 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -4$
  - If $\alpha = 5\%$ then $p_3 < \alpha$, $p_3 + p_2 = \alpha$ and $p_3 + p_2 + p_1 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -8$

Value-at-Risk for discrete random variables: Examples

- Consider again
$$X = \begin{cases} 8 & [p_1 = 95\%] \\ 4 & [p_2 = 4\%] \\ -3 & [p_3 = 1\%] \end{cases}$$
- We obtain based on the previous algorithm:
  - If $\alpha = 0.5\%$ then $p_3 > \alpha$ and $\mathrm{VaR}_\alpha(X) = 3$
  - If $\alpha = 3\%$ then $p_3 < \alpha$ and $p_3 + p_2 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -4$
  - If $\alpha = 4\%$ then $p_3 < \alpha$ and $p_3 + p_2 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -4$
  - If $\alpha = 5\%$ then $p_3 < \alpha$, $p_3 + p_2 = \alpha$ and $p_3 + p_2 + p_1 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -8$
  - If $\alpha = 8\%$ then $p_3 < \alpha$, $p_3 + p_2 < \alpha$ and $p_3 + p_2 + p_1 > \alpha$ so $\mathrm{VaR}_\alpha(X) = -8$.

VaR: important remarks

Drawbacks:

- note that by the very definition the VaR at the confidence level α does not give any information about the severity of losses which occur with a probability less than $1 - \alpha$
- discussing VaR from the point of view of coherence some further problems appear: no account for diversification
- problem: based on historical data one can make statements of the probability distribution

VaR: important remarks

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- discussing VaR from the point of view of coherence some further problems appear: no account for diversification
- problem: based on historical data one can make statements of the probability distribution
- Simon Johnson (MIT): "VaR misses everything that matters when it matters"

Mathematicians: drawback noticed!

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- Paper "An academic response to Basel II" of Jon Danielsson, Paul Embrechts, etc. (LSE 2001)
- Coherent risk measures: P. Artzner, F. Delbaen, etc. (1997, 1999)

In spite of the criticism, the VaR approach is still used, mainly due to the Basel requirements (Basel Committee for Banking Supervision).
$\rightarrow$ Questionable point of the international regulation!

An academic response to Basel II: some quotation

- "the proposed regulations fail to consider the fact that risk is endogenous, Value-at-Risk can destabilize an economy and induce crashes when they would not otherwise occur.
- statistical models used for forecasting risk have been proven to give inconsistent and biased forecasts, notably under-estimating the joint downside risk of different assets. The Basel Committee has chosen poor quality measures of risk when better risk measures are available.
- Heavy reliance on credit agencies for the standard approach to credit risk is misguided as they have been shown to provide conflicting and inconsistent forecasts of individual clients' creditworthiness. They are unregulated and the quality of their risk estimates is largely unobservable."

Example 2: Expected Shortfall

- Fix a level $\alpha \in (0,1)$, and define the acceptance set $\mathscr{A}^{\alpha}$ by

$$\mathscr{A}_{\mathrm{ES}_\alpha} := \left\{ X \, ; \, \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\beta(X)\, d\beta \leq 0 \right\} .$$

  Typically α is small, like above.

- If X has a continuous distribution, then $X \in \mathscr{A}_{\mathrm{ES}_\alpha}$ is equivalent to

$$\mathbb{E}[X 1_{\{X \leq -\mathrm{VaR}_\alpha(X)\}}] \geq 0 \, .$$

  This means that the expected shortfall of X beyond the VaR_α level is nonnegative.

Example 2: Expected Shortfall

- The corresponding risk measure $\rho_{\mathscr{A}_{\mathrm{ES}_\alpha}}$ is called **Expected Shortfall**

$$\mathrm{ES}_\alpha(X) := \rho_{\mathscr{A}_{\mathrm{ES}_\alpha}}(X) = \inf\{m \in \mathbb{R}\,;\; X + m \in \mathscr{A}_{\mathrm{ES}_\alpha}\}\,.$$

- It holds

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\,d\beta\,.$$

- Expected Shortfall is implemented in the Swiss Solvency Test. Open discussions about Basel III.

Expected Shortfall for normal random variables

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then it holds

$$\mathrm{ES}_\alpha(X) = -\mu + \frac{\sigma}{\alpha}\,\Phi'(\Phi^{-1}(\alpha))\,.$$

- *Proof.* Since $\mathrm{VaR}_\beta(X) = -\mu - \sigma\,\Phi^{-1}(\beta)$, we have

$$\begin{aligned}
\mathrm{ES}_\alpha(X) &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\,d\beta \\
&= \frac{1}{\alpha}\int_0^\alpha (-\mu - \sigma\,\Phi^{-1}(\beta))\,d\beta \\
&\overset{\beta=\Phi(z)}{=} -\mu - \frac{\sigma}{\alpha}\int_{-\infty}^{\Phi^{-1}(\alpha)} z\,\Phi'(z)\,dz \\
&\overset{\Phi'(z)=\frac{1}{\sqrt{2\pi}}\exp(-z^2/2)}{=} -\mu - \frac{\sigma}{\alpha}\big[-\Phi'(z)\big]_{-\infty}^{\Phi^{-1}(\alpha)} \\
&= -\mu + \frac{\sigma}{\alpha}\,\Phi'(\Phi^{-1}(\alpha))\,.
\end{aligned}$$

Value-at-Risk vs Expected Shortfall

- VaR is a **frequency-based** risk measure.
- A position X is acceptable for VaR if, and only if,

$$\mathbb{P}(X < 0) \leq \alpha$$

$\rightsquigarrow$ no information about the magnitude of a potential loss!

- ES is a **severity-based** risk measure.
- A (continuous) position X is acceptable for ES if, and only if,

$$\mathbb{E}[X 1_{\{X \leq -\mathrm{VaR}_\alpha(X)\}}] \geq 0$$

$\rightsquigarrow$ positions with fat left tail are likely to be unacceptable!

- There are less acceptable positions in the ES sense. Indeed, $\mathscr{A}_{\mathrm{ES}_\alpha} \subsetneq \mathscr{A}_{\mathrm{VaR}_\alpha}$ and so

$$\mathrm{ES}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)\,.$$

In other words, ES defines **higher risk capital**.

Coherent acceptance sets

- Artzner, Delbaen, Eber, and Heath (1999) introduced the concept of a coherent acceptance set.
- An acceptance set $\mathscr{A}$ is said to be **coherent** if
  - $X, Y \in \mathscr{A}, 0 < \lambda < 1 \implies \lambda X + (1-\lambda)Y \in \mathscr{A}$ (convexity);
  - $X \in \mathscr{A}, \lambda \geq 0 \implies \lambda X \in \mathscr{A}$ (conicity).
- Financial interpretation:
  - convexity = portfolios of acceptable positions are still acceptable, i.e. acceptability is preserved by diversification;
  - conicity = acceptability is independent of the position size.

Coherent risk measures

- If $\mathscr{A}$ is a coherent acceptance set, we say that $\rho_{\mathscr{A}}$ is a **coherent risk measure**. It has the following properties:
  - for all X, Y and $0 < \lambda < 1$

    $$\rho_{\mathscr{A}}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_{\mathscr{A}}(X) + (1-\lambda)\rho_{\mathscr{A}}(Y) \quad \text{(convexity);}$$

  - for all X and $\lambda \geq 0$

    $$\rho_{\mathscr{A}}(\lambda X) = \lambda \rho_{\mathscr{A}}(X) \quad \text{(positive homogeneity).}$$

- Financial interpretation:
  - convexity $=$ the risk capital for an aggregated portfolio is controlled by the risk capital of its components;
  - positive homogeneity $=$ risk capital is proportional to the position size.

A remark on subadditivity

- Coherent risk measures are sometimes defined to be subadditive.
- Let $\mathscr{A}$ be an acceptance set satisfying conicity. Then convexity of $\mathscr{A}$ is equivalent to

$$X, Y \in \mathscr{A} \implies X + Y \in \mathscr{A} \quad \text{(closedness under addition).}$$

- Let $\mathscr{A}$ be an acceptance set satisfying conicity. Then convexity of $\rho_{\mathscr{A}}$ is equivalent to

$$\rho_{\mathscr{A}}(X+Y) \leq \rho_{\mathscr{A}}(X) + \rho_{\mathscr{A}}(Y) \;\; \text{for all } X, Y \;\; \text{(subadditivity).}$$

- Financial interpretation:
  - closedness under addition = acceptability is preserved by merging;
  - subadditivity = the risk capital of a merged position is controlled by the risk capital of the individual positions.

Value-at-Risk vs Expected Shortfall (2)

- VaR and ES are both positively homogeneous;
- VaR is not convex, hence not coherent:
  $\rightsquigarrow$ VaR may penalize diversification!
- ES is convex, hence coherent:
  $\rightsquigarrow$ ES captures a diversification benefit.

Value-at-Risk is not convex (hence not coherent)

- VaR is positively homogeneous.
- *Proof.* Recall $\mathscr{A}_\alpha = \{X\,;\ \mathbb{P}[X < 0] \leq \alpha\}$. Take $X \in \mathscr{A}_\alpha$ and $\lambda > 0$. Then
$$\mathbb{P}[\lambda X < 0] = \mathbb{P}[X < 0] \leq \alpha\,,$$
showing that $\lambda X \in \mathscr{A}_\alpha$. Moreover, $0 = 0X \in \mathscr{A}_\alpha$.
- We will show that VaR is not convex, i.e. for some X, Y and $0 < \lambda < 1$
$$\mathrm{VaR}_\alpha(\lambda X + (1-\lambda)Y) > \lambda\,\mathrm{VaR}_\alpha(X) + (1-\lambda)\,\mathrm{VaR}_\alpha(Y)\,.$$
Hence Value-at-Risk may penalize diversification!
- Lack of convexity typically depends on:
    - skew distributions;
    - fat-tailed distributions;
    - copula structure.

Expected Shortfall is coherent

- It is possible to prove that the acceptance set $\mathscr{A}_{\mathrm{ES}_\alpha}$ is coherent, hence
- the risk measure $\mathrm{ES}_\alpha = \rho_{\mathscr{A}_{\mathrm{ES}_\alpha}}$ is also coherent.
- The proof is not difficult but long. We refer to Section 4.4 in Föllmer and Schied (2011).

A journey through citations (2)

The Economist, issue Feb. 11, 2010: *Number-crunchers crunched*!

- "Thanks to Black-Scholes, ... quants poured into the industry. By 2005 they accounted for 5% of all finance jobs, against 1.2% in 1980, and probably a much higher proportion of pay.
  By 2007 finance was attracting a quarter of all graduates from the California Institute of Technology."
- "Models increased risk exposure instead of limiting it"' says Mr Taleb. "They can be worse than nothing, the equivalent of a dangerous operation on a patient who would stand a better chance if left untreated".
- "Not all models were useless. Those for interest rates and foreign exchange performed roughly as they were meant to."

A journey through citations (3)

S. Shreve, *Don't blame the quants*, 2008

- "When a bridge collapses, no one demands the abolition of civil engineering. [...]
  If engineering is to blame the solution is

A journey through citations (3)

S. Shreve, *Don't blame the quants*, 2008

- "When a bridge collapses, no one demands the abolition of civil engineering. [...]
  If engineering is to blame the solution is
  better – not less – engineering.

  Furthermore, it would be preposterous to replace the bridge with a slower, less efficient ferry rather than to rebuild the bridge and overcome the obstacle"

What to do now?

We need:

- transparent financial products and robust procedures
- stronger buffers for risk
- an intelligent regulation of financial markets on international level
- appropriates stimulating mechanisms

$\Longleftarrow$ this requires more quantitative analysis and mathematics, not less!

What to do now?

We need:

- transparent financial products and robust procedures
- stronger buffers for risk
- an intelligent regulation of financial markets on international level
- appropriates stimulating mechanisms

$\Longleftarrow$ this requires more quantitative analysis and mathematics, not less!
… and this especially for the project of a stronger international regulation!

More or less?

We need more quantitative analysis:

- even a regulatory measure which seems very plausible at the beginning could be an open door for arbitrage-strategies; here one needs careful analysis supported also by mathematical models
- even for incentive schemes mathematics is needed! In the economics literature it belongs to the so-called *principal-agent* problem and to the *mechanism design*

More or less?

We need more quantitative analysis:

- even a regulatory measure which seems very plausible at the beginning could be an open door for arbitrage-strategies; here one needs careful analysis supported also by mathematical models
- even for incentive schemes mathematics is needed! In the economics literature it belongs to the so-called *principal-agent* problem and to the *mechanism design*

We need less quantitative analysis:

- there is no "correct" mathematical model for the financial markets
- any model is somehow "naive" and can become dangerous when it replaces the reality

Closing

- This talk aimed to the conclusion that rules will have to be both tightened and better enforced to avoid future crises but that all the reforms in the world will never guarantee total safety :)

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THANK YOU
FOR YOUR ATTENTION !

http://www.math.ethz.ch/~farkas

Invitation:
Risk Day at ETH Zürich: Friday, 11. September 2015

Reading material (Selection)

- Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk, *Mathematical Finance*, 9: 203-228 (1999)
- Farkas, W., Koch-Medina, P., Munari, C.: Capital requirements with defaultable securities, *Insurance: Mathematics and Economics*, 55, 58-67, (2014)
- Föllmer, H., Schied, A.: *Stochastic finance: an introduction in discrete time*, De Gruyter (2011)
- Kalkbrenner, M., Lotter, H., Overbeck, L.: Sensible and efficient capital allocation for credit portfolios, *RISK*, 19-24 (2004)
- Leippold, M., Trojani, F., Vanini, P.: Equilibrium impact of Value-at-Risk regulation, *Journal of Economic Dynamics and Control*, 30: 1277-1313 (2006)
- Mc Neil, A.J., Frey, R., Embrechts, P.: *Quantitative Risk Management: Concepts, Techniques, Tools*, Princeton University Press (2005)

Value-at-Risk is not convex: Example on credit portfolios (1)

- You are a bank and you give a loan of 100 CHF.
  - The loan interest rate is $r = 2\%$.
  - The default probability of the counterpart is $p = 0.8\%$.
  - The VaR-level is $\alpha = 1\%$.
- The corresponding position at maturity is

$$X = \begin{cases} 100r = 100(1+r) - 100 & [\,1 - p = 99.2\%\,] \\ -100 & [\,p = 0.8\%\,] \end{cases}$$

- Recall

$$\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X + m < 0) \le \alpha\}\,.$$

- How to compute $\mathrm{VaR}_\alpha(X)$?

Value-at-Risk is not convex: credit portfolios (2)

- Recall $\alpha = 1\%$ and

$$X = \begin{cases} 100r & [1-p = 99.2\%] \\ -100 & [p = 0.8\%] \end{cases}$$

- It holds $\mathrm{VaR}_\alpha(X) = -100r$.
- *Proof.* We have

$$\mathbb{P}(X+m<0) = \begin{cases} 1 & \text{if } m < -100r \\ p = 0.8\% & \text{if } -100r \le m < 100 \\ 0 & \text{if } m \ge 100 \end{cases}$$

Hence $\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R}\,;\ \mathbb{P}(X+m<0) \le \alpha\} = -100r$.

- "Trick": look at the probabilities defining X, starting from the lowest. As soon as you find a probability which is strictly greater than α, take the corresponding value of X, with the opposite sign.

Value-at-Risk is not convex: credit portfolios (3)

- Assume you want to diversify, and you give two 50 CHF loans.
  - The default probability of each counterpart is $p = 0.8\%$.
  - Defaults are independent.
- Take $Y, Z \sim X$ with Y, Z independent. The new position at maturity is

$$\frac{1}{2}Y + \frac{1}{2}Z = \begin{cases} 100r & [\,(1-p)^2 = 98.4064\%\,] \\ 50r - 50 & [\,2p(1-p) = 1.5872\%\,] \\ -100 & [\,p^2 = 0.0064\%\,] \end{cases}$$

- Given $\alpha = 1\%$, what is $\mathrm{VaR}_\alpha(\frac{1}{2}Y + \frac{1}{2}Z)$?
- Using the "trick" we get $\mathrm{VaR}_\alpha(\frac{1}{2}Y + \frac{1}{2}Z) = -(50r - 50)$.

Value-at-Risk is not convex: credit portfolios (4)

- Finally we show that Value-at-Risk is not convex.
- Indeed, on one side

$$\mathrm{VaR}_\alpha\left(\frac{1}{2}Y + \frac{1}{2}Z\right) = 50 - 50r = 49\,,$$

while on the other

$$\begin{aligned} \frac{1}{2}\mathrm{VaR}_\alpha(Y) + \frac{1}{2}\mathrm{VaR}_\alpha(Z) &= \frac{1}{2}\mathrm{VaR}_\alpha(X) + \frac{1}{2}\mathrm{VaR}_\alpha(X) \\ &= \mathrm{VaR}_\alpha(X) \\ &= -100r \\ &= -2\,. \end{aligned}$$

- As a result,

$$\mathrm{VaR}_\alpha\left(\frac{1}{2}Y + \frac{1}{2}Z\right) > \frac{1}{2}\mathrm{VaR}_\alpha(Y) + \frac{1}{2}\mathrm{VaR}_\alpha(Z)\,.$$